Convex sets, Separating hyperplane theorem, Farkas alternative

1. Consider two plane sets: $X = \{(x, y) : y \geq x\}$ and $Y = \{(x, y) : x > 0 \text{ and } x^2 - y^2 \geq 1\}$ Are these sets closed and convex? Does there exist a hyperplane (i.e. a line, as we are in $\mathbb{R}^2$) that strictly separates them (without intersecting either one)? Modify, without proof, the statement of the Separating hyperplane theorem to accommodate this example.

2. For any points $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^s \in \mathbb{R}^n$, consider a set
   $X = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \sum_{\alpha=1}^{s} \theta_\alpha \boldsymbol{x}^\alpha, \forall\, \theta \geq 0 : \sum_{\alpha=1}^{s} \theta_\alpha = 1\}$, called the *convex hull* of these points.
   (a) Show that the convex hull is a convex set.
   (b) Draw some examples of this set on the plane for $s = 2, 3, 4, 5$.
   (c) Show by induction in s that the convex hull of $s+1$ points is obtained as the union of line segments, connecting the point $\boldsymbol{x}^{s+1}$ with each point in the convex hull of the points $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^s$.
   (d) Hence, argue that $\boldsymbol{x}$ is an extreme points of the convex hull of the points $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^s \in \mathbb{R}^n$ only if it is one of these points.

3. By introducing slack variables, find the Farkas alternative of the statement $A\boldsymbol{x} \leq \boldsymbol{b}$ has a solution $\boldsymbol{x} \geq 0$. Now drop the condition $\boldsymbol{x} \geq 0$ (so you will have to enlarge the problem by letting $x = \mathbf{u} - \mathbf{v}$, where $\mathbf{u}, \mathbf{v} \geq 0$.) Find the Farkas alternative again.

4. Show that the Farkas alternative of the statement "$A\boldsymbol{x} = 0$, $\boldsymbol{x} \geq 0$ has a *non-trivial* (i.e. nonzero) solution" is "$A^T\boldsymbol{y} > 0$ has a solution".

5. `Optional:` Show that the Farkas alternative of the statement "$A\boldsymbol{x} \geq 0$ has a solution $\boldsymbol{x} \in \mathbb{R}^n$, such that at least one of the inequalities is slack" is: "for some $\boldsymbol{y} \in \mathbb{R}^m_{++}$ (i.e. $\boldsymbol{y} > 0$), $A^T\boldsymbol{y} = 0$." *HINT: on the front side of the alternative, argue that the existence of slack can be mathematically interpreted as $A\boldsymbol{x} \cdot \boldsymbol{e} = 1$, where $\boldsymbol{e} = (1, \ldots, 1) \in \mathbb{R}^m$.* (This statement is known in finance as "The main asset pricing theorem", see the "Application of Farkas..." set of notes.)

Zero-sum two-person games

1. Find optimal strategies for the games with the following payoff matrices:
   $$\text{(i)} \begin{pmatrix} 5 & -9 \\ -7 & 4 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} 2 & 1 & -1 \\ -1 & -2 & 3 \end{pmatrix}.$$
   As the games are small, avoid doing the simplex method for the underlying LPs, solving them e.g. graphically (or just guessing) and then using complementary slackness to solve the dual and verify optimality.

2. Consider the following zero-sum two-person game. Players P and Q simultaneously show each other fingers. The player P may show one, two, or three fingers. The player Q may show one or two fingers. Let s be the total number of fingers shown at an instance of the game. If s is even, then P pays Q the amount $s-1$, in sterling. Otherwise, if s is odd, then Q pays P the amount $s-1$.
   (a) Set up the payoff matrix for P as column player.
   (b) Find the expected payoff to P, if he plays the mixed strategy $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and Q plays the mixed strategy $\boldsymbol{q} = (\frac{1}{2}, \frac{1}{2})$.
   (c) Find the game's value and best strategies for P and Q. Warning: Calculations are slightly cumbersome here. After setting up the corresponding dual pair of linear programmes, it may be easier to solve the one for Q first and then the one for P. Using the Simplex method is not advisable.
   (d) Describe "punishing" strategies that P should play *knowing* that Q plays the mixed strategy $\boldsymbol{q} = (\frac{1}{2}, \frac{1}{2})$ and find the corresponding expected payoff to P.

3. `Optional:` A variant of the *Morra* game (Google it!). As a single act of the game, two players will independently show each other one or two fingers. Before this is done, each tries to guess how many fingers the opponent will show. The payoff is zero if *both* players guess right or wrong. Otherwise, the player who guessed in the wrong pays his opponent the sum equal to the total number of fingers shown by both players.

   (a) Describe the pure strategies and the payoff matrix A.

   (b) Show that the strategy $(0, 7/12, 5/12, 0)$ is optimal – use the fact the game is symmetric.

   (c) Show that any strategy $(0, p_2, p_3, 0)$ with $4/3 \leq p_2/p_3 \leq 3/2$) is, in fact, optimal, and there are no others.

   (d) How much would you expect to win per game if your opponent plays a mixed strategy $(.1, .4, .3, .2)$?

4. `Optional:` A game of "hide and seek" is played as follows. Player $\mathcal{H}$ chooses a place to hide in the matrix below— that is, chooses a particular entry. Player $\mathcal{S}$ chooses either a row or a column in which to seek player $\mathcal{H}$. The choices are made simultaneously and independently. If it turns out that Player $\mathcal{H}$ is in Player $\mathcal{S}$'s row or column, then Player $\mathcal{H}$ pays Player $\mathcal{S}$ x pence, where x equals the value of the entry in which Player $\mathcal{H}$ is hiding; if not, Player $\mathcal{S}$ pays Player $\mathcal{H}$ 1 pence.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 3 \end{array}$$

   Identify pure strategies, set up the payoff matrix, the LP, and then get the optimal strategies using the on-line solver. What is the value of this game?