On distance measures for well-distributed sets

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Abstract

In this paper we investigate the Erdös/Falconer distance conjecture for a natural class of sets statistically, though not necessarily arithmetically, similar to a lattice. We prove a good upper bound for spherical means that have been classically used to study this problem. We conjecture that a majorant for the spherical means suffices to prove the distance conjecture(s) in this setting. For a class of non-Euclidean distances, we show that this generally cannot be achieved, at least in dimension two, by considering integer point distributions on convex curves and surfaces. In higher dimensions, we link this problem to the question about the existence of smooth well-curved hypersurfaces that support many integer points.

1 Introduction

In this paper we study the Erdös/Falconer distance problems, introduced in [5] and [6], respectively. In the discrete and continuous settings they ask whether an appropriate cardinality/Hausdorff dimension condition on a subset of the Euclidean space guarantees that the set of pair-wise distances determined by the set is also suitably large.

Both problems have been studied rather intensely in recent years using a diverse set of methods and ideas. As a result, the best results available today do not necessarily relate to each other immediately. In this paper we make an effort to address this problem.

In a precise form, the Erdös conjecture says that for a finite point set $E \subset \mathbb{R}^d$, the distance set

$$|\Delta(E)| \geq C^{-1}(|E|)|E|^{\frac{2}{d}}, \tag{1.1}$$

where

$$\Delta(E) = \{\|x-y\| : x, y \in E\} \tag{1.2}$$

and the "constant" quantity $C(|E|)$ is only allowed to grow asymptotically slower than any power of the cardinality $|E| \to \infty$. In the sequel $\|\cdot\|$ is the Euclidean norm, the notation $|E|$ is used to denote the cardinality of a finite set or the Lebesgue measure of $E \subset \mathbb{R}^d$.

The Falconer conjecture says that for a Borel set $E \subset \mathbb{R}^d$,

$$\dim_H E > \frac{d}{2} \Rightarrow |\Delta(E)| > 0, \tag{1.3}$$

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where $\dim_H$ is the Hausdorff dimension (and $|\cdot|$ is the Lebesgue measure).

The "critical exponent" $\frac{d}{2}$ and its reciprocal, featuring in both conjectures cannot be improved, as can be demonstrated using constructions based on the Euclidean lattice. These are at least implicitly present in the body of the paper.

In order to reformulate the Falconer conjecture, a problem in geometric measure theory and harmonic analysis, by means of geometric combinatorics and thereby relate it to the question of Erdös, an increasingly refined sequence of discretizations of the continuum set engaged in the Falconer conjecture is necessary. Roughly speaking, instead of a continuous set $E \subset \mathbb{R}^d$, of Hausdorff dimension α, one would deal with a finite union of some $\delta^{-\alpha}$ balls of small radius δ. The distance set, therefore, turns out to be a finite union of intervals of length δ, and one would like to identify the largest possible discrete δ-separated subset therein, to get an approximation of $|\Delta(E)|$.

This issue was studied in depth by Katz and Tao ([11]) who observe that a single discretization in the most general situation does not suffice to support the quantitative parallel between the Falconer and Erdös conjectures, due to a counterexample. In order to bypass this issue, Katz and Tao suggested multi-linear versions of the discretized Falconer conjecture and some other related questions addressed in their paper. The state-of-the-art of this approach combines ideas of ([11]) and Bourgain's sum-product estimates ([1]), culminating in the following conjecture (see [10]). There exists $\epsilon_d > 0$, such that

$$\dim_H E \geq \frac{d}{2} \;\Rightarrow\; \dim_H \Delta(E) \geq \frac{1}{2} + \epsilon_d. \tag{1.4}$$

On the other hand, the best result in the context of the Erdös conjecture, accessible by methods of discrete geometry, is due to Katz and Tardos ([12]) and says that for a discrete $E \subset \mathbb{R}^2$,

$$|\Delta(E)| \geq c|E|^{c_2}, \tag{1.5}$$

for some constant c and $c_2 \approx .86$. A direct comparison between the two results, considering that today it is possible to vindicate only very small values of ϵ_d in (1.4) suggests that to settle the conjectures, one should increase c_2 by some .14, while ϵ_d almost by $\frac{1}{2}$. The comparison suggests that the discrete-combinatorial methods that are partially based on the Szemerédi-Trotter incidence theorem and its corollaries do not immediately transfer to the discretized setting. The latter setting, however, may offer more structural information about the distance set and the distances' distribution as well. It can also be extended to embrace the case of non-Euclidean metrics that are interesting in regard to lattice point distributions.

If one takes the above numerology as evidence to claim that the discretized results are somewhat weaker than the discrete ones, this paper shows that Fourier analytic methods, developed in some generality for the continuous setting, can be used successfully to study the discrete problem as well. Applying the analytic methods does require some additional structure imposed *on the discretization stage.* The multi-linear set-up of Katz and Tao ([11]) gives an example of such a structure, and they show that eventually, the multi-linear formulation *does* enables one to come back to the general formulation of the Falconer conjecture (in the setting they consider.) In fact, Katz ([10]) shows that the assumption of the distance set being sufficiently small, added on top of the multi-linear formulation imposes an impossibly strong non-degeneracy condition on the discretized set.

The assumption we use is well-distributivity, the condition which guarantees that sets in question are statistically, though not necessarily arithmetically, analogous to the integer lattice. Such sets were studied by several authors in the recent years. In particular, it is observed in [9] and [7] that in this context the estimate for th continuous problem can be converted into a corresponding estimate for its discrete predecessor. Likewise the multi-linear formulation, the well-distributed one rules out the above-mentioned counterexample. We shall see below that the structure of well-distributed sets lends itself to elegant and relatively straightforward approach using Fourier analytic methods. We shall also use arithmetic considerations to indicate limitations of these methods for Euclidean and non-Euclidean metrics, especially the latter. The paper is concluded with an explicit synthesis of analytic, combinatorial and number theoretic considerations in the context of incidence problems.

In the context of Fourier analytic methods, the main object in the approach to the distance set problem, initiated by Mattila ([13]) is the L^2 spherical average of the Fourier transform. The two theorems of this paper deal with this object. In the positive direction we obtain a good upper bound for these averages in the context of measures obtained by thickening well-distributed sets. These estimates are stronger than the corresponding results previously obtained by Wolff ([22]), Erdoğan ([4]), and others in the context of general measures.

Our results provide estimates of the exponential sum representing the spherical average. Coarse upper bounds for this sum enable us to match the generally optimal bound of Wolff ([21]) in dimension two. In higher dimensions, they yield better estimates than the best known general bounds due to Erdoğan ([5]). As we note above, our gain, or rather its technical transparency, is due to special features of the well-distributed set-up, whereas the bounds of Wolff and Erdoğan apply to general Frostman measures. To this end, one of the key motives of this paper is that well-distributed sets provide a natural discretization scenario that enables one to link together the conjectures of Erdös and Falconer and provide a reasonably non-technical arena for testing the limits of some methods developed to study them. Hence, we do expect that our improved bound in higher dimension should hold in the general case, yet we do not have evidence on whether or not they should be optimal.

The fact that Wolff's general bound in two dimensions cannot be improved is supported by a counterexample of Sjölin ([17]) although the latter is highly not well-distributed. (Observe that Katz and Tao ([11]) had to resort to multi-linearity precisely to rule out the same counterexample.) There is no evidence that in the well-distributed setting, the spherical average does not satisfy sufficient good bounds to imply the Erdös distance conjecture, and we conjecture that this is indeed the case. Our conjecture is supported by the integer lattice case, when the coarse upper bound for the aforementioned sum can be easily refined by using the Poisson summation formula and elementary number theory. We conclude that the well-distributed Erdös conjecture may well follow from obtaining sufficiently sharp estimates for the sum in question.

