Simplex Method in different guises

The Furniture problem

Max $60x_1 + 30x_2 + 20x_3$, subject to

$$\boldsymbol{x} \geq 0, \quad 8x_1 + 6x_2 + 2x_3 \leq 48, \quad 4x_1 + 2x_2 + 1.5x_3 \leq 20, \quad 2x_1 + 1.5x_2 + .5x_3 \leq 8.$$

Canonical form: slack variables $\boldsymbol{s} = (s_1, s_2, s_3) \geq 0$. Constraints now are

$$\boldsymbol{x}, \boldsymbol{s} \geq 0, \quad 8x_1 + 6x_2 + 2x_3 + s_1 = 48, \quad 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \quad 2x_1 + 1.5x_2 + .5x_3 + s_3 = 8.$$

Introduce the objective variable $z = 60x_1 + 30x_2 + 20x_3$. Write everything as the system of equations, or *tableau:*, where the right-hand side is called *Value,*

BV\V	z	x_1	x_2	x_3	s_1	s_2	s_3	Val
s_1	0	8	6	2	1	0	0	48
s_2	0	4	2	1.5	0	1	0	20
s_3	0	2	1.5	0.5	0	0	1	8
z	1	-60	-30	-20	0	0	0	0

This is the *initial tableau,* where the basic variables are $\boldsymbol{s}, z$, free variables $\boldsymbol{x}$. So, "BV\ V" in the tableau simply reads "Basic Variables \ Variables". Importantly, the columns, corresponding to basic variables are columns of the unit matrix. I.e., the system equations represented by the tableau gives expressions for basic variables via free variables, as free parameters. And above, the rows have been marked by basic variables, according to this.

Now, the presence of negative numbers in the bottom row, reading $z = 0 + 60x_1 + 30x_2 + 20x_3$ implies that making either of x's positive, the rest being retained zero, will increase the objective. So, the BFS, provided by the above tableau is not optimal. One sees that if one starts increasing, say, x_1, keeping $x_2, x_3 = 0$, every extra unit of x_1 will increase the objective by 60. The question now is – what is the largest feasible value of x_1, provided that $x_2, x_3 = 0$, and still, each $\boldsymbol{s} \geq 0$. This is seen from the first three equations: s_3 becomes zero when $x_1 = 8/2 = 4$; if this occurs, $s_1 = 48 - 8 \cdot 4 = 16, s_2 = 20 - 4 \cdot 4 = 4$, and $z = 0 + 4 \cdot 60 = 240$. This is a new, better BFS.

Now, we need a new tableau, where the columns, corresponding to the new basic set of variables s_1, s_2, x_1, z have become columns of the unit matrix. This tableau is obtained from the above tableau by *pivoting* the entry 2, sitting in the x_1 column/s_3 row – it is x_1 that is going to knock s_3 out of the set of basic variables. The pivot entry sits at the intersection of the *pivot column* and *pivot row.* The pivot column has been chosen by the negative-most entry in the objective row. The pivot-row has been identified by the *minimum positive ratio* of the entry in the value column to the coefficient in the pivot column:

BV\V	z	x_1	x_2	x_3	s_1	s_2	s_3	Val	Rat
s_1	0	8	6	2	1	0	0	48	48/8
s_2	0	4	2	1.5	0	1	0	20	20/4
s_3	0	**2**	1.5	0.5	0	0	1	8	8/2
z	1	-60	-30	-20	0	0	0	0	

After the pivot, i.e. the sequence of EROs has been done, we have the new tableau:

BV\V	z	x_1	x_2	x_3	s_1	s_2	s_3	Val
s_1	0	0	0	0	1	0	-4	16
s_2	0	0	-1	.5	0	1	-2	4
x_1	0	1	.75	.25	0	0	.5	4
z	1	0	15	-5	0	0	30	240

Observe that the submatrix, corresponding to the new basic variables x_1, s_1, s_2, z is again the unit matrix. I.e. now the new basic variables have been expressed via new free variables x_2, x_3, s_3 as free parameters.

The negative -5 in the bottom row tells us we have to proceed. Indeed, the last equation now is $z = 240 - 15x_2 + 5x_3 - 30s_3$. If x_3 is made positive, while $x_2, s_3 = 0$, z will increase. So, x_3 is to become basic. To substitute which variable? Again, the first three equations above tell us that as $x_2, s_3 = 0$, and x_3 is being increased from zero, then as soon as it reaches the value 8, s_2 will become zero. When this happens, we will have $s_1 = 16 - 0 \cdot 8 = 16,\ x_1 = 4 - .25 \cdot 8 = 2,\ z = 240 + 5 \cdot 8 = 280$. This is the new BFS, and we now need a tableau for it.

