# An inverse problem of Hamiltonian dynamics

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**Abstract**

We study the question whether for a natural Hamiltonian system on a two-dimensional compact configuration manifold, a single trajectory of sufficiently high energy is almost surely enough to reconstruct a real analytic potential.

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Consider a compact configuration manifold $M^n$ equipped with a finite Borel measure (essentially we deal with the dimension $n = 2$) and a natural Hamiltonian system thereon, with the Hamiltonian

$$H(p, q) = \langle p, p \rangle_q + U(q), \quad (p, q) \in T^* M^n.$$

Above, $\langle \cdot, \cdot \rangle_q$ is a Riemannian metric on $M^n$ and $U$ a potential. The direct problem of dynamics on $M^n$ is finding the trajectory $q(t) \subset M^n$, with initial conditions $q(0) = q_0$ and $\dot{q}(0) = v_0$, moving in the known force field $f(q) = -\nabla_q U(q)$ on $M^n$, where the gradient $\nabla_q$ has been associated with the metric $\langle \cdot, \cdot \rangle_q$.

Let us call the inverse problem of dynamics the problem of reconstruction of the potential by observing the system's trajectories $q(t)$. The first problem of this type was explored in Newton's Principia, in quest for a physical law determining the planetary motion compatible with observational data[1]. In the general case, knowledge of infinitely many trajectories is required to completely solve the problem. In this note we show that in the special case when $M^n$ is two-dimensional, compact and topologically non-trivial, a single trajectory with sufficiently large energy would almost surely suffice to reconstruct the potential.

In the sequel, we assume that $M^n$ as well as all the quantities involved are real-analytic. Also suppose, there is an a-priori estimate $|U(q)| < C_0$, $\forall q \in M^n$, and we consider only the trajectories $q(t)$ with total energy $E \geq C_0$.

**Theorem.** *Let $n = 2$, suppose $M^2$ is not diffeomorphic to $S^2$ or $\mathbb{R}P^2$. Almost every trajectory $q(t)$, $t \geq 0$, with energy $E \geq C_0$, suffices to reconstruct the potential $U$ as a real-analytic function on $M^2$.*

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[1] Note that the term "inverse problem of mechanics" has also been used to address the problem of deciding whether a given system of second order ODEs on $M^n$ has a Lagrangian, see e.g. [5].

Let us recall the definition of a *key set*, or set of uniqueness, see e.g. [3].

**Definition.** *Let $D$ be a domain in $\mathbb{R}^n$ and $C^\omega(D)$ the class of real-analytic functions in $D$. A set $K \subset D$ is a key set if any $f \in C^\omega(D)$ vanishing identically on $K$, vanishes identically on $D$.*

The theorem will follow from the following lemma.

**Lemma.** *If a real-analytic dynamical system $A: \dot{x} = F(x)$ on a compact phase space $P$ is non-singular (i.e. for no $x \in P$, $F(x) = 0$) and the set of its closed orbits has positive measure, then all its orbits are closed.*

### Proof of the lemma

Since the manifold $P$ is compact and the vector field $F$ is non-singular, there exists a Riemannian metric $g$ on $P$, such that in this metric the vector field $F$ has unit length at every $x \in P$. Furthermore, by compactness of $P$, the curvature (associated with $g$) of integral trajectories of $A$ is bounded from above, and therefore there exists some $T_m > 0$, such that any periodic orbit of $A$ has period not smaller than $T_m$. This represents a particular case of the general result of Yorke ([7]).

Let us partition the range $[T_m, \infty)$ of possible periods (henceforth periods stand for minimum periods) for closed orbits into intervals of some small length $\delta_1$ to be specified. Let $I_k = [T_m + k\delta_1, T_m + (k+1)\delta_1) \equiv [T_k, T_{k+1})$, for $k = 0, 1, \ldots$. Let $\Gamma_k$ be the set of all closed orbits, whose periods lie in $I_k$. Then for some $k = k_*$ the set $\Gamma_* = \Gamma_{k_*}$, considered as a subset of $P$, has positive measure. (We refer to the sets $\Gamma$ either as point sets or sets of orbits, depending on the context. As there are only measure-theoretical considerations involved, this should not cause confusion.)

Let $D_{\delta_2}(x)$ be a codimension one disk in $P$, centered at some $x \in P$, with radius $\delta_2$ and perpendicular (in the sense of metric $g$) to the vector field $F(x)$ at the point $x$. Let $\delta_2$ be small enough, so that the vector field is transversal to $D_{\delta_2}(x)$ at every point of the disk. Clearly, $\delta_2$ can be taken as a universal constant independent of $x$.

Let $x_0$ be a Lebesgue point of the set $\Gamma_*$. Recall that at a Lebesgue point, the density of the set is one. Let $\gamma_0$ be the closed trajectory passing through $x_0$, so $\gamma_0 \in \Gamma_*$. Take a disk $D_{\delta_2}(x_0)$.