Estimates for the spherical average provide lower bounds for the number of distinct distances regarding the Erdös conjecture. To this end, our coarse bound enables us to match earlier results of Moser ([14]) Solymosi and Vu ([19]), and one of the authors ([8]) obtained by purely combinatorial methods. Our result may be somewhat stronger, because it establishes the existence of a separated set of distances and applies to the case of non-Euclidean distances as well.

Our second result, a lower bound, provides evidence that in a broader setting of non-isotropic distances generated by well-curved smooth convex bodies, majorants for the analogs of the spherical average alone, which are the Fourier L^2 averages over dilates of the boundary of the dual body, do not generally imply the Erdös conjecture even in the well-distributed case. This makes the Euclidean distance special, as it is in the case of the single distance conjecture in the plane which is generally false for non-Euclidean distances. Moreover, since the methods used in the aforementioned papers of Mattila, Bourgain, Wolff and Erdogan do not distinguish between spheres and smooth surfaces with non-vanishing curvature, our results will show that even the bounds for spherical averages in the Euclidean case are not likely to be obtained using current methods. This is because our counterexample we present are built on the integer lattice and show that sufficiently good upper bounds on the generalized spherical averages imply deep unknown results in number theory pertaining to the distribution of lattice points in the vicinity of dilated convex domains. A result of a similar nature pertaining to the special case of a paraboloid, not based on the integer lattice, was independently obtained in [2]. In the last section of the paper, these results are discussed in some detail in regard to how the distance conjectures can be related to the problem of lattice points distributions on dilates of convex curves and hypersurfaces.

2 Main result: a bound for the spherical average

Spherical averages and Mattila's method for the Falconer conjecture

To motivate the results of this paper we give a brief summary of the Fourier analytic technique developed by Mattila ([13]). Further C, c denote positive constants whose values may change from one appearance of the specific notation to the other. To emphasize the nature of the constants, apart from almost all of them depending on the dimension d (sometimes made explicit like in (1.5) they are often given relevant subscripts.

For a set $E \subset \mathbb{R}^d$, supporting a Borel probability measure μ, the distance measure ν_μ is defined as the push-forward of $\mu \times \mu$ under the distance map $E \times E \mapsto \Delta(E) \subset \mathbb{R}_+$.

Such a set E, with $\dim_H(E) = \alpha$, supports for all $s < \alpha$ a Frostman measure μ (we do not use the more explicit notation μ_s for μ to avoid further accumulation of indices) so that

$$\int_{B(x,\delta)} d\mu \;\leq\; C_\mu \delta^s, \forall\, x \text{ and all } \delta \ll 1, \tag{2.1}$$

and

$$I_s(\mu) = \int\int \frac{d\mu_x d\mu_y}{\|x-y\|^s} < \infty. \tag{2.2}$$

If $\mathcal{M}_s$ is a class of such measures and $\mu \in \mathcal{M}_s$, an important sub-problem in the Falconer conjecture is to establish general asymptotic bounds for the spherical average

$$\sigma_\mu(t) \;=\; \int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega, \tag{2.3}$$

in the form

$$\sigma_\mu(t) \;\leq\; C_{\mu,\beta} t^{-\beta}, \;\; \forall\, t \geq 1. \tag{2.4}$$

Given s and $\mu \in \mathcal{M}_s$, the best known results are as follows: the bound (2.4) holds for all

$$\beta \;<\; \begin{cases} s, & \text{for} \quad 0 < s \leq \frac{d-1}{2}, \\ \frac{d-1}{2}, & \text{for} \quad \frac{d-1}{2} \leq s \leq \frac{d}{2}, \\ \frac{d+2s-2}{4}, & \text{for} \quad \frac{d}{2} \leq s \leq \frac{d+2}{2}, \\ s-1, & \text{for} \quad \frac{d+2}{2} \leq s < d. \end{cases} \tag{2.5}$$

These results are due to Falconer ([6]), Mattila ([13]), Sjölin ([17]), Wolff ([21], Erdoğan ([5]), and others, see, for examples, the references contained in [5]. The crucial interval of the values of s where one would like to improve over (2.5) is for $s \in [\frac{d}{2}, \frac{d+1}{2}]$, and first and foremost at the "critical" value $s = \frac{d}{2}$.

Mattila observed that the Falconer conjecture would follow by the Cauchy-Schwartz inequality if the distance measure ν_μ has L^2 density. The latter can be evaluated by using the Plancherel theorem in polar coordinates, and a calculation shows that after (inconsequential) scaling $\nu_\mu \to \nu_\mu t^{\frac{1-d}{2}}$, one has

$$\|\nu_\mu\|^2_{L^2(\mathbb{R}_+)} = c_d \int_0^\infty \sigma_\mu^2(t) t^{d-1} dt. \tag{2.6}$$

Denoting the integral in the right-hand side of (2.6) as $F(\mu)$, the sufficient condition for the Falconer conjecture is $F_\mu < \infty$.

Using the Plancherel theorem and then passing to polar coordinates yields the following energy estimate:

$$I_s(\mu) \;=\; \int\int \frac{d\mu_x d\mu_y}{\|x-y\|^s} \;=\; c_d \int_0^\infty \sigma_\mu(t) t^{s-1} dt, \tag{2.7}$$

Hence, $\sigma_\mu(t)$ is on average $O(t^{-s})$, but this is not enough for the integral $F(\mu)$ to converge.

However, if one has the bound (2.4), it follows that

$$F(\mu) \;\le\; c_\mu \int_0^\infty \sigma_\mu(t) t^{d-\beta-1} dt \;\le\; C_\mu I_{d-\beta}(\mu) \;<\infty, \;\; \text{for } \; d-\beta \;\le\; s. \tag{2.8}$$

It follows from (2.1), (2.7) that with the notation

$$\bar\beta(s) \;=\; \sup_{\mu \in \mathcal{M}_s} \{\beta : \text{(2.4) holds}\}, \tag{2.9}$$

for $s < \alpha$ one always has $\bar\beta(s) \le \alpha = \dim_H(E)$. In addition, by (2.8) the Falconer conjecture holds if

$$\alpha \;>\; d - \bar\beta(\alpha). \tag{2.10}$$

An estimate $\bar\beta(\alpha) \ge \alpha - 1$ is implicit in [6], [13] and explicit in [17]). It implies that the Falconer conjecture is true for $\alpha > \frac{d+1}{2}$, and hence the upper bound (2.4) is of major interest for $s \in [\frac{d}{2}, \frac{d+1}{2}]$, as was pointed out earlier.

Well-distributed sets

The Falconer conjecture can be regarded as the "continuous version" of the Erdös conjecture, though a quantitative link, obtained in [7] and [9] is only known in the context of well-distributed sets.

We say that an infinite point set $A \subset \mathbb{R}^d$ is class $\mathcal{A}$ well-distributed (sometimes also known as homogeneous, or Delaunay which some authors spell as *Delone*) if it is separated in the sense that for some $c_{\mathcal{A}}$, one has $\|a - a'\| \ge c_{\mathcal{A}}, \forall\, a, a' \in A : a \ne a'$, as well as any cube of side length $C_{\mathcal{A}}$ has a non-empty intersection with A. Constants in the ensuing estimates related to $A \in \mathcal{A}$ will bear the subscript $_A$ and in fact depend on $c_{\mathcal{A}}$, $C_{\mathcal{A}}$ or their ratio.

