OPT2 Problem Sheet 8. Lagrange multipliers method

Equality constraints: "standard" Lagrange multipliers

1. Find the constrained extrema of the following functions:

   (a) $z = x^2 + y^2 - xy + x + y - 4$, s.t. $x + y + 3 = 0$;

   (b) `Optional` $z = \frac{1}{x} + \frac{1}{y}$, s.t. $x + y = 2$;

   (c) $u = x^2 + y^2 + z^2$, s.t. $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$;

2. Find the dimensions of a rectangular open tin tub (a rectangular parallelepiped, without the upper facet) of the given volume V, whose production would require the minimum amount of tin (minimize the tub's surface area).

Inequaltiy constraints: Kuhn-Tucker conditions

Note: Kuhn-Tucker (KT), or Lagrange/Kuhn-Tucker conditions arise as a modification of "standard" Largange multipliers for the case of inequality constraints. The acronym CQ refers to the "Constraint Qualification" nondegeneracy assumption, which is again an adaptation of the non-degeneracy condition of linear independence of constraints' gradients to the case of inequality constraints, requiring a bit more jargon. KT conditions are necessary conditions for some $\boldsymbol{x}$ to be an extremum *only if* the assumption CQ is satisfied at $\boldsymbol{x}$. The assumption itself is that for each *feasible direction* $\boldsymbol{v}$ (i.e a nonzero vector $\boldsymbol{v}$ beginning at $\boldsymbol{x}$), which forms an angle of no more than $\frac{\pi}{2}$ with the gradient of every constraint, which is tight at $\boldsymbol{x}$, there is a feasible path, beginning at $\boldsymbol{x}$, whose tangent vector at $\boldsymbol{x}$ is $\boldsymbol{v}$. In other words, every *feasible direction* $\boldsymbol{v}$ at $\boldsymbol{x}$ is a *true feasible direction.* (Otherwise KT conditions may or may not succeed, as the problems below indicate.) This jargon can be used in the standard Lagrange multipliers as well: there *true feasible* would be directions tangent to the feasible set, and just *feasible* – those, perpendicular to all gradients of constraints. A *true feasible* direction is always a *feasible one.* The non-degeneracy assumption is reversing it: it assumes that all feasible directions, which are easy to describe in terms of the constraints' gradients, are, in fact, *true* ones. That is, tangent to the feasible set in the case of equality constraints and pointing into the feasible set for the case of inequality constraints.

1. Problem: Max $f(x_1, x_2) = x_1$ for $(x_1, x_2) \in F$, where $F = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} \geq 0,\ (1 - x_1)^5 - x_2 \geq 0.$

   (a) Sketch F, guess the maximizer.

   (b) Show CQ is not satisfied at the maximiser. (I.e identify a feasible direction which is not a true feasible direction.)

   (c) Set up the Lagrangian and verify that the system of Kuhn-Tucker conditions is inconsistent.

   (d) Add an additional constraint $x_1 \leq 1$. Set up the Lagrangian again, write down the Kuhn-Tucker conditions, find their solutions, hence identifying the maximiser. Argue that CQ is now satisfied as well.

   (e) Return to the initial problem, only now the objective is Max x_2. Guess the maximizer and verify your guess by solving the system of Kuhn-Tucker conditions. Argue that CQ is satisfied.

2. Let P be a problem in $\mathbb{R}^2$ of `min` x, subject to a constraint $y^2 \leq x^3$.

   (a) Sketch the feasible set F and identify the minimizer.

   (b) Set up the Lagrangian and the KT conditions for P, and show that they are inconsistent. What does this tell us about CQ at the minimiser?

(c) Let P' be a problem, obtained by adding to P an extra constraint $x \geq 0$. Proceed as in (b) and show that now the KT conditions have a one-parameter family of solutions, which identifies the minimizer uniquely. Verify explicitly that not withstanding the success of KT method in (c), CQ still fails at the minimiser.

3. `Optional:` Consider a problem P: minimize x such that $-x^5 \leq y \leq x^5$. Do a sketch, find minimizer. Show that CQ are not satisfied. Write down the Lagrange/Kuhn-Tucker conditions and show that they are inconsistent.

   Now modify the problem by adding a constraint $x \geq 0$. The sketch is the same. Show that CQ are now satisfied and Lagrange/Kuhn-Tucker conditions yield the minimiser.

4. `Optional:` Consider the *quadratic* program Max/Min $\frac{1}{2}\boldsymbol{x}^T C\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x}$, s.t. $\boldsymbol{x} \geq 0$, $A\boldsymbol{x} = \boldsymbol{b}$. Above, C is an $n \times n$ matrix and $\boldsymbol{c} \in \mathbb{R}^n$.

   (a) Write the Kuhn-Tucker conditions for this problem.

   (b) What assumption on C will be sufficient to ensure the existence and uniqueness of the global maximizer/minimizer.