Then there is a well defined analytic Poincaré map $S$ from a disk $D_\epsilon(x_0)$ contained in $D_{\delta_2}(x_0)$, where $\epsilon < \frac{1}{C_1 T_{k_*}}\delta_2$, for some $C_1 = C_1(\delta_1)$, and $x_0$ is a fixed point of $S$. On the disk $D_\epsilon(x_0)$, points that are initial conditions for orbits from $\Gamma_*$ form a set of positive measure, as $x_0$ is a Lebesgue point. Besides, the quantity $\delta_1 < T_m$ can be chosen small enough to ensure that any $x \in D_\epsilon(x_0) \cap \Gamma_*$ is also a fixed point of the map $S$.

The union of all $x \in D_\epsilon(x_0) \cap \Gamma_*$ is a positive measure subset of $D_\epsilon(x_0)$, and hence is a key set (see [6] for the proof that every set of positive measure is a key set). Therefore, every point of $D_\epsilon(x_0)$ is an equilibrium of the map $S$, and hence an initial condition for a periodic orbit of $A$. Let $\Gamma_\epsilon$ be the union of all such orbits, with initial conditions in $D_\epsilon(x_0)$. Let $\Gamma \subseteq P$ be the maximum connected open set, which contains $\Gamma_\epsilon$ and is a union of periodic orbits.

To complete the proof, let us show that the set $\Gamma$ does not have a boundary, i.e. $\Gamma = P$. To show it, we use the following Gronwall type estimate.

**Proposition.** *Let $\phi(s) \subset \Gamma$ be a curve of length $L$, where $s$ is a natural parameter with respect to the metric $g$. Then for all $s \in [0, L]$ and some absolute constant $C_2$,*

$$T(\phi(s)) \leq T(\phi(0))e^{C_2 L},$$

*where $T(\phi(s))$ is the period of the closed orbit passing through the point $\phi(s)$.*

Indeed, the proposition follows immediately from the following infinitesimal estimate: for some $C_2$,

$$\left|\frac{d}{ds}T(\phi(s))\right| \leq C_2 T(\phi(s)).$$

Returning to the proof of the lemma, suppose the boundary $\partial\Gamma$ is non-empty. As $\partial\Gamma$ is a compact set, the distance (in the sense of metric $g$) between $\partial\Gamma$ and the above mentioned point $x_0$ attains its minimum at some point $y \in \partial\Gamma$. Connect $x_0$ and $y$ by a geodesic segment. Let the latter segment have length $L$; clearly all its points, except $y$ belong to $\Gamma$. Let $\gamma_y$ be the trajectory of $A$ with initial condition $y$.

For any point $x_1 \neq y$ on the above geodesic segment, there is a uniform bound for the period of the corresponding closed orbit, by the proposition. Hence, for any such $x_1$, there exists a uniform $\varepsilon$ (one can take $\varepsilon = \epsilon e^{-C_2 L}$, where $\epsilon$ has been defined earlier) such that an analytic Poincaré map can be defined in exactly the same way as $S$ above, but now with the domain $D_\varepsilon(x_1)$. Choosing $x_1$ such that the intersection $\gamma_y \cap D_\varepsilon(x_1)$ is not empty and repeating the key set argument leads to contradiction: all orbits in some tubular neighborhood of $\gamma_y$, including $\gamma_y$ itself, are periodic. *Q.E.D.*

**Proof of the theorem**

Consider a randomly chosen trajectory $\gamma$ on some energy level $H^{-1}(E)$, $E \geq C_0$, which is obviously a non-critical level.

According to the lemma, either (i) all the trajectories on $H^{-1}(E)$ are periodic, or (ii) a randomly chosen initial condition $(p_0, q_0) \in H^{-1}(E)$ results in a phase trajectory of infinite length almost surely.

The former case (i) may occur only if $M^2$ is a so-called $P$-manifold. Indeed, according to the Maupertuis principle, the phase trajectories of motions with total energy $E$ project onto $M^2$ as geodesic lines of the corresponding (Riemannian) Jacobi metric. $P$-manifolds are Riemannian manifolds, all whose geodesics are closed, see [1]. Topological properties of $P$-manifolds are characterized in great detail in various dimensions, within the framework of the Bott-Samelson theorem. In our (simplest possible) case, it is easy to see that $M^2$ can only be diffeomorphic to either $S^2$ or $\mathbb{R}P^2$.

Indeed, the proof of the lemma implies that all the (closed) phase trajectories on the energy level $E$ are homotopic to one another. Then their images on $M^2$ (under natural projection) are also homotopic. On the other hand, if $M^2$ is different from $S^2$ or $\mathbb{R}P^2$, the number of generators for its fundamental group equals at least two. As for any Riemannian metric there are closed geodesics in each free homotopy class, we would have a contradiction, unless $M^2$ is $S^2$ or $\mathbb{R}P^2$.

In the case (ii), let $q(t) = (q_1(t), q_2(t))$ be a randomly chosen trajectory: it almost surely has infinite length. Clearly, we can easily derive the gradient of $U$ at every point of the trajectory from the Euler-Lagrange equations.