For the truncations $A_q = A \cap B(0, q)$ of $A \in \mathcal{A}$, with $q \gg 1$ and $B(x, \delta)$ denoting the Euclidean ball of radius δ centered at x, the Erdös conjecture says that

$$|\Delta(A_q)| \ge C^{-1}(q) q^2, \tag{2.11}$$

where the quantity $C(q)$ grows slower than any power of q as $q \to \infty$. Further q plays the rôle of the asymptotic parameter, and none of the constants, explicit or implicit in the standard notations $X = O(Y)$, $X = \Omega(Y)$, and $X = \Theta(Y)$, (indicating that a quantity X is asymptotically bounded by a positive constant times Y from above, below, and on both sides) are allowed to depend on q.

Note that the distance conjecture (2.11) in the well-distributed setting in $d > 2$ would follow from the case $d = 2$ by restricting the set $A_q \in \mathbb{R}^d$ to a "horizontal" slab of thickness $O(1)$ in $\mathbb{R}^{d-1}$. We do not know whether or not the general Falconer conjecture in any $d > 2$ should follow from the case $d = 2$.

Let us scale the truncation A_q into the unit ball and then thicken each point of A_q to a ball of radius $\delta = q^p$, for some $p > 1$. The resulting set represents a δ-discretization of a Cantor-like set E of dimension $s = d/p$, $p > 1$. The construction of such a set is described in [6], [7] and [9].

We denote the resulting thickening of the set A_q, compressed into the unit ball, as E_q. It follows that the distance set $\Delta(A_q)$ has a q^{-p+1}-separated subset, whose cardinality is $\Omega(q^p|\Delta(E_q)|)$.

To put a smooth probability measure on E_q, on the technical level, let in what follows $\phi(x), x \in \mathbb{R}^d$ be a radial test function, whose support is contained in the unit ball. Suppose that ϕ is positive in the interior of its support, $\phi(0) = 1$, $\int \phi = 1$, and the Fourier transform $\widehat{\phi}$ is non-negative. Let $\phi_{q^p}(x) = q^{pd}\phi(q^p x)$ and define

$$d\mu_s(x) = \rho_s(x)dx, \text{ with } \rho_s(x) = c_A q^{-d} \sum_{a \in q^{-1}A_q} \phi_{q^p}(x - a), \tag{2.12}$$

where c_A is the normalization constant, depending on how well-distributed the set A is which is reflected in the values of the constants $c_{\mathcal{A}}$ and $C_{\mathcal{A}}$. Heuristically,

$$\rho_s(x) \sim q^{(p-1)d} \sum_{a \in q^{-1}A_q} B_{a,q^{-p}}(x), \tag{2.13}$$

where the notation $B_{a,q^{-p}}$ for the ball centered at a, of radius q^{-p} has been identified with its characteristic function $B_{a,q^{-p}}(x)$.

The Lebesgue measure $|\Delta(E_q)|$ can now be bounded from below by the above described method for the Falconer distance problem. Our main theorem gives a bound on the spherical average for the measure μ_s, specified by (2.12).

Theorem 2.1. *Let $A \in \mathcal{A}$ be a well-distributed set, let the measure μ_s be defined by (2.12), with $p \in (1, 2]$. Then $\mu_s \in \mathcal{M}_{s=\frac{d}{p}}$ and for some rapidly decaying cut-off function $\eta(\tau) \leq C_n(1 + |\tau|)^{-n}$, for any large n, one has the following bound:*

$$\sigma_{\mu_s}(t) \;=\; O\left[q^{-d+1}\eta\left(\frac{t}{q^p}\right)\right]. \tag{2.14}$$

In addition, for $s = \frac{d}{2}$ and $t = \Omega(q)$,

$$\sigma_{\mu_{d/2}}(t) \;=\; O\left(t^{1-d}\Sigma_{d/2}(t)\right), \tag{2.15}$$

where the quantity $\Sigma_{d/2}(t)$, to be defined explicitly, satisfies the coarse bound

$$\Sigma_{d/2}(t) \;\leq\; C_A t^{\frac{d-1}{2}}\eta\left(\frac{t}{q^2}\right). \tag{2.16}$$

Remark 2.2. The bound (2.16) for the quantity $\Sigma_{d/2}(t)$ does not appear to be optimal, an one can expect that it can be improved, by the factor of $\sqrt{t}$. This would then imply the Erdös conjecture for well-distributed sets. The explicit expression for $\Sigma_{d/2}(t)$ as well as a conditional bound, which, as suggested by the integer lattice example discussed at the final section of the paper, can indeed beat (2.15) by the factor $\sqrt{t}$ are given by (2.39) below.

By (2.14) and the decay of the cut-off η therein, for $s \geq \frac{d}{2}$ we have the following improvement over the spherical average bounds (2.5):

$$\sigma_{\mu_s}(t) \ \leq \ C_A t^{-\frac{d-1}{d}s}, \ \ \forall\, t \geq 1. \tag{2.17}$$

This implies the following corollary.

Corollary 2.3. *The set A_q determines $\Omega\left(|A_q|^{\frac{2}{d}-\frac{1}{d^2}}\right)$ distinct distances, separated by $q^{\frac{1-d}{d}}$.*

Observe that $|A_q| = c_A q^d$, hence the exponent in Corollary (2.3) is $\frac{1}{d^2}$ off its conjectured value. We remind the reader that the symbols c_A, C_A may indicate different constants throughout their appearances. These constants depend, however, on the well-distributedness constants $c_{\mathcal{A}}$ and $C_{\mathcal{A}}$ and the dimension d only.

2.1 Proof of Theorem2.1

We start out with a simple calculation showing that μ_s defined by (2.12) is in $\mathcal{M}_s$.

Lemma 2.4. *For $s = \frac{d}{p}$, we have $I_s(\mu_s) = O(1)$.*

Clearly, the approximate expression (2.13) is good enough to substitute for μ_s in the energy computation, see (2.2). For any x in the support of μ_s, let us split

$$\int \frac{d\mu_s(y)}{\|x-y\|^s} = I_1 + I_2,$$

where the integral I_1 is taken over the ball $B(x, \frac{c}{q})$ and I_2 over its complement. Then

$$I_1 \ \leq \ C_A q^{(p-1)d} \int_{B(0,q^{-p})} \frac{dy}{\|y\|^{\frac{d}{p}}} = O(1).$$

Besides, as A is well-distributed, and the μ_s-mass of each peak centered at $a \in q^{-1}A$ in (2.13) is approximately q^{-d}, one has

$$I_2 \leq C_A q^{-d} \sum_{a \in q^{-1}A_q \setminus B(0,cq^{-1})} \frac{1}{\|a\|^s} = C_A q^{-d+s} \sum_{a \in A_q \setminus B(0,c)} \frac{1}{\|a\|^s} = O(1).$$

Let us further in the proof drop the $_s$ subscript for μ_s and ρ_s from (2.12), to avoid having too many indices. The proof of Theorem 2.1 contains three steps, and we start out with two preliminary observations. Since

$$\widehat{\rho}(\xi) = c_A \widehat{\phi}(q^{-\frac{d}{s}}\xi) \sum_{a \in q^{-1}A_q} e^{-2\pi i a \cdot \xi}, \tag{2.18}$$

the "dimension" s characterizing the thickening μ_s of the atomic measure $\sum_{a \in q^{-1}A_q} \delta(x-a)$ appears only in the cut-off $\widehat{\phi}(q^{-\frac{d}{s}}\xi)$. Hence, given s, it suffices to consider $t = \Theta(q^{\frac{d}{s}})$ only.

