1. See Figure 1. Answers: (a) unique minimum $(1/2, 0)$, value $1/2$; (b),(d) unbounded; (c) unfeasible; (e) alternative maxima $P = (12, 0)$ and $Q = (0, 14)$ and any point $S = \lambda P + (1 - \lambda)Q$, $0 < \lambda < 1$ in between, value 84.

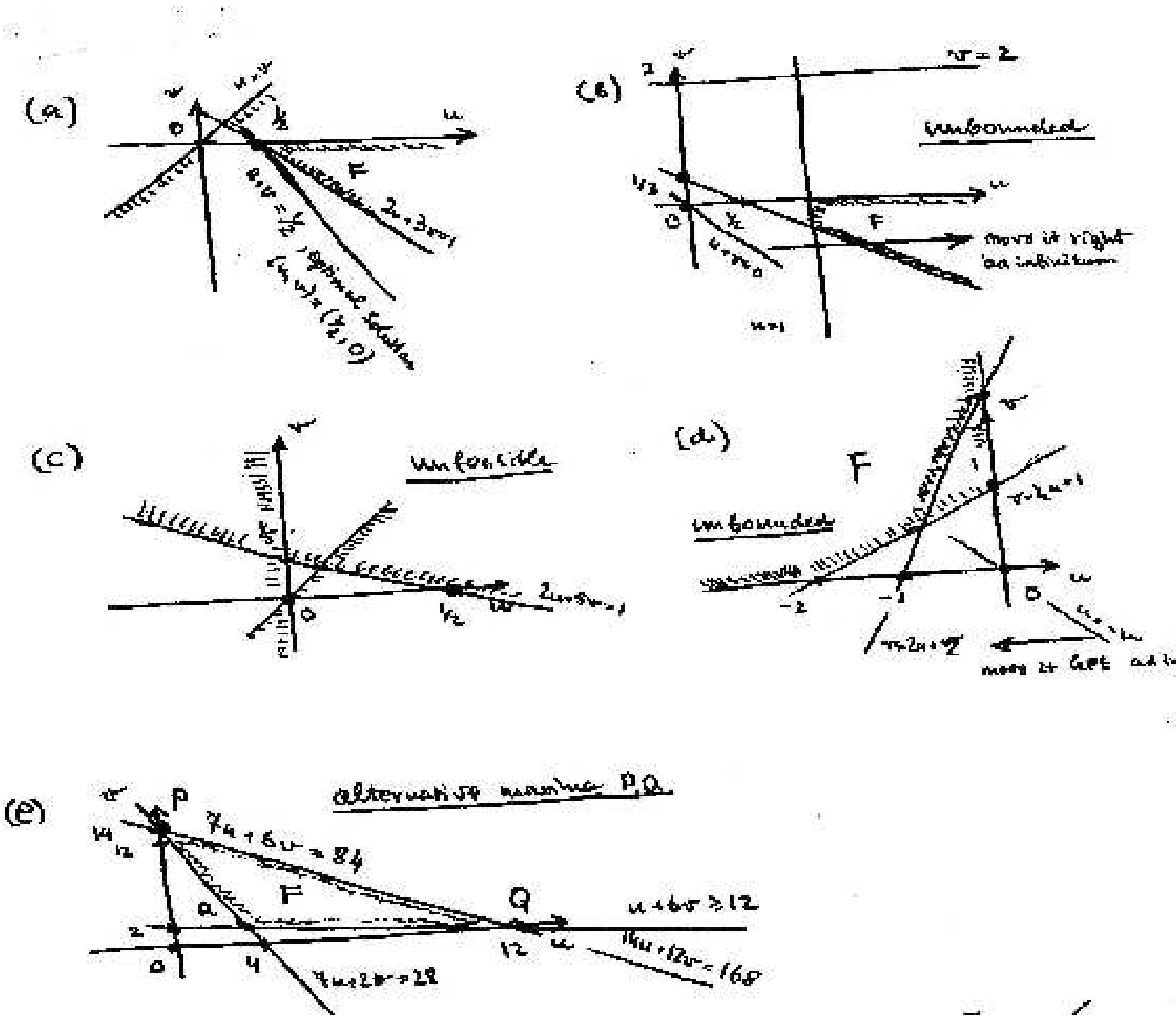

Figure 1: Figures for Problem 1

2. The canonical forms are

   (a) Minimise $u - \tilde{v}$ subject to $2u - 3\tilde{v} - s_1 = 1$, $u + \tilde{v} - s_2 = 0$; $u, \tilde{v}, s_1, s_2 \geq 0$. Above, $\tilde{v} = -v$, s_1, s_2 - excess variables.