Thus to complete the proof of the theorem, let us show that any non-closed trajectory is a key set. Consider orbit segments $\{q(t),\, t \in [k, k+1)\}$, $k = 0, 1, \ldots$ in $M^2$. (Note that time can always be scaled to ensure that each segment is a simple curve in $M^2$, or shorter time intervals can be considered.) As $M^2$ is compact, there is a limit point $q_*$ of the point sequence $\{q(k + 1/2)\}$. Consider a sufficiently small circle centered at $q_*$. There are two options. Either the circle intersects the trajectory $q(t)$ at infinitely many points, or at some point on the circle, the trajectory $q(t)$ intersects itself (transversely as it is a geodesic) infinitely many times. In the later case, take the point of infinite self-intersection for $q_*$, otherwise leave $q_*$ as it is. In either case, there exists a point

$q_*$, with the property that any sufficiently small circle centered at $q_*$ is intersected by the trajectory $q(t)$ infinitely many times.

Therefore, the force $f$ and the potential $U$ can be uniquely reconstructed on any sufficiently small circle centered at $q_*$ (an infinite point set on a circle is a key set for the circle), and therefore in some neighborhood of $q_*$, and hence on the whole configuration manifold $M^2$. *Q.E.D.*

In conclusion, let us make several remarks.

1. The theorem can be stated in terms of analytic geodesic flows on compact Riemannian 2-manifolds. Namely, if $M^2$ is not diffeomorphic to a sphere or real projective plane, a randomly chosen geodesic suffices to reconstruct the metric, almost surely. Indeed, any Riemannian metric is locally conformal to the Euclidean one, i.e. can be locally associated with the Hamiltonian $H(p,q) = e^{\rho(q_1,q_2)}(p_1^2 + p_2^2)$. Our theorem enables one to reconstruct the real-analytic function $\rho(q_1, q_2)$ locally near $q_*$ from the Hamilton equations, with subsequent analytic continuation to get the metric globally on $M^2$.

2. The theorem is essentially two-dimensional, as for $n \geq 3$ the fact that a random trajectory has infinite length does not suffice to reconstruct the potential. Consider for instance the Euler top, where $M^n = SO(3)$. In this case, the phase space is foliated by invariant two-tori, where the trajectories are in general conditionally periodic. Clearly, a projection of a single invariant two-torus onto the three-dimensional configuration space is not a key set.

3. The lemma does not apply to the case of invariant tori of dimension higher than one. Indeed, a particular case of the KAM theorem states that a sufficiently small perturbation of a non-degenerate Liouville-integrable Hamiltonian system in $T^*M^n$ yields a positive measure set of invariant $n$-tori that do not fill the whole energy surface.

4. In the special case $M^2 = \mathbb{R}P^2$, by the theorem of L. Green ([4]), the only metric for which all geodesics are closed is the standard metric. The case $M^2 = S^2$ has been a subject of extensive research for over 100 years, arguably beginning with the doctoral thesis of O. Zoll ([8]). The reader is referred to the excellent book by L. Besse ([1]), which gives the issue a thorough treatment.

5. The condition $E \geq C_0$ seems unavoidable. For small energies the domain of possible motions can be a disk, with the Jacobi metric degenerate on the boundary, in which case one may expect a scenario similar to the case $M^2 = S^2$.

6. Observe that in the exceptional case when $M^2$ is a $P$-manifold, the geodesic flow thereon is completely integrable (see e.g. [2] for the proof of this fact). Hence our theorem implies that if $n = 2$, it is sufficient for restoration of the potential almost surely from a single trajectory that the system possess no other analytic integrals of motion but energy. It seems likely that in the latter weaker formulation the theorem should extend to the case $n > 2$, however we do not know how to prove it.

## References


[1] A. Besse. Manifolds all of whose Geodesics are Closed. *Springer Verlag, Berlin, Heidelberg, New York,* 1978.

[2] A.V. Bolsinov, B. Jovanović. Noncommutative integrability, moment map and geodesic flows. *Ann. Global Anal. Geom.* **23** (2003), no. 4, 305–322.

[3] V.I. Arnold, V.V. Kozlov, A.I. Neishtadt. Encyclopedia of Mathematical Sciences. Dynamical Systems III. Mathematical Aspects of Classical and Celestial Mechanics. *Springer-Verlag, Berlin*, 1988.

[4] L.W. Green. Auf Wiedersehensflächen. *Ann. of Math.* **78** (1963), no 2, 289–299.

[5] M. de León, P.R. Rodrigues. Methods of differential geometry in analytical mechanics. North-Holland Mathematics Studies **158**. *North-Holland Publishing Co., Amsterdam*, 1989.

[6] V.V. Ten. Analytic invariants of dynamical systems with positive entropy. (Russian). *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* (1997) no. 3, 40–43.

[7] J.A. Yorke. Periods of periodic solutions and the Lipschitz constant. *Proc. Amer. Math. Soc.* **22** (1969), 509–512.

[8] O. Zoll. Über Flächen mit Scharen geschlossener geodätischer Linien. (German). *Math. Ann.* **57** (1903), 108-133.