This is assumed throughout Step 1 of the proof. Indeed, instead of considering $t \ll q^{\frac{d}{s}}$, one can rather increase s (it is assumed that $t \gg q$). In Step 2 we verify the estimate (2.15) for $s = \frac{d}{2}$ and $t = O(q^2)$. Technically, in the end, we will consider separately the "endpoint case" given $s \in [\frac{d}{2}, d)$ and $t = Nq^{\frac{d}{s}}$, for $N \to \infty$. In this case, $\widehat{\phi}(q^{-\frac{d}{s}}\xi)$, with $\|\xi\| = t$ satisfies the standard decay estimate $O((1+N)^{-n})$, for any n, and this accounts for the pre-factor η in the estimates of Theorem 2.1. This is carried out in Step 3 of the proof.

The second standard preliminary observation is that the density ρ in (2.12) can be multiplied by any test function that equals one in the unit ball, reflecting the fact that μ is compact. This implies that $\widehat{\mu}$ changes slowly on the length scale $\Theta(1)$. Namely, if $A(t,c)$ denotes the spherical shell of radius t and width $2c$, we have (for some C)

$$\begin{aligned} C^{-1}\, t^{d-1}\sigma_\mu(t) &\leq \int_{A(t,c)} |\widehat{\rho}(\xi)|^2 d\xi \\ &= \sup_{f:\, \|f\|_2=1,\, \operatorname{supp} f \in A(t,c)} \left(\int \widehat{\rho}(\xi) f(\xi) d\xi\right)^2 \\ &\equiv (M_\mu[f])^2 \end{aligned} \tag{2.19}$$

Step 1. Take any such f. Let P_q be a maximum $\frac{c_1 q}{t}$ separated set on S^{d-1}, for some sufficiently small c_1. For $p \in P_q$, let f_p be the restriction of f on the intersection of $A(t,c)$ with the cone, emanating from the origin and built upon the Voronoi cell of S^{d-1} centered at p[1]. (The latter is defined as the set of all points on S^{d-1} that are closer to p than to any other point of P_q.) Decompose

$$f = \sum_p f_p, \tag{2.20}$$

clearly,

$$|P_q| = c_d \left(\frac{t}{c_1 q}\right)^{d-1}. \tag{2.21}$$

By choosing a small c_1, we can ensure that for c_2 as small as necessary (in terms of the bounding constants $c_{\mathcal{A}}$, $C_{\mathcal{A}}$ characterizing the well-distributed set class $\mathcal{A}$) the support of f_p is contained in some d-dimensional rectangle (henceforth simply *rectangle*) of the size $c_2(q^2/t \times q \times \ldots \times q)$, which is centered at tp and the first measurement is taken in the direction of p.

By orthogonality,

$$1 = \|f\|_2^2 = \sum_p \|f_p\|_2^2. \tag{2.22}$$

Then

$$(M_\mu[f])^2 \leq \left(\sum_p |M_\mu[f_p]|\right)^2. \tag{2.23}$$

To prove (2.14), we are going to show that for each $p \in P_q$,

$$|M_\mu[f_p]| \leq C_A \|f_p\|_2, \;\; \text{uniformly in } p, \tag{2.24}$$

as a coarse estimate. This will imply by Cauchy-Schwartz and (2.22) that

$$(M_\mu[f])^2 \; \leq C_A \|f\|_2^2 \cdot \sum_{p \in P_q} 1 = O\left[\left(\frac{t}{q}\right)^{d-1}\right]. \tag{2.25}$$

[1]We warn the reader that the symbol p that in the main body of the paper appears as $p = \frac{d}{s}$ is used throughout the proof of Theorem 2.1 as the summation index over the partition of S^{d-1}.

Let us first prove (2.24). Without loss of generality, assume that $p = (1, 0, \ldots, 0)$, relative to the coordinates (ξ_1, ξ_2), where ξ_1 is one-dimensional and ξ_2 is $(d-1)$-dimensional.

By the Plancherel theorem we have

$$M_\mu[f_p] = \int \widehat{f_p}(x, y)\rho(x, y)dxdy, \tag{2.26}$$

where x is one-dimensional and y is $(d-1)$-dimensional.

The function f_p is supported in the translate by tp of the rectangle $\bar{R}_p$, where $\bar{R}_p$ – with the above choice of $p = (1, 0)$ – is a "vertical" rectangle centered at the origin in the (ξ_1, ξ_2) "plane", of width c_2q^2/t and height (meaning the ξ_2-directions) c_2q. Let us write $f_p(\xi) = h_p(\xi - tp)$, i.e. h_p is supported in $\bar{R}_p$. All the rectangles involved are further identified with their characteristic functions.

By the uncertainty principle, as $h_p = h_p \cdot \bar{R}_p$, its Fourier transform $\widehat{h_p}$ is approximately constant in the translates of the dual to $\bar{R}_p$ rectangle R_p of size $C_2(t/q^2 \times q^{-1} \times \ldots \times q^{-1})$, relative to the coordinates (x, y). More precisely, if $\bar{r}_p(\xi_1, \xi_2)$ is a test function which is one in $\bar{R}_p$ and vanishes outside, say $2\bar{R}_p$, then $\widehat{h_p} = \widehat{h_p} * \widehat{\bar{r}}_p$.

Accordingly, let us decompose

$$\widehat{h_p} \;=\; \sum_j \widehat{h_p} R_{p,j} \;\equiv\; \sum_j \widehat{h}_{p,j}: \quad \|\widehat{h_p}\|_2^2 \;=\; \sum_j \|\widehat{h}_{p,j}\|_2^2. \tag{2.27}$$

Above, $R_{p,j}$ are the translates of R_p, covering $\mathbb{R}^d$. In the remaining part of the proof, however, we will need only those values of j, such that $R_{p,j}$ intersects the support of μ. In other words, we are not claiming that the Fourier transform of a compactly supported function has compact support, we are simply restricting its support by integrating it against the compactly supported measure μ. As before, $R_{p,j}$ is identified with its characteristic function. The constant C_2 can be made as large as necessary by decreasing c_1 above. We shall further use well-distributedness of the set A, by claiming that each $R_{p,j}$ supports $q^d\Theta(|R_p|)$ members of $q^{-1}A$.

By Young's inequality

$$\|\widehat{h}_{p,j}\|_\infty \;\leq\; c_d \frac{1}{\sqrt{|R_p|}} \|\widehat{h}_{p,j}\|_2, \tag{2.28}$$

moreover as $\widehat{h_p} = \widehat{h_p} * \widehat{\bar{r}}_p$, we can write

$$\widehat{h}_{p,j} \;=\; \left(\frac{1}{\sqrt{|R_p|}} \|\widehat{h}_{p,j}\|_2\right) R_{p,j}\psi_{p,j}. \tag{2.29}$$

Above, $\psi_{p,j}$ is a smooth function which is $O(1)$ and can be made to vanish outside $2R_{p,j}$; in addition one has uniform bounds

$$|\partial_x \psi_{p,j}(x, y)| \;=\; O\left(\frac{q^2}{t}\right), \quad |\partial_y \psi_{p,j}(x, y)| \;=\; O(q). \tag{2.30}$$

Clearly

$$\widehat{f_p} \;=\; e^{-2\pi i t x}\widehat{h_p}. \tag{2.31}$$

I.e. $\widehat{f_p}$ is the rapid phase $e^{-2\pi i t x}$ that does not depend on y times $\widehat{h_p}$ (this is specific for the spherical average, versus non-isotropic ∂K-averages) which is approximately constant in each rectangle $R_{p,j}$, with the sharp bound (2.28).