   (b) Maximise $3 + \tilde{u} - \tilde{v}$ subject to $-2\tilde{u} + 3\tilde{v} + s_1 = 7$, $\tilde{u} + \tilde{v} - s_2 = 1$; $\tilde{u}, \tilde{v}, s_1, s_2 \geq 0$. Above, $\tilde{u} = u - 1$, $\tilde{v} = 2 - v$, s_1 - a slack and s_2 - an excess variable.

   (c) Minimise $-\tilde{u}+x-y$ subject to $-2\tilde{u}+5x-5y-s_1 = 1$, $\tilde{u}+x-y+s_2 = 0$; $\tilde{u}, x, y, s_1, s_2 \geq 0$. Above, $\tilde{u} = -u$, $v = x - y$, s_1 - an excess and s_2 - a slack variable.

(d) Minimise $-\tilde{u}+v$ subject to $v+2\tilde{u}-s_1=2, \quad \tilde{u}+2v-s_2=2; \; \tilde{u}, v, s_1, s_2 \geq 0$. Above, $\tilde{u}=-u$, s_1, s_2 - excess variables.

(e) Maximise $7a-7b+6c-6d$ subject to $7a-7b+2c-2d-s_1=28, \; a-b+6c-6d-s_2=12, \; 14a-14b+12c-12d+s_3=168; \; a,b,c,d,s_1,s_2,s_3 \geq 0$. Above, $u=a-b$, $v=c-d$, s_1, s_2 - excess variables, s_3 - a slack variable.

3. Let $A=[\boldsymbol{a}^1 \; \boldsymbol{a}^2 \; \boldsymbol{a}^3]$.

- Basic solutions:

  $\boldsymbol{b}=(2,0)$: no solution, $\qquad \boldsymbol{b}=(3,3)$: $\boldsymbol{x}=(3,0,0)$,

  $\boldsymbol{b}=(3,6)$: $\boldsymbol{x}=(3,3,0)$ or $\boldsymbol{x}=(4.5,0,1.5)$, $\qquad \boldsymbol{b}=(6,3)$: no solution,

  $\boldsymbol{b}=(-2;4)$: $\boldsymbol{x}=(0,3,3)$ or $\boldsymbol{x}=(1,0,3)$, $\qquad \boldsymbol{b}=(0,7)$: $\boldsymbol{x}=(0,7,0)$ or $\boldsymbol{x}=(3.5,0,3.5)$,

  $\boldsymbol{b}=(-5,-5)$: no solution.

  See Figure 2.

  Feasible solutions will exist only for those $\boldsymbol{b}$, which lie within the shaded region on the plane. Indeed, given a pair of vectors $(\boldsymbol{a}^\alpha, \boldsymbol{a}^\beta)$, representing a pair of columns of A, the basic solution $x_\alpha \boldsymbol{a}^\alpha + x_\beta \boldsymbol{a}^\beta = \boldsymbol{b}$, depending on these columns, with $x_{\alpha,\beta} \geq 0$ will exist if and only if $\boldsymbol{b}$ lies within a sector of the plane, bounded by the rays in the directions of $\boldsymbol{a}^\alpha$ and $\boldsymbol{a}^\beta$. Then, the values of $\boldsymbol{b}=(2,0),(6,3),(-5,5)$ will not have feasible solutions, as they are outside the union of all the feasible sectors. On the other hand, $\boldsymbol{b}=(3,3)$ lies only in the sector spanned by $(\boldsymbol{a}^1, \boldsymbol{a}^2)$ (on the boundary) hence there is a unique BFS, corresponding to it. The values of $\boldsymbol{b}=(3,6),(-2,4)$ each belong to two sectors simultaneously, hence there are two corresponding different BFSs. Finally, $\boldsymbol{b}=(0,7)$ lies in all the three sectors (spanned by the pairs $\boldsymbol{a}^{1,2}, \boldsymbol{a}^{2,3}$ and $\boldsymbol{a}^{1,3}$). However, due to the degeneracy of the intersection of the first two sectors, which is only the vertical line, whereupon there sits the point $\boldsymbol{b}=(0,7)$, there are only two (rather than three) BFSs, corresponding to this value of $\boldsymbol{b}$.

