Last time:

$M|K$ finite $\Rightarrow$ $X_{M|K} = G/H$ G-set

$$G \left(\begin{array}{c} F \\ | H \\ M \\ | \\ K \end{array} \right.$$

$\rightsquigarrow \mathbb{C}[X_{M|K}]$ representation of G

Thm $\zeta_{M|K}(s) = L(\mathbb{C}[G/H], s)$

$\zeta_{M|K}$: (relative) Dedekind ζ-fnc.

$L(\mathbb{C}[G/H], s)$: Artin L-fnc

§9 Characters & Induction

Character theory: G finite, $\rho : G \to GL(V)$ representation

Def The character of V [of ρ]

$$\chi_\rho = \chi_V : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \operatorname{tr} \rho(g)$$

- $\chi_V(e) = \dim V$.
- ρ 1-dimensional "$\chi_\rho = \rho$".

$\operatorname{tr} \rho(hgh^{-1}) = \operatorname{tr} \rho(g) \Rightarrow \chi_V$ constant on conjugacy classes.

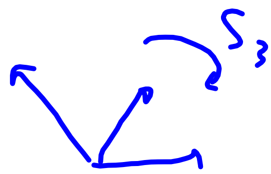
__Def__ __Inner product__ $\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)}$

__Ex__ $V = \mathbb{C}[X]$ permutation rep.

$\chi_V(g) = \#\{x \in X \mid g \cdot x = x\}$ $\qquad \rho(g) = \begin{pmatrix} 0 & 0 & & 1 \\ 0 & 1 & & 0 \\ 1 & 0 & \ddots & \vdots \\ 0 & | & & \vdots \\ \vdots & | & & 0 \; 1 \\ 0 & 0 & & \end{pmatrix}$

$= \#$fixed pts of g on X.

<u>Ex</u> $G = S_3 \circlearrowleft X = \{1,2,3\}$, $V = \mathbb{C}[X]$
(3-dim)



$\mathcal{C} = \{\text{conj. classes}\} =$

$= \{[e], [(12)], [(123)]\}$

$\chi_V = (3, 1, 0) : \mathcal{C} \longrightarrow \mathbb{C}$

$$\langle \chi_V, \chi_V \rangle = \frac{1}{6}\left[3 \cdot \overline{3} \cdot \underline{1} + 1 \cdot \overline{1} \cdot \underline{3} + 0\right]$$

$= 2$

sizes of conj. classes.

<u>Thm</u> G finite, $\mathcal{C} = \{\text{conj. classes}\}$,

$\mathcal{I} = \{\text{irr. reps } V_1, V_2, \ldots\}/\cong$

└ irreducible means no

$\tilde{V} \subseteq V_i$ G-stable except $0, V_i$

- $|\mathcal{I}| = \overset{k}{\overset{\shortparallel}{|\mathcal{C}|}}$, $\dim V_i \mid |G|$, $\sum_{i=1}^{k} \dim V_i^2 = |G|$
- $\{$Every $V \cong V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}$ some $n_i \geq 0$, unique "complete reducibility"

- If $W = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$ then

$$\langle \chi_W, \chi_V \rangle = \langle \chi_V, \chi_W \rangle = \sum_{i=1}^{k} n_i m_i$$

$$= \dim_{\mathbb{C}} \mathrm{Hom}_G(V, W),$$

in particular

- $\langle \chi_V, \chi_V \rangle = \sum n_i^2$
- $\langle \chi_{V_i}, \chi_{V_j} \rangle = \delta_{ij}$

- V irr. $\Leftrightarrow$ $\langle \chi_V, \chi_V \rangle = 1.$

orthogonality.

- $\chi_V + \chi_W = \chi_{V \oplus W}$ (trivial)

$$\chi_V \chi_W = \chi_{V \otimes W}$$

$$\overline{\chi_V} = \chi_{V^*}$$

dual rep

$g \longmapsto (\rho(g)^t)^{-1}$

Ex G abelian. $\Leftrightarrow$ $|\mathcal{C}| = G$, $|\mathcal{I}| = |G|$.

$\sum \dim^2 = |G| \Rightarrow$ all $V_i \in \mathcal{I}$ 1-dimensional

$\{$irr. reps. of $G\} = \hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$.

For any G

$$\left\{\begin{matrix}1\text{-dim reps}\\ \text{of } G\end{matrix}\right\} = \hat{G} = \widehat{\frac{G}{[G,G]}}$$

maximal abelian quotient of G.

so $\#\{1\text{-dim. reps}\} = (G : [G,G])$.