By (2.26) we have then

$$
\begin{aligned}
M_\mu[f_p] &= \textstyle\sum_j \int \widehat{h}_p R_{p,j}(x,y) e^{-2\pi i t x} \rho(x,y) dy dx \\
&\equiv \textstyle\sum_j \left(\frac{1}{\sqrt{|R_p|}}\right) \|\widehat{h}_{p,j}\|_2 \, \widehat{\tilde{\mu}}_{p,j}(t),
\end{aligned}
\tag{2.32}
$$

where

$$
\tilde{\mu}_{p,j}(x) = \int R_{p,j}(x,y)\psi_{p,j}(x,y)\rho(x,y)dy. \tag{2.33}
$$

Now the desired inequality

$$
|M_\mu[f_p]|^2 = O(\|f_p\|_2^2) \tag{2.34}
$$

follows by the Cauchy-Schwartz inequality from the trivial bound

$$
\forall j, \quad \int R_{p,j}(x,y)\rho(x,y)dxdy \;\le\; C_A|R_p|, \tag{2.35}
$$

by well-distributedness of A and the fact that there are $O(|R_p|^{-1})$ terms in the summation in j. This proves (2.24).

Step 2. Naturally, cf. (2.32), similar to (2.33), one is tempted to define

$$
\mu_{p,j}(x) \;=\; \int R_{p,j}(x,y)\rho(x,y)dy, \tag{2.36}
$$

and have $\widehat{\mu}_{p,j}(t)$ substitute $\widehat{\tilde{\mu}}_{p,j}(t)$ in the second line of (2.32). The two can be related point-wise however only if the x-measurement $C_2\frac{t}{q^2}$ of the rectangle R_p is $\Omega(1)$, to ensure $|\partial_x\psi_{p,j}(x,y)| \;=\; O(1)$ rather than the first bound in (2.30).

It is easy to achieve this by changing the partition (2.20), (2.21) and essentially repeating the argument up to this point. In this part of the proof, we assume $s = \frac{d}{2}$ and $t = O(q^2)$. Let us use a slightly different decomposition of the sphere S^{d-1}, with $P_{\sqrt{t}}$ denoting a maximum $\frac{c_1}{\sqrt{t}}$ separated subset of S^{d-1}. Similar to (2.20) and (2.21), decompose

$$
f = \sum_p f_p, \quad p \in P_{\sqrt{t}}, \quad |P_{\sqrt{t}}| = \Theta\left(t^{\frac{d-1}{2}}\right).
$$

Now f_p is supported inside the rectangle $\bar{R}_p$ of the size $c_2(1 \times \sqrt{t} \times \ldots \times \sqrt{t})$. Accordingly, its dual R_p has the size $C_2(1 \times \frac{1}{\sqrt{t}} \times \ldots \times \frac{1}{\sqrt{t}})$.

We repeat the argument from (2.27) through (2.33), with the same notations, relative to the new partition $\{f_p\}$, only now we can write

$$
\tilde{\mu}_{p,j}(x) \;=\; \varphi_{p,j}(x)\mu_{p,j}(x), \tag{2.37}
$$

for some test function $\varphi_{p,j}(x)$ of a single variable, which is supported on $[-C_2, C_2]$, and is $O(1)$, together with is derivative. Above, the quantities $\tilde{\mu}_{p,j}(x)$ and $\mu_{p,j}(x)$ have been defined respectively by (2.33) and (2.36), only relative to the rectangles $R_{p,j}$ of the size $C_2(1 \times \frac{1}{\sqrt{t}} \times \ldots \times \frac{1}{\sqrt{t}})$, hence the desired properties of $\varphi_{p,j}(x)$ that arise after integration in y in (2.33), in view of the bound $|\partial_x\psi_{p,j}(x,y)| \;=\; O(1)$.

Thus we have $\widehat{\tilde{\mu}}_{p,j} = \widehat{\mu}_{p,j} * \widehat{\varphi}_{p,j}$, and this implies the bound

$$
|\widehat{\tilde{\mu}}_{p,j}(t)| \;\le\; c_d \sup_\tau |\widehat{\mu}_{p,j}(t-\tau)|\, \eta(cC_2\tau), \tag{2.38}
$$

where c is independent of the governing constants c_1, C_2, and the quantity η has been defined in the statement of Theorem 2.1.

In view of this, we can give a more refined bound than (2.34) following (2.32). Using (2.38) and the fact that now $|R_p| = \Theta(t^{\frac{1-d}{2}})$, we obtain, essentially repeating the argument in Step 1, that

$$\begin{aligned} |M_\mu[f]|^2 \;\le\; \Sigma_{d/2}(t) \;&\equiv\; c_d \sum_{p\in P_{\sqrt{q}}} \left(\tfrac{1}{|R_p|} \sum_j |\widehat{\mu}_{p,j}(t)|^2 \right) \\ &\le\; C_d \sum_{p\in P_{\sqrt{q}}} \left(t^{\frac{d-1}{2}} \sum_j \sup_\tau |\widehat{\mu}_{p,j}(t-\tau)|^2 \, \eta(cC_2\tau) \right). \end{aligned} \tag{2.39}$$

A coarse bound (2.16) follows in exactly the same way as (2.14) on Step 1. I.e. for both partitions of S^{d-1}, we have

$$|M_\mu[f]|^2 \;\le\; C_d \sum_p \left(\frac{1}{|R_p|} \sum_j |\widehat{\mu}_{p,j}(t)|^2 \right). \tag{2.40}$$

Step 3. So far, the bounds (2.14) – (2.16) of Theorem 2.1 have been justified only for $t = O(q^{\frac{d}{s}})$, with $s \in [\frac{d}{2}, d)$ on Step 1 and $s = \frac{d}{2}$ on Step 2. Suppose now that $t = \Theta(Nq^{\frac{d}{s}})$, where N increases. The impact of this shall be compensated by the choice of the constant c_1, increasing the number of Voronoi cells on S^{d-1}, to ensure that C_2 remains sufficiently large. Hence, the constants hidden in (2.34) as well as in (2.39) will increase as N^{d-1}. On the other hand, built into (2.18), we have the decay of $\widehat{\phi}(q^{-\frac{d}{s}}\xi)$. This clearly enables one to use $|R_p|\widehat{\phi}(q^{-\frac{d}{s}}\xi)$, with $\|\xi\| = t$ as a coarse bound for $|\widehat{\mu}_{p,j}(t)|$, i.e. multiply $|R_p|$ by $C_n(1+N)^{-n}$ for any n, and $t = \Theta(Nq^{\frac{d}{s}})$. This accounts for the presence of the quantity η in (2.14) and (2.16) and completes the proof of Theorem 2.1.

