Simplex Method: how to start?

The simplex algorithm in essence takes a BFS and analyses whether it is optimal or not. If it is not, it either yields a better FS, or decides that the problem is unbounded. The question is – where does one start? How does one find the initial BFS for the algorithm to initiate? With the Manufacturing problem, this is obvious, but how about a general $A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq 0$, where A is $m \times n,\ n > m$? One can try to choose, say, the leftmost $m \times m$ square submatrix A_1 of A and solve $A_1\boldsymbol{x} = \boldsymbol{b}$. But what's the guarantee that we get $\boldsymbol{x} \geq 0$? If one tries such a naive approach, there is a chance go being unlucky "choose m out of n times", which may be too many.

There are at least two ways to do it more cleverly, which both use the same trick: introduce more variables, which will later be required to become zero. (Such technique in maths goes a long way...) They are the "big M method" and the "Two-phase method". I prefer the latter.

Example: `Min` $2x_1 + 3x_2$, subject to $x_1, x_2 \geq 0$ and $.5x_1 + .25x_2 \leq 4,\ x_1 + 3x_2 \geq 20,\ x_1 + x_2 = 10$.

Canonical form: min $2x_1 + 3x_2$ for $x \geq 0$, such that

$$\begin{cases} .5x_1 + .25x_2 + x_3 = 4 \\ x_1 + 3x_2 - x_4 = 20 \\ x_1 + x_2 = 10 \end{cases}$$

Problem: there is no initial BFS readily available to start SM. I.e. the matrix in the left-hand side does not have a square unit submatrix, whereof a BFS could be read. Solution to this: introduce extra, "*artificial,* variables", as few as possible, to create the unit submatrix:

$$\begin{cases} .5x_1 + .25x_2 + x_3 = 4 \\ x_1 + 3x_2 - x_4 + x_5 = 20 \\ x_1 + x_2 + x_6 = 10 \end{cases},$$

The original problem is equivalent now to the following task. For $x \geq 0$ let minimize $2x_1 + 3x_2$ *provided that* $x_5 = x_6 = 0$.

`Note:` The number of artificial variables to be used is *at most* the number of "bad" constraints, which are '$\geq$' or '$=$' constraints. But in principle, it can be less. All we need is to generate the unit submatrix.

Now, how to take care of "*provided that* $x_5 = x_6 = 0$"? There are at least two ways. One is, let M be a very large number. Do the usual SM computation for the objective minimize $2x_1 + 3x_2 + M(x_5 + x_6)$, and carry the symbol M through until is disappears from the objective value. The quantity M "works against", the objective. M is huge. So, if the original problem is feasible, the optimal solution will not involve M. This is called the "big M" method.

However, it is technically less messy to split the procedure in two steps: Phase I and Phase II. On the first step, one will be dealing with artificial variables and the objective to *minimize the sum of all artificial variables.* AS all of them are non-negative, if one achieves the minimum value zero, the optimal solution for this auxiliary problem will be feasible for the initial problem. Then, on Phase II one will delete all the artificial variables and return to the original objective.

- Phase I: Consider an auxiliary problem - Minimize $\tilde{z} = x_5 + x_6$.

BV\V	$\tilde{z}$	x_1	x_2	x_3	x_4	x_5	x_6	Val
x_3	0	.5	.25	1	0	0	0	4
x_5	0	1	3	0	-1	$\mathbf{1}$	0	20
x_6	0	1	1	0	0	0	$\mathbf{1}$	10
$\tilde{z}$	1	0	0	0	0	-1	-1	0

Important: The x_5, x_6 columns are not yet columns of the unit matrix. Before anything is done, the *basic* variables x_5, x_6 should be eliminated from the objective $\tilde{z}$.

BV\V	$\tilde{z}$	x_1	x_2	x_3	x_4	x_5	x_6	Val
x_3	0	.5	.25	1	0	0	0	4
x_5	0	1	$\mathbf{3}$	0	-1	1	0	20
x_6	0	1	1	0	0	0	1	10
$\tilde{z}$	1	2	4	0	-1	0	0	30

BV\V	$\tilde{z}$	x_1	x_2	x_3	x_4	x_5	x_6	Val
x_3	0	$\frac{5}{3}$	0	4	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{28}{3}$
x_2	0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$
x_6	0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$
$\tilde{z}$	1	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{4}{3}$	0	$\frac{10}{3}$

BV\V	$\tilde{z}$	x_1	x_2	x_3	x_4	x_5	x_6	Val
x_3	0	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$
x_2	0	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5
x_1	0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5
$\tilde{z}$	1	0	0	0	0	-1	-1	0

Conclusion: $x_1 = x_2 = 5$, $x_3 = .25$ is a feasible solution for the original problem.

Remark: If at this stage one finds the optimal value $\tilde{z} > 0$, it means it is impossible to zero the artificial variables, so the original problem would be *unfeasible.*

- Phase II: Now throw away the columns x_5 and x_6 as well as the row $\tilde{z}$; add row $z - 2x_1 - 3x_2 = 0$, corresponding to the initial objective function. Now one starts out with the BFS $x_1 = x_2 = 5$, $x_3 = .25$ and the tableau

BV\V	z	x_1	x_2	x_3	x_4	Val
x_3	0	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$
x_2	0	0	**1**	0	$-\frac{1}{2}$	5
x_1	0	**1**	0	0	$\frac{1}{2}$	5
z	1	-2	-3	0	0	0

  Important: The x_1, x_2 columns are not yet columns of the unit matrix. Before we proceed, the *basic* variables x_1, x_2 should be eliminated from the objective z.

BV\V	z	x_1	x_2	x_3	x_4	Val
x_3	0	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$
x_2	0	0	**1**	0	$-\frac{1}{2}$	5
x_1	0	**1**	0	0	$\frac{1}{2}$	5
z	1	0	0	0	$-\frac{1}{2}$	25

  This tableau is now ready for the simplex method. But it is simultaneously the final tableau: there are no positive reduced costs (entries, corresponding to free variables) in the objective row.

  Conclusion: $x_1 = x_2 = 5$, $x_3 = .25$ is the optimal solution for the original problem.

Remark: Using short tableaus should not be a problem, and is again really shorter. E.g., after Phase I, removing the artificial objective row and erasing the free artificial variables' columns we would just have

BV\V	x_4	Val
x_3	$-\frac{1}{8}$	$\frac{1}{4}$
x_2	$-\frac{1}{2}$	5
x_1	$\frac{1}{2}$	5

and bringing in the objective $z = 2x_1 + 3x_2$ would mean adding an extra row $z \;\; -\frac{1}{2} \;\; 5$, which is merely $2{\cdot}x_1$-row + $3{\cdot}x_2$-row.