<u>Ex</u> $G = S_4$

$\mathcal{C} = \{ e, [(12)], [(123)], [(1234)], [(12)(34)] \}$

$\Rightarrow |\mathcal{I}| = 5$; every rep. of S_4

$\cong V_1^{\oplus n_1} \oplus \cdots \oplus V_5^{\oplus n_5}$.

5 irr. reps ρ_i of dim $\underbrace{1, 1}$, ?, ?, ?

from $G/[G,G] = S_4/A_4 = C_2$

$$\sum_{i=1}^{5} \dim \rho_i^2 = |G| = 24 \Rightarrow 1, 1, 2, 3, 3.$$

- $\rho_1 = \mathbb{1} : S_4 \longrightarrow GL_1(\mathbb{C})$
  $\forall g \longmapsto 1$
  $\boxed{\chi_{\rho_1} = (1,1,1,1,1)}$

- $\rho_2 = \text{sign} : S_4 \longrightarrow GL_1(\mathbb{C})$
  $A_4 \longmapsto 1$
  $\text{rest} \longmapsto -1$
  $\boxed{\chi_{\rho_2} = (1,-1,1,-1,1)}$

- $S_4 \circlearrowright \{1,2,3,4\} \Rightarrow \pi \quad \chi_\pi = \{\#\text{fixed pts}\}$

$$= (4,2,1,0,0)$$

$$\langle \chi_\pi, \chi_\pi \rangle = 2 \;,\; \langle \chi_\pi, \chi_{\mathbb{1}} \rangle = 1$$

$$\Rightarrow \pi \cong \mathbb{1} \oplus \rho_4 \quad (\text{say}).$$

$$\boxed{\chi_{\rho_4} = \chi_\pi - \chi_{\mathbb{1}} = (3,1,0,-1,-1)}$$

- $\boxed{\chi_5 = \chi_2\chi_4 = (3,-1,0,1,-1)}$

- $\boxed{\chi_3 = (2,0,-1,0,2)}$

Exc Get it by

a) lifting from $S_4/V_4 \cong S_3$

b) from $\chi_1,\dots,\chi_4$ using $\langle \chi_i, \chi_j \rangle = \delta_{ij}$

c) from $\chi_{\mathbb{C}[G]} = \sum_{i=1} \dim \rho_i \cdot \chi_{\rho_i}$

d) from $\chi_5 \chi_5$

Alternatively, use induction:

<u>Thm</u> $H < G$ index d. There are maps

$$\begin{tikzcd} \text{Reps of } H \arrow[r, bend right, "\operatorname{Ind}_H^G"'] & \text{Reps of } G \arrow[l, bend right, "\operatorname{Res}_H"'] \end{tikzcd}$$

n dim $\leftarrow$ n dim

n dim $\mapsto$ dn-dim

$\rho : G \longrightarrow GL(V)$

such that for all reps $\sigma : H \longmapsto GL(W)$

- $\langle V, \operatorname{Ind} W \rangle_G = \langle \operatorname{Res} V, W \rangle_H$

$\nwarrow$ Frobenius reciprocity

- $\mathrm{Res}_H V$ = same V with H action $(H < G)$

$$\chi_{\mathrm{Res}_H V}(h) = \chi_V(h)$$

- $\mathrm{Ind}_H^G W = \{ f : G \to W \mid f(hg) = \sigma(h) f(g) \ \forall h \in H, g \in G \}$

$\circlearrowleft$

$g \in G$ acts by $f(x) \mapsto f(xg)$.

$$\chi_{\mathrm{Ind}_H^G W}(g) = \frac{1}{|G|} \sum_{x \in G} \chi_W^\circ(xgx^{-1})$$

χ_W on H, 0 else.

- $\mathrm{Ind}_H^G \mathbb{1} \cong \mathbb{C}[G/H]$

§ 10 Artin formalism

Thm (L-fncs are invariant under induction)

$$\begin{tikzcd} F \arrow[d, dash, "d \mid H"] \\ M \arrow[d, dash] \\ K \arrow[uu, bend left, dash, "G"] \end{tikzcd}$$

If $\rho : H \longrightarrow GL_n(\mathbb{C})$ is an Artin rep. then

$$L(\underbrace{\rho}_{\substack{\text{rep. of } G_M \\ \text{of dim } n}}, s) = L(\underbrace{\mathrm{Ind}_H^G}_{\substack{\text{rep. of } G_K \\ \text{of dim } n \cdot d}} \rho, s)$$

