Convexity and Farkas

1. In this example both sets X, Y are unbounded, and the infimum of the distance between points of X, Y is zero. The role of the separating hyperplane is now played by the line $y = x$. Hence, a safe restatement of the theorem should be: if X, Y are two disjoint, close and convex sets, there is a hyperplane $\boldsymbol{n} \cdot \boldsymbol{x} = \alpha$, such that for all $\boldsymbol{x} \in X$, $\boldsymbol{n} \cdot \boldsymbol{x} \geq \alpha$, for all $\boldsymbol{y} \in Y$, $\boldsymbol{n} \cdot \boldsymbol{y} \leq \alpha$.

2. (a) By definition of a convex set, if $\boldsymbol{y}^1 = \sum \theta_i \boldsymbol{x}^i$, $\boldsymbol{y}^2 = \sum \eta_i \boldsymbol{x}^i$, then the convex combination

$$\lambda \boldsymbol{y}^1 + (1-\lambda)\boldsymbol{y}^2 = \sum (\lambda\theta_i + (1-\lambda)\eta_i)\boldsymbol{x}^i,$$

and all the coefficients $(\lambda\theta_i + (1-\lambda)\eta_i) \geq 0$ and sum up to 1:

$$\sum (\lambda\theta_i + (1-\lambda)\eta_i) = \lambda \cdot 1 + (1-\lambda)\cdot 1 = 1.$$

So, a convex hull is a convex set.

(b) Two points: line segment, three points not on the same line - a triangle, etc. Four points, not on the same line: either a quadrangle or a triangle (if one of the points lies in the convex hull of the remaining three).

(c) For two points: $s = 2$, $\theta_2 = (1 - \theta_1)$, the convex hull X is the line segment, connecting them. Now the claim is as follows. Consider the convex hull X' of $s - 1$ points $\boldsymbol{x}^2, \ldots, \boldsymbol{x}^s$. Then X contains all the points that lie between $\boldsymbol{x}^1$ and some point $\boldsymbol{x}' \in X'$, including the endpoints. `Proof:` Let $\boldsymbol{x} = \theta_1 \boldsymbol{x}^1 + \theta_2 \boldsymbol{x}^2 + \ldots \theta_s \boldsymbol{x}^s$. Then $\theta_2 + \ldots + \theta_s = 1 - \theta_1$. If $\theta_1 = 1$, then $\boldsymbol{x} = \boldsymbol{x}^1$. If $\theta_1 = 0$, then $\boldsymbol{x} \in X'$. If $\theta_1 \in (0, 1)$, do the following:

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}^1 + (1-\theta_1)\left[\frac{\theta_2}{1-\theta_1}\boldsymbol{x}^2 + \ldots + \frac{\theta_s}{1-\theta_1}\boldsymbol{x}^s\right].$$

Notice that $\dfrac{\theta_2}{1-\theta_1} + \dfrac{\theta_3}{1-\theta_1} + \ldots + \dfrac{\theta_s}{1-\theta_1} = 1$, so in square brackets one has some $\boldsymbol{x}' \in X'$.

(d) Use (iii) and induction. For the convex hull of two distinct points, they are extreme points, the endpoints of the line segment. Assume the statement for the convex hull of $s - 1$ points. When the sth point is added, it will only become an extreme point, provided that it does not lie in the convex hull of the first $s-1$ points. If it does, nothing happens. Otherwise, it is the only extra extreme point added, although there may be a situation when one of the extreme points of the convex hull of the first $s-1$ points will thereafter cease beng an extreme point. IN any case, there are no extreme points but "our" points.

3. Add slack variables $\boldsymbol{s} \geq 0$, so with $\tilde{A} = [A, I]$ and $\tilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{s})$, the primal, or positive Farkas alternative becomes $\tilde{A}\tilde{\boldsymbol{x}} = \boldsymbol{b}$ has a solution $\tilde{\boldsymbol{x}} \geq 0$. The dual, or negative, side of the alternative is then $\exists \boldsymbol{n} : \boldsymbol{n} \cdot \boldsymbol{b} < 0$ and $\tilde{A}^T \boldsymbol{n} \geq 0$. The latter condition means that $A^T \boldsymbol{n} \geq 0$ and $\boldsymbol{n} \geq 0$. So, similar to LP duality, the condition $\boldsymbol{n} \geq 0$ gets added to the Farkas alternative for $\geq$ inequalities. If the condition $\boldsymbol{x} \geq 0$ is dropped, then one has to replace $\boldsymbol{x} = \boldsymbol{u} - \boldsymbol{v}$, where $\boldsymbol{u}, \boldsymbol{v} \geq 0$. This means, A gets replaced by $[A \ \ -A]$, so on the dual Farkas side $A^T \boldsymbol{n} \geq 0$ gets replaced by $A^T \boldsymbol{n} \geq 0$ and $(-A)^T \boldsymbol{n} \geq 0$, i.e. $A^T \boldsymbol{n} = 0$.