Proof of Corollary 2.3

Theorem 2.1 implies that the measure μ_s defined by (2.12) satisfies (2.4) for $\beta = \frac{d-1}{d}s$. Hence, by (2.8), the Falconer conjecture is satisfied by the support of μ_s, provided that $s \ge \frac{d^2}{2d-1}$. Therefore, the number of distinct q^{-p+1} separated distances generated by the set A_q, where $p = \frac{d}{s}$, is bounded from below by a constant times

$$q^p = q^{\frac{2d-1}{d}} = \Theta\left(|A_q|^{\frac{2}{d} - \frac{1}{d^2}} \right). \tag{2.41}$$

This proves Corollary 2.3. Let us point out here that this is precisely the lower bound obtained by Moser ([14]) in the case $d = 2$ (see also [8] for higher-dimensional generalization of this method), and Solymosi and Vu ([19]) for well-distributed sets using methods of geometric combinatorics. Recently Solymosi and Tóth ([20]) made further progress in that direction, having improved the margin $\frac{1}{d^2}$ in (2.41) to $\frac{2}{d(d^2+1)}$.

3 The case of K-distances

It is interesting to broaden the scope of the distance conjectures by generalizing the Euclidean distance $\|\cdot\|$ as $\|\cdot\|_K$, the Minkowski functional of a strictly convex body $K \subset \mathbb{R}^d$, with the smooth boundary ∂K. Let $\mathcal{K}$ be described a class of such bodies, whose volume equals the volume of the unit ball, and the Gaussian curvature is bounded in some interval $[c_{\mathcal{K}}, C_{\mathcal{K}}]$.

The spherical average generalizes accordingly by replacing the domain of integration S^{d-1} in (2.3) by ∂K, substituting S^{d-1}.

Mattila's formalism extends to the case of K-distances, concerning the surface average $\sigma_{\mu,K}(t)$, defined in (3.3). Then, see [7], the Mattila formulation of the Falconer conjecture for non-isotropic distances $\|\cdot\|_K$ is equivalent to proving that

$$F_K(\mu) \;=\; \int_0^\infty \sigma^2_{\mu,K^*}(t)t^{d-1}dt \;<\; \infty, \tag{3.1}$$

where

$$K^* = \{x: \sup_{y\in\partial K} x\cdot y \le 1\}$$

is the dual body of $K\in\mathcal{K}$.

By following step-by-step the proof of Theorem 2.1, one concludes that the bound (2.17) and hence Corollary 2.3 are still true in the case of K-distances, with the constants depending now on $c_{\mathcal{K}}, C_{\mathcal{K}}$ as well.

In addition, we have the following conditional result.

Theorem 3.1. *Let $\tau \gg 1$, $\gamma\in[0,1)$, and suppose there exists a convex body $K\in\mathcal{K}$, such that*

$$|\{\tau\partial K\cap\mathbb{Z}^d\}| \;\ge\; C_K\,\tau^{d-2+\gamma}. \tag{3.2}$$

For any $s\in(0,d)$, there exists a measure $\mu\in\mathcal{M}_s$, supported in the unit ball, such that for $p=\frac{d}{s}$ and $t=\tau^{\frac{p}{p-1}}$, one has

$$\sigma_{\mu,K}(t) \;\equiv\; \int_{\partial K}|\widehat{\mu}(t\omega)|^2 d\omega_K \;\ge\; c_K\,t^{-s+(\frac{2s}{d}-1)+\gamma\frac{p-1}{p}}, \tag{3.3}$$

where $d\omega_K$ is the Lebesgue measure on ∂K.

In the case when K is a compact piece of a paraboloid, this result, with the exponents corresponding to $\gamma=1$ is established in [**?**]

Corollary 3.2. *In dimension 2, there exists $K\in\mathcal{K}$, such that for a sequence of values of t going to infinity, there exists a measure $\mu_t\in\mathcal{M}_s$, supported in the unit ball, such that*

$$\int_{\partial K}|\widehat{\mu_t}(t\omega)|^2 d\omega_K \;\ge c_K\,t^{-\frac{1}{2}-\frac{s}{4}}. \tag{3.4}$$

We remark that (3.3), even for $\gamma=0$, is always non-trivial for $s>\frac{d}{2}$, while (3.4) is non-trivial for $s>\frac{2}{3}$. This indicates the impossibility of the equality $\bar{\beta}(\alpha)=\alpha$, see (2.9), in this range of parameters. Such an equality is the case for $\alpha\le\frac{d-1}{2}$, see [13], as well as (2.5).

Proof of Theorem 3.1

Let us modify the measure μ_s in (2.12) slightly, keeping the same notation, with now again $p=\frac{d}{s}$:

$$d\mu_s(x) \;=\; \rho_s(x)dx, \;\text{ with }\; \rho_s(x) \;=\; c_A\,\phi(x)\,q^{-d}\sum_{a\in q^{-1}A}\phi(a)\phi_{q^p}(x-a). \tag{3.5}$$

Lemma 2.4 clearly remains true, although in comparison with the expression (2.13), the prefactor $\phi(x)$ has enabled to extend the summation over the whole $q^{-1}A$; besides each peak at $a\in q^{-1}A_q$ has been weighted by $\phi(a)$.

The analog of (2.18) is now

$$\widehat{\rho}_s(\xi) \;=\; c_A\,\widehat{\phi}(\xi) * \left(\widehat{\phi}(q^{-p}\xi) \sum_{a\in q^{-1}A} \phi(a)e^{-2\pi i a\cdot\xi}\right). \tag{3.6}$$

We now consider the special case $A = \mathbb{Z}^d$ and apply the Poisson summation formula to the sum in a, which results in the summation over the dual to $q^{-1}\mathbb{Z}^d$ lattice $q\mathbb{Z}^d$, at each of whose elements b there sits a bump $\widehat{\phi}(\xi - b)$. I.e.

$$\widehat{\rho}_s(\xi) \;=\; c_A\widehat{\phi}(\xi) * \left[\widehat{\phi}(q^{-p}\xi) \sum_{b\in q\mathbb{Z}^d} \widehat{\phi}(\xi - b)\right]. \tag{3.7}$$

Consider now the average

$$\int_{\partial K} |\widehat{\mu}_s(t\omega)|^2 d\omega_K \;\sim\; t^{1-d}\int_{A_K(t,c)} |\widehat{\mu}_s(\xi)|^2 d\xi,$$

where $A_K(t,c) = (t+c)K \setminus (t-c)K$. Strictly speaking in the above relation one should have the O-symbol in the right-hand side. However, in this particular case the right-hand side suffices for the lower bound as well.

Indeed, if K is such as stated by Theorem 3.1, $t\partial K$ contains $\Omega\left((\frac{t}{q})^\gamma\right)$ points of the lattice $q\mathbb{Z}^d$, and hence by (3.7), there are $\Omega\left((\frac{t}{q})^\gamma\right)$ bumps, each of the hight approximately one and with an $\Omega(1)$-overlap with the shell $A(t,c)$ or the dilated boundary $t\partial K$ itself.

Therefore,

$$\int_{\partial K} |\widehat{\mu}_s(t\omega)|^2 d\omega_K \;=\; \Omega\left[t^{1-d}\left(\frac{t}{q}\right)^\gamma\right],$$

and the proof of Theorem 3.1 is complete by choosing $q = t^{\frac{s}{d}}$, with $\tau = \frac{t}{q}$ in the condition (3.2).

To prove Corollary 3.2 in the case $d = 2$, it is easy to see that some τ-dilates of a piece of the parabola $\{y = \pm\sqrt{x},\, x \in [0,1]\}$ would contain $\Theta(\tau^{\frac{1}{2}})$ integer points. Indeed, the dilate in question can be written as $\{(x, \pm\sqrt{x}\sqrt{\tau}),\, x \in [0,\tau]\}$, and if τ is a square, the dilate obviously contains an integer point whenever x is a square. The above parabola can be made part of the boundary ∂K of the body K determining the metric $\|\cdot\|_K$. This proves the corollary.