  Note that $\boldsymbol{b}=(3,3)$ and $(0,7)$ do not satisfy the non-degeneracy assumption, introduced in class: they can be expressed as linear combinations of *less than two* (namely one) columns of A.

- Most importantly, the equation $x_1+x_2+x_3=b_2$, $\boldsymbol{x} \geq 0$ tells us that if $b_2<0$, there are no solutions, and if $b_2 \geq 0$, each x_j is bounded by b_2. Hence, the feasible set is bounded, and therefore the LP is *not unbounded*, i.e. has optimal solutions (if it is feasible).

  So, to minimise x_2-x_3, for $\boldsymbol{b}=(0,7)$ it is enough to compare the former value for the pair of BFSs, corresponding to this $\boldsymbol{b}$. Clearly, $\boldsymbol{x}=(3.5,0,3.5)$ makes x_2-x_3 smaller (value -3.5) than $\boldsymbol{x}=(0,7,0)$ (value 7).

4. Let $A=[\boldsymbol{a}^1 \; \boldsymbol{a}^2 \; \boldsymbol{a}^3 \; \boldsymbol{a}^4]$. It's the same as for the previous problem with the extra column $\boldsymbol{a}^4=(1,0)$.

- The shaded region for the previous problem in Figure 2, plus an additional sector spanned by the pair of vectors $(\boldsymbol{a}^4, \boldsymbol{a}^1)$.
- BFSs for different values of $\boldsymbol{b}$:

(a) $\boldsymbol{b} = (0,7)$, apart from the solutions in Problem 4 (extended to four components by letting $x_4 = 0$), also acquires a BFS $\boldsymbol{x} = (0,0,7,7)$.

(b) $\boldsymbol{b} = (1,1)$, apart from the solution $(1,0,0,0)$, also acquires BFSs $\boldsymbol{x} = (0,1,0,1)$ and $\boldsymbol{x} = (0,0,1,2)$.

(c) $\boldsymbol{b} = (-1,1)$ yields a unique BFS $\boldsymbol{x} = (0,0,1,0)$, as it would be for the previous problem.

(d) $\boldsymbol{b} = (-1,0)$ yields no solutions, lying outside the union of all the sectors.

- Most importantly, the equation $x_1 + x_2 + x_3 + 0x_4 = b_2$, $\boldsymbol{x} \geq 0$ tells us that if $b_2 < 0$, there are no solutions, and if $b_2 \geq 0$, each $x_{1,2,3}$ is bounded by b_2. Although the feasible set is not bounded (x_4 may go to infinity), this LP is *not unbounded*, as long as x_4 does not appear in the objective function.

  So to minimise x_1, with $\boldsymbol{b} = (7,0)$, check the three BFSs, corresponding to it: the minimum value 0 is supplied by either $\boldsymbol{x}^1 = (0,7,0,0)$ or $\boldsymbol{x}^2 = (0,0,7,7)$. So, we are in the alternative optima case. Then any $\boldsymbol{x} = \lambda \boldsymbol{x}^1 + (1 - \lambda \boldsymbol{x}^2)$ for $0 < \lambda < 1$ would also be an optimal, but non-basic solution.
- To maximize x_2, with $\boldsymbol{b} = (1,1)$, inspecting the three BFSs above, corresponding to it, one sees that the unique optimal solution is $\boldsymbol{x} = (0,1,0,1)$.