Pf Same argument as for $\rho = \mathbb{1}$,

$$\mathrm{Ind}_H^G \rho = \mathbb{C}[G/H]$$

Instead of

as a D-set $G/H = \coprod_{g_i \in D\backslash G/H} D/_{D\cap g_i H g_i^{-1}}$

use Mackey's formula

$$\mathrm{Res}_D \mathrm{Ind}_H^G \rho = \bigoplus_{g_i \in D\backslash G/H} \mathrm{Ind}_{D\cap g_i H g_i^{-1}}^D \rho^{g_i}$$

<u>Thm</u> (Brauer induction) $\rho : G \to Gl_n(\mathbb{C})$ rep.

Then

$$\chi_\rho = \sum_i n_i \operatorname{Ind}_{H_i}^{G} \chi_i$$

for some $n_i \in \mathbb{Z}$, $H_i < G$, $\sigma_i : H_i \to \mathbb{C}^\times$ <u>1-dim reps.</u>

may be taken to be cyclic × p-group

<u>Rmk</u> Used to construct character tables of groups

Cor Every Artin L-fnc can be written as

$$L(\rho, s) = \prod_i L(\sigma_i, s)^{n_i} \leftarrow \text{Hecke L-fncs}$$

$\rho : G_K \longrightarrow GL_n(\mathbb{C})$

$\sigma_i : G_{M_i} \longrightarrow \mathbb{C}^\times$

M_i/K finite

In particular,

$L(\rho, s)$ is meromorphic on $\mathbb{C}$ and satisfies fun. eq. under $s \leftrightarrow 1-s$.

<u>Conj</u> (Artin) $\rho : G_{\mathbb{Q}} \to GL_n(\mathbb{C})$

irr. Artin rep., $\rho \neq \mathbb{1}$. Then $L(\rho, s)$

has <u>analytic</u> cont. to $\mathbb{C}$.

<u>Rmk</u> The two properties

$L(V_1 \oplus V_2, s) = L(V_1, s) L(V_2, s)$

$L(\mathrm{Ind} V, s) = L(V, s)$

that define L-fncs uniquely from those of 1-dim reps are called <u>Artin formalism</u>

<u>Ex</u> $K = \mathbb{Q}$.

$M = \mathbb{Q}(\sqrt[4]{2}) \leftarrow$ root of $x^4 - 2$

$F = \mathbb{Q}(\sqrt[4]{2}, i) \leftarrow$ all 4 roots of $x^4 - 2$

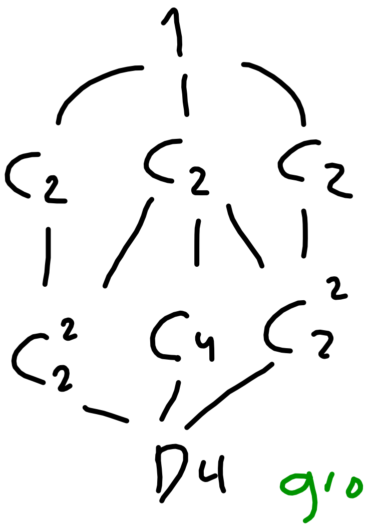
$$\begin{tikzcd}
\sqrt[4]{2} \arrow[r] & i\sqrt[4]{2} \arrow[d] \\
-i\sqrt[4]{2} \arrow[u] & -\sqrt[4]{2} \arrow[l]
\end{tikzcd} \quad \sigma$$

$\tau =$ complex conjugation

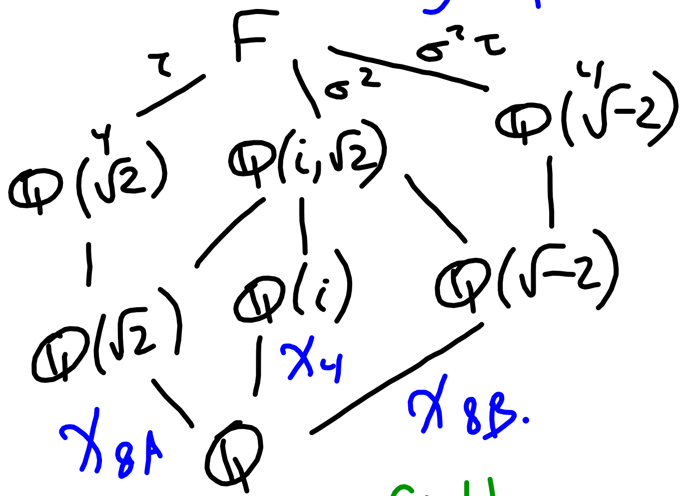
$G = \mathrm{Gal}(F/K) \cong D_4$
$= \langle \sigma, \tau \rangle$

Note $\sqrt[4]{-2} = \zeta_8 \cdot \sqrt[4]{2}$

Note See D_4 on groupnames.org



groups



fields

characters of irr. reps.