4. If $A\boldsymbol{x} = 0$, $x \neq 0$, where $A \in \mathbb{R}^{m\times n}$. We can assume that $x_1+x_2+\ldots+x_n = 1$, because $\boldsymbol{x}$ is defined up to a positive multiple. Then let $\tilde{A}$ be obtained from A by adding an extra row $[1\ 1\ \ldots\ 1]$, and let $\tilde{\boldsymbol{b}} \in \mathbb{R}^{m+1}$ arise from $\boldsymbol{b} = \boldsymbol{0}$ by adding an extra component equal to one. The positive side of Farkas is now $\tilde{A}\boldsymbol{x} = \tilde{\boldsymbol{b}}$ has a solution $\boldsymbol{x} \geq 0$. The opposite side is: there exists $\boldsymbol{n} \in \mathbb{R}^{m+1}$, such that $\boldsymbol{n}\cdot\tilde{\boldsymbol{b}} < 0$ and $\tilde{A}^T\boldsymbol{n} \geq 0$. Denote $\tilde{\boldsymbol{n}} = (\boldsymbol{y}, r)$, where $\boldsymbol{y} \in \mathbb{R}^m$, $r \in \mathbb{R}$. It follows that

$$\boldsymbol{y}\cdot\boldsymbol{0} + r < 0,\ A^T\boldsymbol{y} \geq -r\boldsymbol{1},$$

where $\boldsymbol{1}$ is a vector in $\mathbb{R}^n$, whose every component equals 1. So, the alternative becomes $\exists\,\boldsymbol{y}$ such that $A^T\boldsymbol{y} > 0$.

5. See the handout on Farkas, the asset pricing example. Anyway, here is the solution.

Observe that the condition of positivity of some component $(A\boldsymbol{x})_i$ and non-negativity of the rest of the components, as $\boldsymbol{x}$ there is defined up to a positive multiplier, can be interpreted as $\boldsymbol{e}\cdot A\boldsymbol{x} = 1$, where $\boldsymbol{e} = (1,\ldots,1) \in \mathbb{R}^{m+1}$. Now, as $\boldsymbol{x} \geq 0$ is *not* required, we have to let $\boldsymbol{x} = \boldsymbol{u} - \boldsymbol{v}$, with $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_+$. In addition, the condition $A\boldsymbol{x} \geq 0$ is equivalent to $A\boldsymbol{x} - I\boldsymbol{s} = 0$, where $\boldsymbol{s} \in \mathbb{R}^{m+1}_+$ and I is the identity matrix. Putting it all together, the front side of the Farkas alternative, corresponding is $\tilde{A}\boldsymbol{w} = \boldsymbol{b}$ has a solution $\boldsymbol{w} \geq 0$, with $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{s}) \in \mathbb{R}^{2n+(m+1)}_+$, $\boldsymbol{b} = (1, \boldsymbol{0}) \in \mathbb{R}^{1+m}$, and

$$\tilde{A} = \begin{pmatrix} \boldsymbol{e}^T A & -\boldsymbol{e}^T A & 0 \\ A & -A & -I \end{pmatrix}.$$

The opposite side of Farkas is: there exists $\boldsymbol{n} = (z, \boldsymbol{y}) \in \mathbb{R}^{1+m}$, such that $z < 0$ and

$$\begin{cases} zA^T\boldsymbol{e} + A^T\boldsymbol{y} \geq 0, \\ -zA^T\boldsymbol{e} - A^T\boldsymbol{y} \geq 0, \\ \boldsymbol{y} \leq 0. \end{cases}$$

This means $-A^T(\boldsymbol{y} + z\boldsymbol{e}) = 0$, with $\boldsymbol{y} \leq 0$, $z < 0$. Changing $\boldsymbol{y}$ to $-(\boldsymbol{y} + z\boldsymbol{e}) > 0$ now proves the claim.

Zero-sum two-person games

1. Find optimal strategies for the games with the following payoff matrices:

$$\text{(i)}\ \begin{pmatrix} 5 & -9 \\ -7 & 4 \end{pmatrix},\quad \text{(ii)}\ \begin{pmatrix} 2 & 1 & -1 \\ -1 & -2 & 3 \end{pmatrix}.$$

As the games are small, avoid doing the simplex method for the underlying LPs, solving them e.g. graphically (or just guessing) and then using complementary slackness to solve the dual and verify optimality.