Remark 3.3. Observe that the condition (3.2) can be relaxed by having the points of $q\mathbb{Z}^d$ located c-close to tK, rather than immediately on it. In other words, it suffices to take the right-hand side of (3.2) as the lower bound for the number of integer points located $c\tau^{\frac{1}{1-p}}$ close to $\tau\partial K$. In the case $p = 2$, $s = \frac{d}{2}$, we have $\tau = q$.

4 Implications for distance conjectures and lattice point distributions

This section does not contain new results, but poses some open questions that arise from Theorems 2.1, 3.1.

General bounds for the spherical average

The strongest general spherical average bounds summarized in (2.5) cannot be improved in dimension 2 beyond the endpoint, because Sjölin ([17]) used a Knapp-type example to show that for $s \geq 1$, there are measures in $\mathcal{M}_s$ that satisfy

$$\sigma_\mu(t) \geq c_\mu t^{-\left(\frac{s}{2}+\frac{d-2}{2}\right)} I_s(\mu). \tag{4.1}$$

Namely, in the notation (2.9) one has the following general relation between the Hausdorff dimension α and the "Spherical Fourier dimension" $\bar{\beta}(\alpha)$:

$$\frac{\alpha}{2} \leq \bar{\beta}(\alpha) \leq \alpha, \text{ for } \alpha \geq 1 \text{ and } d = 2. \tag{4.2}$$

Observe that the estimate (4.1) provides non-trivial information only in the range of Hausdorff dimensions $\alpha > d - 2$.

Sjölin's example shows that in dimensions 2 and 3, the Falconer conjecture cannot be resolved in full generality merely by proving a sharp power majorant (2.4) for the spherical average $\sigma_\mu(t)$, but leaves open the question whether this may be possible for $d \geq 4$. This question has been recently asked by Erdoğan ([4]) who generalized Wolff's result to $d \geq 2$, obtaining the best known general upper bounds (2.5). Let us restate the part of the above estimate in the form relevant to this discussion:

$$\bar{\beta}(\alpha) \geq \frac{d + 2\alpha - 2}{4}, \text{ for } \alpha \in \left[\frac{d}{2}, \frac{d}{2} + 1\right]. \tag{4.3}$$

Our bound (2.17) is an improvement over (4.3), and it appears reasonable to ask the following.

Question 4.1. Does the bound (2.17) generalize to the class of Frostman measures $\mu \in \mathcal{M}_s$ (at least in the important case $s \in [\frac{d}{2}, \frac{d+1}{2}]$) in the case $d \geq 3$, and if it does, is it generally best possible?

We believe that the first part of the question can be answered affirmatively, and the presence of different space scales in the general problem would only cause one to lose the endpoint in (2.17). The discretization component of the possible proof should probably take advantage of the relevant techniques of [11], and eventually, after dyadic localization and pigeonholing, one should be able to effectively assume that μ is a density supported on a union of disjoint balls of radius t^{-1}, so that the μ-mass of each ball is $O(t^{-s})$, and the total number of balls is $O(t^s)$.

Then the quantity $q = t^{\frac{s}{d}}$ arises as a natural partition parameter, in the sense that a subcube of diameter q^{-1} contains on average one of the union of balls whereupon μ is supported. Therefore, the partition in the proof of Theorem 2.1 of the sphere of radius t onto pieces of diameter $O(q)$ also arises naturally, and would yield the analog of the double sum given by (2.40) in the general case as well. The trivial estimate that has been applied to the double sum, which claimed that the expression in brackets there was $O(1)$ for each partition angle p, will be no longer applicable however. Indeed, given p, the mass μ may not be distributed between the tiles $R_{p,j}$ uniformly. Then one will have to tackle the whole double sum in (2.40). This creates a reasonably accessible combinatorial problem, which would most importantly still ignore the phases in (2.32). Thus the estimates (2.17), (4.3), and in fact (2.5) are in essence *coarse estimates*.

We see no cogent reason to believe or disbelieve whether the bound (2.17) may be tight in higher dimensions, for the lack of a geometric concept that would underly a possible counterexample. From the point of view of the proof of Theorem 2.1, Sjölin's example can be rendered essentially one-dimensional. The discretized version thereof is as follows (see also [11]). In the plane, one takes points with coordinates $(x,y) = (j/q, 0)$, $j = 0, \pm 1, \ldots, \pm q$, thickens them into rectangles of width $q^{-\frac{d}{s}}$ and height q^{-1}, puts a uniform probability measure thereon and uses the one-dimensional Poisson summation formula to look at the Fourier side at $t = \Theta(q^{\frac{d}{s}})$. In the decomposition framework of Theorem 2.1, it is tantamount to having a single direction p in the double sum (2.40). This alone does not suffice to match the upper bound (2.17) in the case $d = 3$, and even less so in higher dimensions. It also indicates that cancelations for different values of p are inherent in the problem. More precisely, the measures $\mu_{p,j}(x)$, defined by (2.36), and localizing μ in different directions p, cannot all resonate with fast plane waves in these directions, all having the same frequency t. Hence targeting the sharp bounds for the sum in (2.40), one cannot simply ignore the presence of the phase factors in (2.32).

The estimate (2.15) of Theorem 2.1 is conditional on the term $\Sigma_{d/2}(t)$ which is given explicitly by (2.39). Naturally, the estimate (2.15) poses a question of estimating the quantity $\Sigma_{d/2}(t)$ by taking the phase factors into account. Observe that if $s = \frac{d}{2}$, then $t = \Theta(q^2)$, and the underlying well-distributed set A is the lattice $\mathbb{Z}^d$, the number of nonzero terms in the summation over p in (2.39) will be approximately q^{d-2} (modulo a slowly growing function of q in the case $d = 2$). Indeed, any such p would correspond to a point of the lattice $q\mathbb{Z}^d$ lying in the $O(1)$ neighborhood of the sphere of radius $t = \Theta(q^2)$. The number of such points cannot exceed $O(q^{d-2})$ in dimensions three and higher, with an additional slowly growing term in dimension two. This implies that in this specific case, for $s = \frac{d}{2}$, the bound (2.39) improves from $t^{-\frac{d-1}{2}}$ to $t^{-\frac{d}{2}}$ (modulo a slowly growing function of q in the case $d = 2$) which is precisely what one needs to settle the distance conjecture.

Question 4.2. Is it true that for the measures (2.12) on thickenings of well-distributed sets and $s = \frac{d}{2}$, one actually has

$$\sigma_{\mu_{d/2}(t)} \;\le\; C_\mu(t) t^{-\frac{d}{2}}, \tag{4.4}$$

where the quantity $C_\mu(t)$ grows slower than any power of t?

This cannot be true for $s > \frac{d}{2}$ by Theorem 3.1, cf. 3.3 with $K = S^{d-1}$ and $\gamma = 0$. Besides, the answer may possibly be in the positive only for the Euclidean distance, as for the generalized quantity $\sigma_{\mu_s,K}$ this is impossible by Corollary (3.2). By (2.10), the affirmative answer to Question 4.2 would imply the Erdös conjecture for well-distributed sets. In this sense, as a special feature of the Euclidean, or spherical distance, this question is similar to the Erdös single distance conjecture discussed next.