	1	σ^2	τ	σ	$\sigma\tau$
$\mathbb{1}$	1	1	1	1	1
χ_4	1	1	-1	1	-1
χ_{8A}	1	1	1	-1	-1
χ_{8B}	1	1	-1	-1	1
ψ	2	-2	0	0	0

std rep of $D_4 \to GL_2(\mathbb{C})$

L irr. characters; $\dim \rho_i = 1,1,1,1,2$, $\sum \dim^2 = 8$.

Commutator $G' = Z(G) = \{e, \sigma^2\}$

cuts out maximal abelian extension of $\mathbb{Q}$ in F

$$F^{G'} = \mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(\zeta_8)$$

$$Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times \cong C_2 \times C_2$$

has 1-dim reps $\mathbb{1}$, χ_4, χ_{8A}, χ_{8B}

$\left(\frac{-1}{\cdot}\right)$, $\left(\frac{2}{\cdot}\right)$, $\left(\frac{-2}{\cdot}\right)$

$\rightsquigarrow$ Dirichlet L-function.

One other 2-dim. irr.rep. with character ψ

$\rightsquigarrow L(\psi, s)$ of degree 2.

$$L(\psi,s) = \underset{\substack{\mathcal{I}_2 = D_4 \\ \text{no invariants on } \mathbb{C}^2}}{\underset{\mathcal{C}}{1}} \cdot \frac{1}{1-(3^{-s})^2} \; \underbrace{\frac{1}{1+(5^{-s})^2}}_{\substack{\text{Frob}_5 \text{ rotation} \\ \text{char poly } 1+T^2}} \; \underbrace{\frac{1}{1-(7^{-s})^2}}_{\substack{\text{Frob}_7 \text{ reflection} \\ \text{char poly } 1-T^2}} \cdots$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{with } a_p = \psi(\mathrm{Frob}_p)$$

$$(p \nmid \Delta_F)$$

All ζ-fncs of subfields of F are products of these, e.g

$$\zeta_{\mathbb{Q}(\sqrt[4]{2})}(s) = L(\mathbb{C}[G/\langle\tau\rangle], s)$$

G-set $\{1,2,3,4\}$ with natural D_4-action

$$\chi_{\mathbb{C}[G/\langle\tau\rangle]} = (4,0,2,0,0)$$

$$= \underset{\mathbb{1}}{(1,1,1,1,1)} + \underset{\chi_{8A}}{(1,1,1,-1,-1)} + \underset{\psi}{(2,-2,0,0,0)}$$

so $\zeta_{\mathbb{Q}(\sqrt[4]{2})}(s) = L(\mathbb{1},s)\,L(\chi_{8A},s)\cdot L(\psi,s)$

$$= \zeta_{\mathbb{Q}(\sqrt{2})}(s)\cdot L(\psi,s)$$

and similarly

$$\zeta_{\mathbb{Q}(\sqrt[4]{-2})}(s) = L(\mathbb{1}, s)\, L(\chi_{B}, s)\, L(\psi, s)$$
$$= \zeta_{\mathbb{Q}(\sqrt{-2})}(s)\, L(\psi, s)$$

$$\zeta_{\mathbb{Q}(i,\sqrt{2})}(s) = L(\mathbb{1}, s)\, L(\chi_4, s)\, L(\chi_{8A}, s)\, L(\chi_{8B}, s)$$
$$= \frac{\zeta_{\mathbb{Q}(i)}(s)\, \zeta_{\mathbb{Q}(\sqrt{2})}(s)\, \zeta_{\mathbb{Q}(\sqrt{-2})}(s)}{\zeta(s)^2}$$

Rmk This is how $\zeta_K(s)$ are computed.
(e.g. in Magma)

Thm Suppose $\rho, \sigma : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_*(\mathbb{C})$ be two Artin representations.

Then $\rho \cong \sigma \iff L(\rho,s) = L(\sigma,s)$ as anal. fncs on $Re\, s \gg 0$

[L-fnc determines the rep. uniquely].