In other words, we have identified the pivot column by the negative entry in the bottom row as the x_3-column, and now the pivot row has been identified as the second, i.e. s_2-row by looking at the minimum positive ratio of the right-hand-side to the coefficient in the pivot column. This ratio equals 2 and occurs in the s_2-row. Here:

BV\V	z	x_1	x_2	x_3	s_1	s_2	s_3	Val	Rat
s_1	0	0	0	0	1	0	-4	16	16/0
s_2	0	0	-1	**.5**	0	1	-2	4	4/.5
x_1	0	1	.75	.25	0	0	.5	4	4/.25
z	1	0	15	-5	0	0	30	240	

So, we pivot .**5** in **bold** and arrive in the new tableau:

BV\V	z	x_1	x_2	x_3	s_1	s_2	s_3	Val
s_1	0	0	0	0	1	0	-4	16
x_3	0	0	-2	1	0	2	-4	8
x_1	0	1	1.25	0	0	$-.5$	1.5	2
z	1	0	5	0	0	10	10	280

And this is a final tableau: we have $z = 280 - 5x_2 - 10s_2 - 10s_3$, so making either of the free variables $x_2, s_{2,3}$ positive will not increase, but rather decrease the objective.

Unbounded problem: Example

Max $36x_1 + 30x_2 - 3x_3 - 4x_4$, subject to $\boldsymbol{x} \geq 0,\ \ x_3 + 5 \geq x_1 + x_2,\ x_4 + 10 \geq 6x_1 + 5x_2$.

Solution: There is a time-saving strategy: we know that the submatrix in the tableau, corresponding to the basic variables, is the unit matrix. So, why writing it all the time? Not doing this gives *short tableaus.*

Long Tableau

BV\V	z	x_1	x_2	x_3	x_4	x_5	x_6	Val	R
x_5	0	1	1	-1	0	1	0	5	$\frac{5}{1}$
x_6	0	**6**	5	0	-1	0	1	10	$\frac{10}{6}$
z	1	-36	-30	3	4	0	0	0	

Short Tableau

BV\FV	x_1	x_2	x_3	x_4	Val	R
x_5	1	1	-1	0	5	$\frac{5}{1}$
x_6	**6**	5	0	-1	10	$\frac{10}{6}$
z	-36	-30	3	4	0	

BV\V	z	x_1	x_2	x_3	x_4	x_5	x_6	Val	R
x_5	0	0	$\frac{1}{6}$	-1	$\boldsymbol{\frac{1}{6}}$	1	$-\frac{1}{6}$	$\frac{10}{3}$	20
x_1	0	1	$\frac{5}{6}$	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{5}{3}$	–
z	1	0	0	3	-2	0	6	60	

BV\FV	x_6	x_2	x_3	x_4	Val	R
x_5	$-\frac{1}{6}$	$\frac{1}{6}$	-1	$\boldsymbol{\frac{1}{6}}$	$\frac{10}{3}$	20
x_1	$\frac{1}{6}$	$\frac{5}{6}$	0	$-\frac{1}{6}$	$\frac{5}{3}$	–
z	6	0	3	-2	60	

How has the new short tableau been obtained? The variable x_1 has replaced x_6 in the basis. By looking at the short tableau above, one realises that pivoting the 6 will consist of the following independently done EROs: (i) adding to the first row $-1/6$ times the second row; (ii) multiplying the second row by $1/6$; (iii) adding to the third row the "old" second row multiplied by 6. In the long tableau, this has also been done to the x_1 column, which is the column of the unit matrix. The result, which is simply the summary of the three used multipliers: $-1/6, 1/6, 6$ is exactly what becomes the new x_1 column in the short tableau.