Solutions: (i) Add 10 to each entry of A to get

$$\tilde{A} = \begin{pmatrix} 15 & 1 \\ 3 & 14 \end{pmatrix},$$

now consider the problem for the row player: $\tilde{A}^T\boldsymbol{x} \leq 1$, $max\ x_1 + x_2$ i.e. $\boldsymbol{x} \geq 1$, $15x_1 + 3x_2 \leq 1$, $x_1 + 14x_2 \leq 1$. Optimal solution is when both constraints are tight: $x_1 = 11/207, x_2 = 14/207$, so the optimal strategy $\boldsymbol{q} = \frac{\boldsymbol{x}}{x_1+x_2}$, i.e. $q_1 = 11/(11+14) = 11/25$, $q_2 = 14/25$. The value of the game is $V = -10 + \frac{1}{x_1+x_2} = -43/25$.

For the column player we have (tight) dual constraints $15y_1 + y_2 = 1$, $3y_1 + 14y_2 = 1$, so $y_1 = 13/207, y_2 = 12/207$, thus $p_1 = 13/25, p_2 = 12/25$.

(ii) To simplify the solutions, observe that the column player should never paly the second pure strategy, because it is always worse for him than the first one. In other words, $p_2 = 0$. This observation is not necessary, but saves some work. (Equivalently, if this is not used, the second dual constraint in the following row player's LP $A'y \leq 1$, max $y_1 + y_2$ will be slack, and therefore $p_2 = 0$). So, we are in fact dealing with the payoff matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}$$

after adding 2 to each entry, and all it really takes is to solve $\tilde{A}^T \boldsymbol{x} \leq 1$ for the row player and $\tilde{A}\boldsymbol{y} \geq 1$ for the column one, where the constraints, are, in fact, tight. So $\boldsymbol{x} = \boldsymbol{y} = (4/19, 3/19)$, and the value of the game is $V = -2 + \frac{1}{4/19+3/19} = \frac{5}{7}$. The best strategies are $\boldsymbol{q} = \frac{\boldsymbol{y}}{y_1+y_2} = (4/7, 3/7)$ for the row player, and the same for the column player for the pair (p_1, p_3), i.e. $\boldsymbol{p} = (4/7, 0, 3/7)$.

2. (a) The payoff matrix for P as column player.

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 2 & -3 & 4 \end{pmatrix}$$

(b) The expected payoff to P, if he plays the strategy $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and Q plays the strategy $q = (\frac{1}{2}, \frac{1}{2})$ is

$$q \cdot Ap = \frac{1}{6}$$

(c) Add 4 to every element of A, get

$$B = \begin{pmatrix} 3 & 6 & 1 \\ 6 & 1 & 8 \end{pmatrix}$$

Now the problem for Q is Max $y_1 + y_2$, such that $B^T \boldsymbol{y} \leq 1$, and the problem for P is Min $x_1 + x_2 + x_3$, such that $B\boldsymbol{x} \leq 1$. Solve for Q first, then use complementary slackness.
Solve graphically. To avoid drawing pictures here, let's do it by hand, by considering three 2×2 systems as equalities and checking the remaining constraint:

$$3y_1+6y_2 = 1,\ 6y_1+y_2 = 1 \qquad \text{or} \qquad 6y_1+y_2 = 1,\ y_1+8y_2 = 1\text{or} \qquad 3y_1+6y_2 = 1 = 1,\ y_1+8y_2 = 1.$$

The first system has solution $\boldsymbol{y}^1 = (5/33, 3/33)$, with the value $8/33$. Does it satisfy $y_1+8y_2 \leq 1$? Yes, it does.
The second one has solution $\boldsymbol{y}^2 = (7/47, 5/47)$, with the value $12/47 > 8/33$. Does it satisfy $3y_1 + 6y_2 \leq 1$.? No, it does not.
The third system has the solution $\boldsymbol{y}^3 = (1/9, 1/9)$, with the value $2/9 < 8/33$.
So, the optimiser for Q, is the solution $(5/33, 3/33)$, with the value $8/33$, the constraint $y_1 + 8y_2 \leq 1$ being slack. Now the value of the game is $-4 + \frac{1}{8/33} = 1/8$. The best mixed strategy for Q is $(\frac{33}{8}(5/33, 3/33) = (5/8, 3/8)$. The third Q's constraint is slack, hence P never plays the third move.
Now the problem for P is just min $x_1 + x_2$ such that $3x_1 + 6x_2 \geq 1$, $6x_1 + 3x_2 \geq 1$, with the solution, coincidentally, $(5/33, 3/33)$, so the best strategy for P is $(5/8, 3/8, 0)$.

