Analysis 3: Exercises 2

1. Show that if μ is a measure on a σ-algebra $\mathfrak{A}$ and A is a fixed set in $\mathfrak{A}$ then the function λ defined for $E \in \mathfrak{A}$ by $\lambda(E) = \mu(A \cap E)$ is a measure on $\mathfrak{A}$.

2. If $\mu_1, \mu_2, \ldots$ are measures on $\mathfrak{A}$ and $a_1, a_2, \ldots$ are nonnegative real numbers, then the function λ defined for $E \in \mathfrak{A}$ by

$$\lambda(E) = \sum_{j=1}^{n} a_j \mu_j(E)$$

is a measure on $\mathfrak{A}$.

3. Let $\Omega = \mathbb{N}$ and let $\mathfrak{A}$ be the family of all subsets of Ω. If (a_n) is a sequence of non-negative extended real numbers and we define μ by

$$\mu(\emptyset) = 0; \qquad \mu(E) = \sum_{n \in E} a_n, E \neq \emptyset;$$

then μ is a measure on Ω. Conversely, every measure on Ω is obtained in this way for some sequence (a_n) of non-negative numbers in $\overline{\mathbb{R}}$.

4. Let $\Omega = \mathbb{N}$ and $\mathfrak{A}$ be the family of all subsets of Ω. If E is finite, let $\mu(E) = 0$; if E is infinite, let $\mu(E) = +\infty$. Is μ a measure on $\mathfrak{A}$?

5. Let $A, B, C \in \mathfrak{A}$ such that A, B are subsets of C, and μ a measure on $\mathfrak{A}$. Show that if $\mu(A) = \mu(C) < +\infty$ then $\mu(A \cap B) = \mu(B)$.

6. Show that Lemma 3.4 (b) may fail if the finiteness condition $\mu(F_1) < +\infty$ is dropped.

7. Let (A_n) be a sequence of sets in Ω. Put

$$A = \bigcap_{m=1}^{\infty} [\bigcup_{n=m}^{\infty} A_n], \qquad B = \bigcup_{m=1}^{\infty} [\bigcap_{n=m}^{\infty} A_n].$$

The set A is called the **limit superior** of the sets (A_n) and denoted by lim sup A_n. The set B is called the **limit inferior** of the sets (A_n) and denoted by lim inf A_n. A consists of all those $\omega \in \Omega$ that belong to infinitely many of the sets A_n while B consists of all those $\omega \in \Omega$ that belong to all but finitely many of the sets A_n. Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space and let (A_n) be a sequence in $\mathfrak{A}$. Show that

a) $\mu(\liminf A_n) \leq \liminf \mu(A_n)$;

and, if $\mu(\cup A_n) < +\infty$,

b) $\limsup \mu(A_n) \leq \mu(\limsup A_n)$.

Show that the inequality in b) may fail if $\mu(\cup A_n) = +\infty$.

8. The measure space $(\Omega, \mathfrak{A}, \mu)$ is said to be **complete** if whenever $N \in \mathfrak{A}$ with $\mu(N) = 0$ and $N_0 \subseteq N$ then $N_0 \in \mathfrak{A}$. Give an example of a measure space that is not complete.

9. Let $\varphi \in M^+$ be a simple function with (not necessarily standard) representation

$$\varphi = \sum_{k=1}^{m} b_k \chi_{E_k},$$

where $b_k \in \mathbb{R}$ and $E_k \in \mathfrak{A}$. Show that

$$\int \varphi \, d\mu = \sum_{k=1}^{m} b_k \mu(E_k).$$

10. If $\varphi_1, \varphi_2 \in M^+$ are simple functions, show that

$$\psi = \sup\{\varphi_1, \varphi_2\}, \qquad \omega = \inf\{\varphi_1, \varphi_2\}$$

are simple functions in M^+.

11. Let $F \subseteq [0,1]$ be the Cantor ternary set. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{on } F, \\ n & \text{on each complementary interval of length } 3^{-n}, \\ 0 & \text{elsewhere.} \end{cases}$$

Show that f is Borel measurable and that

$$\int_0^1 f\,dm := \int \chi_{[0,1]} f\,dm = 3.$$

12.

a) Let $f_n = (1/n)\chi_{[0,n]}$, $f = 0$. Show that the sequence (f_n) converges uniformly to f, but that

$$\int f\,dm \neq \lim \int f_n\,dm.$$

Why does this not contradict the Monotone Convergence Theorem? Does Fatou's Lemma apply?

b) Let $g_n = n\chi_{[1/n,2/n]}$, $g = 0$. Show that

$$\int g\,dm \neq \lim \int g_n\,dm.$$

Does the sequence (g_n) converge uniformly to g? Does the Monotone Convergence Theorem apply? Does Fatou's Lemma apply?

13. If $(\Omega, \mathfrak{A}, \mu)$ is a finite measure space, and if (f_n) is a real-valued sequence in $M^+(\Omega, \mathfrak{A})$ which converges unifromly to a function f, then f belongs to $M^+(\Omega, \mathfrak{A})$ and

$$\int f\,d\mu = \lim \int f_n\,d\mu.$$