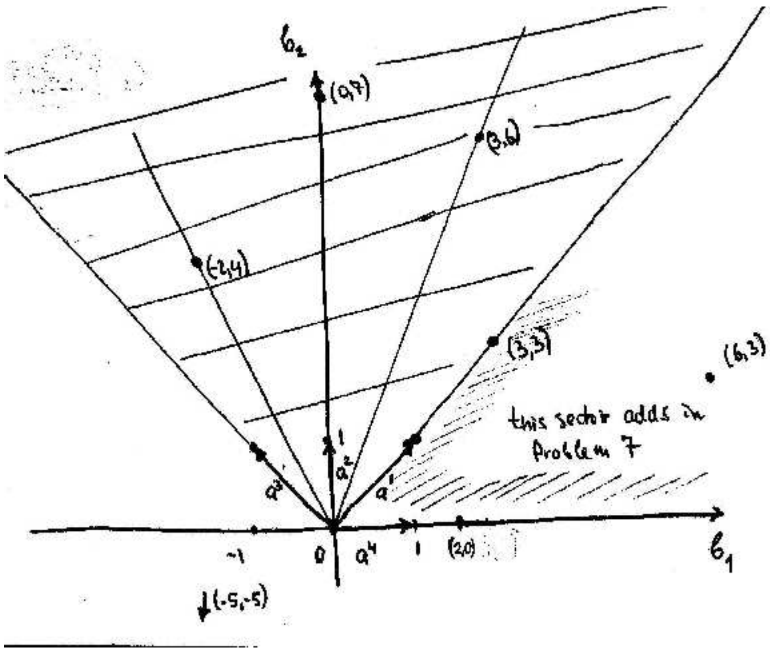

Figure 2: Illustration for Problems 4,5

5. $\boldsymbol{x} = \sum_{k=1}^{N} \theta_k \boldsymbol{x}^k$, where for all $k = 1,2,\ldots,N$, $0 \leq \theta_k \leq 1$ and $\sum_{k=1}^{N} \theta_k = 1$, where $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^N$ list *all* the BFSs of $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$. Compute the quantity $A\boldsymbol{x}$:

$$A\boldsymbol{x} = A\sum_{k=1}^{N} \theta_k \boldsymbol{x}^k = \sum_{k=1}^{N} \theta_k (A\boldsymbol{x}^k) = \left(\sum_{k=1}^{N} \theta_k\right) \boldsymbol{b} = \boldsymbol{b}.$$

So $\boldsymbol{x}$ is feasible indeed, as one clearly has $\boldsymbol{x} \geq 0$..

6. (a) Initial tableau, the columns labeled by $(z, x_1, \ldots, x_5, \text{Val})$ (for *value*).

$$T0 = \begin{bmatrix} 0 & -1 & 2 & 1 & 0 & 0 & 4 \\ 0 & 3 & 2 & 0 & 1 & 0 & 14 \\ 0 & \mathbf{1} & -1 & 0 & 0 & 1 & 3 \\ 1 & -3 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot $T0_{32}$:

$$T1 := \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 7 \\ 0 & 0 & \mathbf{1} & 0 & \frac{1}{5} & \frac{-3}{5} & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 0 & -5 & 0 & 0 & 3 & 9 \end{bmatrix}$$

Pivot $T1_{23}$:

$$T2 = \begin{bmatrix} 0 & 0 & 0 & 1 & \frac{-1}{5} & \frac{8}{5} & 6 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{-3}{5} & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & 4 \\ 1 & 0 & 0 & 0 & 1 & 0 & 14 \end{bmatrix}$$

Optimal value $z = 14$, achieved by $x_4 = 0$, $x_1 = 4 - .4x_5$, $x_2 = 1 + .6x_5$, $x_3 = 6 - 1.6x_5$, $0 \leq x_5 \leq 3.75$: multiple solutions.

(b) Initial tableau, the columns labeled by $(z, x_1, \ldots, x_5, \text{Val})$.

$$T0 = \begin{bmatrix} 0 & 3 & 1 & 0 & 1 & 0 & 18 \\ 0 & 0 & \mathbf{2} & 1 & 0 & 1 & 7 \\ 1 & -2 & 8 & 5 & 0 & 0 & 0 \end{bmatrix}$$

Pivot $T0_{23}$:

$$T1 = \begin{bmatrix} 0 & 3 & 0 & \frac{-1}{2} & 1 & \frac{-1}{2} & \frac{29}{2} \\ 0 & 0 & 1 & \mathbf{\frac{1}{2}} & 0 & \frac{1}{2} & \frac{7}{2} \\ 1 & -2 & 0 & 1 & 0 & -4 & -28 \end{bmatrix}$$

Pivot $T2_{24}$:

$$T2 = \begin{bmatrix} 0 & 3 & 1 & 0 & 1 & 0 & 18 \\ 0 & 0 & 2 & 1 & 0 & 1 & 7 \\ 1 & -2 & -2 & 0 & 0 & -5 & -35 \end{bmatrix}$$

Optimal value -35, with a BOS $x_4 = 18$, $x_3 = 7$, $x_1 = x_2 = x_5 = 0$. Note: pivoting $T0_{24}$ in the original tableau would have completed the procedure in one step, as x_3 had a reduced cost 8, with a minimum ratio 3.5, while x_4 had a minimum ratio 7 with reduced cost 5, i.e. 7 units of x_4 reduce the cost by more than 3.5 units of x_3.