Single distance conjecture and the spherical average

The Erdös single distance conjecture in the case $d = 2$, in the well-distributed setting can be written as

$$\sup_{u \in \Delta(A_q)} |\{(x,y) \in A_q \times A_q : \|x - y\| = u\}| \;\le C(q) q^2, \tag{4.5}$$

where the quantity $C(q)$ asymptotically grows slower than any power of q. Using (4.9), with $d = 2$ and $s = \frac{d}{2} = 1$, it follows that in terms of the measure (2.12) and its spherical average (2.3) it is equivalent to asking for any $\tau \in (0,1)$, whether

$$\int_0^\infty t J_0(\tau t) \sigma_{\mu_{d/2}}(t) dt \le C(q). \tag{4.6}$$

(We have retained the notation $\sigma_{\mu_{d/2}}$ without specifying that $d = 2$ as we would like soon to pass on to all $d \geq 2$.) At the same time, the Mattila criterion (2.6) is

$$\int_0^\infty t^{d-1}\sigma^2_{\mu_{d/2}}(t)dt \leq C(q). \tag{4.7}$$

Note that by (2.18), it is essentially sufficient to integrate up to q^2. The affirmative answer to Question 4.2 implies (4.7), but not (4.6) which requires more regularity than merely the majorant (4.4) for the spherical average $\sigma_{\mu_{d/2}}(t)$. By using the asymptotics of the Bessel function J_0, the well-distributed set single distance conjecture reduces to the estimate, with $d = 2$:

$$\int_{q^2}^{2q^2} t^{\frac{d-1}{2}} e^{-2\pi i\tau t}\sigma_{\mu_{d/2}}(t)dt \ \leq C(q), \forall \tau \in (0,1). \tag{4.8}$$

Hence, (4.8) asserts a special property of the Euclidean distance, which is more stringent than (4.4) in Question 4.2. This certainly adds to the credibility of the conjectured affirmative answer to the latter question.

Observe that the standard counterexample to the single distance conjecture in dimensions $d \geq 4$ is not valid in the well-distributed setting, and therefore (4.5) whose main component in the right-hand side, see also (2.11), does not depend on the dimension should generalize to any $d \geq 2$. Retracing the steps that have lead to (4.8), we conclude that the latter formulation yields the single distance conjecture for thickenings of well-distributed sets, regardless of the dimension. Combining it with the earlier claim that the support of the distance measure $\nu_{\mu_{d/2}}$ should contain almost q^2 separated intervals of length q^{-2}, we may formulate the general conjecture that distance measures $\nu_{\mu_{d/2}}$ generated by thickenings of well-distributed sets are *quasi-uniform*.

Observe that the single distance conjecture, as well as (4.4), are generally not true in the distance class $\|\cdot\|_{K\in\mathcal{K}}$, the counterexample to the former conjecture being constructed in essentially the same way as it has been done to prove Corollary 3.2.

The best known single distance conjecture bound of the form (4.5) is $q^{\frac{8}{3}}$ and is due to Spencer, Szemerédi, and Trotter ([18]). It arises as an immediate corollary of the Szemerédi-Trotter theorem. In light of the discussion in this paper, the latter theorem can be formulated as follows. Given a set P of points and the set J of translates $T_{j\in J}$ of K, with $|P| = \Theta(|J|) = \Theta(q^2)$, one has the following bound for the number of incidences:

$$|\{(p,j) \in P \times J : p \in T_j\partial K\}| \ \leq \ Cq^{\frac{8}{3}}. \tag{4.9}$$

The single distance conjecture then claims that the bound in (4.9) can be improved to q^2 (modulo a slowly growing function of q) in the special case $\partial K = S^1$. If one believes in optimality of the parabola example used to prove Corollary 3.2, a reasonable question to ask is as follows.

Question 4.3. Is this true that for a general $K \in \mathcal{K}$, the bound $q^{\frac{8}{3}}$ in (4.9) can be improved to $q^{\frac{5}{2}}$ (modulo a slowly growing function of q)?

The parabola example mentioned above shows that the exponent $\frac{5}{2}$ cannot be improved.

Non-isotropic surface means and lattice point distributions

Theorem 3.1 implies that the Falconer conjecture for a class of distances $\|\cdot\|_{K\in\mathcal{K}}$ cannot be resolved by proving best possible majorants for the quantity $\sigma_{\mu,K}$ alone, provided that for any fixed $\varepsilon > 0$ and arbitrarily large τ, there exists $K \in \mathcal{K}$ such that the condition (3.2) holds for any $\gamma > 0$. Let us address the issue how large γ can possibly be. Much more is known to this end in $d = 2$ than in higher dimensions.

In the case $d = 2$, by the result of Bombieri and Pila ([3]) there are no C^∞ bodies K, such that (3.2) holds for $\gamma = \frac{1}{2} + \epsilon$, for any $\epsilon > 0$. (More recently, the result of Bombieri and Pila has been given more refinement under additional assumptions which are beyond the scope of this paper.) The conjecture of Schmidt ([16]) states that this is actually the case for the class C^2. Observe that so far, for the analysis of ∂K means in the literature dedicated to the Falconer conjecture a finite order of differentiability of ∂K suffices.

In higher dimensions, to our best knowledge, there are no explicit examples of K satisfying (3.2) with $\gamma > 0$. An upper bound for γ can be derived, for instance, from the results concerning the lattice point distributions error term. We now quote the estimate due to Müller ([15]). Let $N(\tau) = |\{\tau K \cap \mathbb{Z}^d\}|$, $d > 2$, suppose $N(\tau) = |K|\tau^d + E(\tau)$. Then

$$|E(\tau)| = O\left(\tau^{d-2+\gamma_d}\right), \text{ with } \gamma_d = \begin{cases} \frac{20}{43}, & d = 3, \\ \\ \frac{d+4}{d^2+d+2}, & d \geq 4. \end{cases} \tag{4.10}$$

Clearly (4.10) implies $\gamma \leq \gamma_d$ for the condition (3.2) for otherwise one could construct an immediate counterexample to (4.10).

Returning to L^2 surface averages, the above quoted upper bounds (4.2) and (4.3) of Wolff and Erdoğan, as well as the coarse bounds (2.14) and (2.17) of Theorem 2.1 are applicable to the quantity $\sigma_{\mu,K}$ defined in (3.3) as well (with the bounding constants now also depending on K.) If one attempts to use these bounds for the specific case $A = \mathbb{Z}^d$, they lead to a trivial estimate $\gamma \leq 1$. In other words, $\tau\partial K$ contains no more than τ integer points thereon or in its $\frac{1}{\tau}$-vicinity.

Observe that the affirmative answer to Question 4.2 would imply that τS^{d-1} contains no more than $C(\tau)$ – the quantity growing slower than any power of τ – integer points, which is indeed known to be true. Such an improvement could in principle come from taking the phase factors in (2.32), (2.39) into account. The fact that these phase factors appear in their present form is the special feature of the Euclidean case. We conclude the paper with the following generalization of Question 4.2.

Question 4.4. Are the bounds for the number of lattice points on or near the dilates of the boundaries of $K \in \mathcal{K}$ a particular case of general asymptotic bounds for the quantity $\sigma_{\mu,K}(t)$, for measures arising as thickenings of well-distributed sets and not necessarily lattices? Is it true, in particular, that in the case $d = 2$, cf. (2.13),

$$\sigma_{\mu_{d/2},K}(t) \;\leq\; C(t)t^{-\frac{3}{4}}, \tag{4.11}$$

where $C(t)$ grows slower than any power of $t \to \infty$?

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