OPT2 Problem Sheet 7

Unconstrained extrema of functions of several variables

Find and classify the critical points of the following functions. Identify all local and global (if exist) extrema.

1. $f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 - 3x_1 - 6x_2$;
2. `Optional` $f(x_1, x_2) = x_1^3 + 3x_1x_2^2 - 15x_1 - 12x_2$;
3. $f(x_1, x_2) = (2x_1^2 + x_2^2)e^{-(x_1^2+x_2^2)}$;
4. $f(x_1, x_2, x_3) = x_1 + \dfrac{x_2}{x_1} + \dfrac{x_3}{x_2} + \dfrac{2}{x_3}$.
5. `Optional` $f(x_1, x_2) = 2 - \sqrt[3]{x_1^2 + x_2^2}$;
6. `Optional` $f(x_1, x_2, x_3) = x_1x_2^2x_3^3(1 - x_1 - 2x_2 - 3x_3),\ \boldsymbol{x} > 0$;
7. `Optional` $2x^2 + 2y^2 + z^2 + 8yz - z + 8 = 0$, for an implicit function $z(x, y)$. HINT: differentiate implicitly first w.r.t x and then y, find critical points by letting $z_x = z_y = 0$. Then differentiate implicitly one more time, and find second the partials *at critical points* – do not forget that at critical points the first partials are zero.

Convex Functions

1. What convexity properties (either convex, or concave, or none of the above) do the following functions have:
   (a) $f(x) = x^2 - 10x + 2,\ x \in \mathbb{R}$;
   (b) $f(x) = \ln x,\ x > 0$;
   (c) `Optional` $f(x) = e^x,\ x \in \mathbb{R}$;
   (d) $f(x_1, x_2) = x_1^2 + 3x_2^2 - x_1x_2,\ \boldsymbol{x} \in \mathbb{R}^2$;
   (e) `Optional` $f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - 2x_3^2 + \dfrac{1}{2}x_1x_2,\ \boldsymbol{x} \in \mathbb{R}^3$;
2. Show that the set $\{(x, y) : e^{x^2+2y^2} \le 100\}$ is convex.
3. True or false (all the functions are of several variables, well defined on an open domain):
   (a) If a function is convex, it cannot be concave.
   (b) The sum of two convex functions is convex.
   (c) The product of two convex functions is convex.
4. Prove the following inequalities either by appealing directly to the convexity properties of the function involved or using one of the classical inequalities (Jensen, Cauchy-Schwartz, Cauchy, etc.):
   (a) $\dfrac{1}{2}(x^p + y^p) \ge \left(\dfrac{x+y}{2}\right)^p, \quad x, y > 0,\ p > 1$;
   (b) `Optional:` $\dfrac{1}{3}\left(x^{\frac{1}{p}} + y^{\frac{1}{p}} + z^{\frac{1}{p}}\right) < \left(\dfrac{x+y+z}{3}\right)^{\frac{1}{p}}, \quad x, y, z > 0,\ x \ne y,\ p > 1$;
   (c) $\frac{4}{a^{-1}+b^{-1}+c^{-1}+d^{-1}} \le \sqrt[4]{abcd} \le \dfrac{a+b+c+d}{4}, \quad a, b, c, d > 0$;

(d) $\dfrac{5(1-a^{-1})}{1-a^{-5}} \le 5a^2 \le 1+a+a^2+a^3+a^4 = \dfrac{1-a^5}{1-a}, \quad a>0;$

(e) $\displaystyle\sum_{i=1}^n x_i \le \sqrt{n\sum_{i=1}^n x_i^2}, \quad x_i>0,\ i=1,\dots,n;$

(f) `Optional` $\displaystyle\sum_{i=1}^n x_i \le n^{\frac{1}{p}}\left(\sum_{i=1}^n x_i^q\right)^{\frac{1}{q}}, \quad x_i>0,\ i=1,\dots,n,\ p>1,\ q=\frac{p}{p-1};$

(g) `Optional` $\displaystyle\ln\left(\int_0^1 g(x)dx\right) \ge \int_0^1 \ln[g(x)]dx,$ for a continuous $g(x)>0$ for $x\in[0,1]$.

When do these inequalities become strict?

5. `Optional:` Let A be a set of N positive reals. Let $A+A$ denote the set of all pair-wise sums of elements of A:
$$A+A=\{a_1+a_2:\ a_1,a_2\in A\}.$$
The number of elements X in $A+A$ can be anything between $2N-1$ and $N(N+1)/2$, depending on A. Suppose, however, we know that the number of *ordered* quadruples (a_1,a_2,a_3,a_4) that satisfy
$$a_1+a_2=a_3+a_4$$
is bounded by some number E. Show that the Cauchy-Schwartz inequality implies that $X\ge\dfrac{N^4}{E}$.

6. `Optional -- I once explained it to Bristol six-formers:` Prove the "fat elephant inequality": a set S of N points in $\mathbb{R}^3$ has a projection on one of the coordinate planes, whose size is not less than $N^{2/3}$. (A fat elephant cannot look thin from all the three directions – it must have at last one fat projection.) Assume for simplicity that all the points have integer coordinates in the interval $[1,..,M]$.

Hint: introduce the characteristic function $S(x,y,z)$ of the set S, which equals 1 if the point $(x,y,z)\in S$ and $S(x,y,z)=0$ otherwise. Let $S_1(x,y)$, $S_2(y,z)$, $S_3(z,x)$ be characteristic functions of the projections of the set S onto the xy, yz, zx-planes, respectively. Then
$$S(x,y,z)\le S_1(x,y)S_2(y,z)S_3(z,x).$$
(Why?) Besides, $\sum_{x,y,z}S(x,y,z)=N$. Use this and Cauchy-Scwartz applied twice to the above inequality.