MT&I: Exercises 3

1. Let $f \in L$ and $a > 0$. Show that $\{|f| \geq a\}$ has finite measure. Show that $\{f \neq 0\}$ has σ-finite measure.

2. Let f be an $\mathfrak{A}$-measurable $\mathbb{R}$-valued function such that $f = 0$ μ-a.e. Show that $f \in L$ and $\int f \, d\mu = 0$.

3. Let $f \in L$ and g an $\mathfrak{A}$-measurable $\mathbb{R}$-valued function such that $f = g$ μ-a.e. Show that $g \in L$ and $\int f \, d\mu = \int g \, d\mu$.

4. Let $f \in L$ and $\varepsilon > 0$. Show that there exists an $\mathfrak{A}$-measurable simple function φ such that

$$\int |f - \varphi| \, d\mu < \varepsilon.$$

5. Let $f \in L$ with indefinite integral

$$\lambda(E) = \int_E f \, d\mu, \qquad E \in \mathfrak{A}.$$

Show that $\lambda(E) \geq 0$ for all $E \in \mathfrak{A}$ if and only if $f \geq 0$ μ-a.e. Moreover, $\lambda(E) = 0$ for all $E \in \mathfrak{A}$ if and only if $f = 0$ μ-a.e.

6. Let (f_n) be a sequence in L that converges uniformly on Ω to f. If $\mu(\Omega) < \infty$ show that

$$\int f \, d\mu = \lim \int f_n \, d\mu$$

Show that this may fail if $\mu(\Omega) = \infty$.

7. Let $f_n \in L$ such that

$$\sum_{n=1}^{\infty} \int |f_n| \, d\mu < \infty.$$

Show

a) $\sum f_n$ converges μ-a.e. to a function $f \in L$;

b) $\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$.

8. Let $f_n \in L$ such that $f_n \to f \in L$. Suppose that

$$\lim \int |f_n - f| \, d\mu = 0.$$

Show that

$$\int |f| \, d\mu = \lim \int |f_n| \, d\mu.$$