OPT2 **Problem Sheet 5**

Duality and Sensitivity

1. Suppose, you have found an optimal basis for a canonical LP Max $\boldsymbol{c}\cdot\boldsymbol{x}$: $A\boldsymbol{x}=\boldsymbol{b}, \boldsymbol{x}\geq 0$. True or false:
   (a) Changing the objective function coefficients (i.e. $\boldsymbol{c}$) may affect feasibility of the basis.
   (b) If $\boldsymbol{b}$ changes the basis remains optimal as long as it is still feasible.
   (c) If a feasible solution (FS) of the primal LP gives the same objective value as some FS of the dual, then both FSs are optimal.
2. In the manufacturing problem, show that the shadow prices of the constraints are equal to the reduced costs of the corresponding slack variables. (Hence, in the final tableau, the entries in the slack variables' columns represent the optimal solution of the dual problem.)
3. Optional – another exercise in writing duals: Given that the dual for the canonical maximization LP is $A^T y \geq \boldsymbol{c}$, min $\boldsymbol{y}\cdot\boldsymbol{b}$, find the dual problem for the manufacturing and the diet problems (we've done it the other way around once).
4. Optional – another exercise in writing duals: Starting from the canonical LP, show that the dual of the dual is primal.
5. For LP max $x_1+x_2+3x_3-2x_4+8x_5$, $x_i\geq 0$, and
$$x_1+3x_2-x_3+2x_4+2x_5=7,\ -x_1-x_2+3x_3+x_5\leq 16,\ 3x_1+2x_2+x_3-x_4-2x_5\geq 4$$
one suggests a feasible solution $x_1=2.75$, $x_3=4.75$, $x_5=4.5, x_2=x_4=0$. Use complementary slackness to check its optimality. Then find the reduced costs of the non-basic variables and shadow prices of the constraints (i.e. the optimal solution for the dual).
6. Optional – on Dual Simplex method: For an LP you've solved in Problem 1 in the first group find shadow prices of the constraints. (HINT: after completing Phase I, retain the artificial variables keeping them thence free; in the final tableau the artificial variables' reduced costs will yield shadow prices of the constraints for which they were introduced. As a matter of fact, Solutions 4 do it. So, just have a look at what all this means.)

Geometry of LP

The first four problems here are optional: look at them if you need practice with the notion of closed sets.

1. `Optional:` Prove that the plane set $\{(x,y): x^2+y^2\geq 1\}$ is closed. Identify the set's interior and boundary.
2. `Optional:` Identify the set of all the limit points of the set $\{(x,y): x^2-y^2>1\}$.
3. `Optional:` Identify the closure of the infinite plane set which in polar coordinates (r,ϕ) is given as $\{(r,\phi): r=1,\ \phi=n,\ \text{with } n=0,1,2,\ldots\}$. Identify the interior and the boundary of this set.
4. `Optional:` Prove the equivalence of the definitions of the closure $\bar{X}$ of a set $X\subseteq\mathbb{R}^n$ as the set of all limit points of X and the smallest closed set containing X.
5. Prove by definition that the set $\{(x,y,z)\in\mathbb{R}^3: |x|+|y|+|z|<3\}$ is convex.
6. Prove by definition that the set $\{(x,y)\in\mathbb{R}^2: x^2+y^2\leq 1\}$ is convex. *HINT: you will need to use the deep fact that a dot product of two vectors does not exceed in absolute value the product of their lengths.*
7. Identify the extreme points of the two sets above.

8. True or false? Give (very short) arguments for/against the following statements:

(a) The union of any two convex sets is convex.

(b) The intersection of any two convex sets is convex.

(c) The set $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq 0\}$ is convex. What about $\mathbb{R}^n_{++}$, i.e. when $\boldsymbol{x} > 0$?

(d) The set $\mathcal{C}_A = \{\boldsymbol{y} \in \mathbb{R}^m : \boldsymbol{y} = A\boldsymbol{x}, \forall\, \boldsymbol{x} \in \mathbb{R}^n_+$ is convex. A is an $m \times n$ matrix.

(e) The set $\{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}\}$ is closed and convex.

(f) `Optional:` Let X be any closed and convex set in $\mathbb{R}^n$, A an $m \times n$ matrix, let now

$$Y = \{\boldsymbol{y} \in \mathbb{R}^m : \boldsymbol{y} = A\boldsymbol{x}, \text{ for some } \boldsymbol{x} \in X\}.$$

Prove that the set Y is closed and convex.