BV\V	z	x_1	x_2	x_3	x_4	x_5	x_6	Val	R
x_4	0	0	1	-6	1	6	-1	20	–
x_1	0	1	1	-1	0	1	0	5	–
z	1	0	2	-9	0	12	4	100	

BV\FV	x_6	x_2	x_3	x_5	Val	R
x_4	-1	1	-6	6	20	–
x_1	0	1	-1	1	5	–
z	4	2	-9	12	100	

Conclusion: The last tableau indicates that a free x_3 can be taken arbitrarily large without violating the feasibility of basic x_1 and x_4, yielding an arbitrarily large z. Indeed, there are *no positive ratios:* if $x_6 = 2_2 = x_5 = 0$, we have $x_4 = 20 + 6x_3,\ x_1 = 5 + x_3,\ z = 100 + 9x_3$. Making $x_3 \to +\infty$ creates a family of feasible solutions, effecting arbitrarily large objective. Note, that these solutions are *not basic.*

Alternative solutions: Example

This is the same old furniture problem, after table have grown in price up to £35. Now it makes sense to manufacture them, but instead of desks or chairs? Max $60x_1 + 35x_2 + 20x_3$, subject to $\boldsymbol{x} \geq 0$ and $8x_1 + 6x_2 + 2x_3 \leq 48,\ 4x_1 + 2x_2 + 1.5x_3 \leq 20,\ 2x_1 + 1.5x_2 + .5x_3 \leq 8,\ x_2 \leq 5$.

Long Tableau

BV\V	z	x_1	x_2	x_3	x_4	x_5	x_6	Val	R
x_4	0	8	6	2	1	0	0	48	6
x_5	0	4	2	1.5	0	1	0	20	5
x_6	0	**2**	1.5	.5	0	0	1	8	4
z	1	-60	-35	-20	0	0	0	0	

Short Tableau

BV\FV	x_1	x_2	x_3	Val	R
x_4	8	6	2	48	6
x_5	4	2	1.5	20	5
x_6	**2**	1.5	.5	8	4
z	-60	-35	-20	0	

BV\V	z	x_1	x_2	x_3	x_4	x_5	x_6	Val	R
x_4	0	0	0	0	1	0	-4	16	∞
x_5	0	0	-1	**.5**	0	1	-2	4	8
x_1	0	1	.75	.25	0	0	.5	4	16
z	1	0	10	-5	0	0	30	240	

BV\FV	x_6	x_2	x_3	Val	R
x_4	-4	0	0	16	∞
x_5	-2	-1	**.5**	4	8
x_1	.5	.75	.25	4	16
z	30	10	-5	240	

Long Tableau

BV\V	z	x_1	x_2	x_3	x_4	x_5	x_6	Val	R
x_4	0	0	0	0	1	0	-4	16	∞
x_3	0	0	-2	1	0	2	-4	8	$-$
x_1	0	1	**1.25**	0	0	$-.5$	1.5	2	$\frac{8}{5}$
z	1	0	0	0	0	10	10	280	

Short Tableau

BV\FV	x_6	x_2	x_5	Val	R
x_4	-4	0	0	16	∞
x_3	-4	-2	2	8	$-$
x_1	1.5	**1.25**	$-.5$	2	$\frac{8}{5}$
z	10	0	10	280	

Alarm: accidentally, x_2 is a free variable, and as long as other free variables, $x_5 = x_6 = 0$, we have $z = 280 - 0 \cdot x_2$. In other words, x_2 can be made positive, and z will not change. Then, as soon as x_2 knocks out a basic variable x_1, which happens for $x_2 = 8/5$, we'll have a new BFS, where in addition $x_3 = 8 + 2 \cdot 1.6 = 11.2$, $x_4 = 16$. And still $z = 280$. So, if in the final tableau there is a free variable, such that the coefficient in the objective row is zero, this indicates that there are alternative solutions. Here is the new tableau, after x_2 has been brought into the basis:

BV\V	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Val
x_4	0	0	0	0	1	0	-4	0	16
x_3	0	1.6	0	1	0	1.2	-1.6	0	11.2
x_2	0	.8	1	0	0	$-.4$	1.2	0	1.6
z	1	0	0	0	0	10	10	0	280

BV\FV	x_6	x_1	x_5	Val
x_4	-4	0	0	16
x_3	-1.6	1.6	1.2	11.2
x_2	1.2	.8	$-.4$	1.6
z	10	0	10	280

Conclusion: A pair $(\boldsymbol{x}^1, \boldsymbol{x}^2)$ of basic solutions $x_1 = 2$, $x_2 = 0$, $x_3 = 8$ and $x_1 = 0$, $x_2 = 1.6$, $x_3 = 11.2$ both yield the optimal $z = 280$. So is any convex combination $\boldsymbol{x}_\theta = \theta \boldsymbol{x}^1 + (1-\theta)\boldsymbol{x}^2$, $0 \leq \theta \leq 1$.