MT&I: Exercises 1

1. Show that

(i) $[a, b] = \bigcap_{n=1}^{\infty} (a - 1/n,\, b + 1/n)$;

(ii) $(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n,\, b - 1/n]$.

Conclude that any σ-algebra of subsets of $\mathbb{R}$ which contains all open (resp. closed) intervals also contains all closed (resp. open) intervals.

2. Show that the Borel σ-algebra $\mathfrak{B}$ is generated by the collection of half-open intervals $(a, b]$. Show that it is also generated by the collection of half-rays (a, ∞), $a \in \mathbb{R}$.

3. Let $A \subset \Omega$. Describe $\sigma(\{A\})$.

4. Let f be a measurable function on $(\Omega, \mathcal{A})$ and $A > 0$. The **truncation** f_A of f is defined by

$$f_A(\omega) = \begin{cases} f(\omega) & \text{if } |f(\omega)| \leq A, \\ A & \text{if } f(\omega) > A, \\ -A & \text{if } f(\omega) < -A. \end{cases}$$

Show that f_A is measurable.

5. Let $f : (\Omega, \mathcal{A}) \to \mathbb{R}$ (or $\overline{\mathbb{R}}$). Show that the following are equivalent.

(i) f is $\mathcal{A}$-measurable;

(ii) $\{f > q\} \in \mathcal{A}$ for each $q \in \mathbb{Q}$;

(iii) $\{f \leq q\} \in \mathcal{A}$ for each $q \in \mathbb{Q}$.

6. Give an example of an $\mathbb{R}$-valued function f on some measurable space $(\Omega, \mathcal{A})$ which is not $\mathcal{A}$-measurable, but is such that $|f|$ and f^2 are $\mathcal{A}$-measurable.

7. Consider the Borel σ-algebra $(\mathbb{R}, \mathfrak{B})$. Show that any monotone function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

8. Let $x \in (0, 1]$ have the expansion $x = 0.x_1x_2x_3\ldots$ in base 2, the non-terminatng expansion being used in cases of ambiguity. Show that $f_n(x) = x_n$ is a Borel measurable function of x for each n.

9. Let f be a non-negative measurable function on $(\Omega, \mathcal{A})$ which is bounded (so $0 \leq f(\omega) \leq K$ for all $\omega \in \Omega$). Show that the sequence of simple measurable functions φ_n constructed in Lemma 2.17 converges to f uniformly on Ω.

10. Let $f : \Omega_1 \to \Omega_2$ be a function. For $E \subseteq \Omega_2$ define

$$f^{-1}(E) = \{\omega \in \Omega_1 : f(\omega) \in E\}$$

Show that

(i) $f^{-1}(\emptyset) = \emptyset$,

(ii) $f^{-1}(\Omega_2) = \Omega_1$,

(iii) $f^{-1}(E \setminus F) = f^{-1}(E) \setminus f^{-1}(F)$ for $E, F \subseteq \Omega_2$,

for any non-empty collection $\{E_\alpha\}$ of Ω_2,

(iv) $f^{-1}(\cup_\alpha E_\alpha) = \cup_\alpha f^{-1}(E_\alpha)$,

(v) $f^{-1}(\cap_\alpha E_\alpha) = \cap_\alpha f^{-1}(E_\alpha)$.

Show that if $\mathcal{A}_2$ is a σ-algebra of subsets of Ω_2, then $\{f^{-1}(E) : E \in \mathcal{A}_2\}$ is a σ-algebra of subsets of Ω_1.

11. Let $f : \Omega_1 \to \Omega_2$ be a function. Let $\mathcal{A}_1$ be a σ-algebra of subsets of Ω_1. Put $\mathcal{A}_2 = \{E \subseteq \Omega_2 : f^{-1}(E) \in \mathcal{A}_1\}$. Show that $\mathcal{A}_2$ is a σ-algebra.

12. Let $(\Omega_1, \mathcal{A}_1)$ be a measurable space and $f : \Omega_1 \to \Omega_2$. Let $\mathcal{E}$ be a collection of subsets of Ω_2 such that $f^{-1}(E) \in \mathcal{A}_1$ for every $E \in \mathcal{E}$. Show that $f^{-1}(F) \in \mathcal{A}_1$ for any set F that belongs to the σ-algebra generated by $\mathcal{E}$. *Hint:* Use Question 11.