(d) To punish Q for not playing his best strategy, but rather $q = (\frac{1}{2}, \frac{1}{2})$, P should maximize $p \cdot (1/2, -1/2, 1/2)$ (the latter vector is $A^T q$, with the given $q = (\frac{1}{2}, \frac{1}{2})$. This clearly happens if he plays the first or the third strategy, or any combination of the two; with the expected payoff $\frac{1}{2}$.

3. `Optional:` A variant of the *Morra* game (Google it!). As a single act of the game, two players will independently show each other one or two fingers. Before this is done, each tries to guess how many fingers the opponent will show. The payoff is zero if *both* players guess right or wrong. Otherwise, the player who guessed in the wrong pays his opponent the sum equal to the total number of fingers shown by both players.

(a) Describe the pure strategies and the payoff matrix A.

The pure strategies are pairs (S, G) (show,guess) where both $G, S \in \{1, 2\}$. So there are four pure strategies, and the payoff matrix, with the rows/columns labeled by $(1,1), (1,2), (2,1), (2,2)$ is

$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}$$

(b) Show that the strategy $\boldsymbol{p} = (0, 7/12, 5/12, 0)$ is optimal – use the fact the game is symmetric.

All it takes is to verify that $A\boldsymbol{p} \geq 0$, as the game is symmetric, so its value is zero.

(c) Show that any strategy $(0, p_2, p_3, 0)$ with $4/3 \leq p_2/p_3 \leq 3/2)$ is, in fact, optimal, and there are no others.

The above solves the inequalities $-2p_2 + 3p_3 \geq 0,\ \ 3p_2 - 4p_3 \geq 0$. This is what $A\boldsymbol{p} \geq 0$ boils down to as long as $p_1 = p_4 = 0$.

On the other hand, the second and third constraints read $2p_1 - 3p_4 \geq 0, \ \ -3p_1 + 4p_4 \geq 0$, which is inconsistent, except $p_1 = p_4 = 0$.

(d) How much would you expect to win per game if your opponent plays a mixed strategy $(.1, .4, .3, .2)$?

To punish the row player for playing a wrong strategy $\boldsymbol{q}$, the column player should solve $\max \boldsymbol{q} \cdot A\boldsymbol{p}$ for $\boldsymbol{p}$, such that $\sum p_i = 1$.

So the objective with $\boldsymbol{q} = (.1, .4, .3, .2)$ becomes

$$\boldsymbol{q} \cdot A\boldsymbol{p} = (A^T \boldsymbol{q}) \cdot \boldsymbol{p} = -.1p_1 + .4p_2 - .5p_3,$$

which is clearly maximised by a pure strategy $p_2 = 1$, with the value .4.

4. `Optional:` A game of "hide and seek" is played as follows. Player $\mathcal{H}$ chooses a place to hide in the matrix below— that is, chooses a particular entry. Player $\mathcal{S}$ chooses either a row or a column in which to seek player $\mathcal{H}$. The choices are made simultaneously and independently. If it turns out that Player $\mathcal{H}$ is in Player $\mathcal{S}$'s row or column, then Player $\mathcal{H}$ pays Player $\mathcal{S}$ x pence, where x equals the value of the entry in which Player $\mathcal{H}$ is hiding; if not, Player $\mathcal{S}$ pays Player $\mathcal{H}$ 1 pence.

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 4 & 3 \end{matrix}$$

Identify pure strategies, set up the payoff matrix, the LP, and then get the optimal strategies using the on-line solver. What is the value of this game?

Solution.

Player $\mathcal{H}$ has pure strategies which can be labeled ij (meaning choose the entry in the ith row and jth column). Write them in the order 11,12, 13,21,22,23. List Player $\mathcal{S}$'s pure strategies as $r1$,$r2$,$c1$,$c2$,$c3$. Then Player $\mathcal{S}$'s payoff matrix is

$$\begin{pmatrix} 1 & 2 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 4 & 3 \\ 1 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 4 & -1 \\ -1 & -1 & 3 & -1 & -1 & 3 \end{pmatrix}.$$

Add 1 to each entry to give a matrix A' with non-negative entries, and solve the resulting LP online. The optimal mixed strategy for Player $\mathcal{S}$ is then $p = \frac{1}{45}(22, 12, 8, 0, 3)$.

with value 1/3. Likewise Player $\mathcal{H}$'s optimal mixed strategy is $q = \frac{1}{3}(2, 0, 0, 0, 1)$.