UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level 3)

MEASURE THEORY AND INTEGRATION

MATH 34000
(Paper Code MATH-34000)

April 2013, 2 hours 30 minutes

This paper contains **five** *questions*
The best **FOUR** *answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. Let X be a non-empty set.

   (a) **(5 marks)**
   Explain what is meant by "$\mathbb{X}$ is a sigma-algebra of subsets of X".

   (b) **(5 marks)**
   Let $X = \mathbb{N}$ and let $\mathbb{X}$ be the set of all finite subsets of X and all subsets of X where the complement is finite. Show that $\mathbb{X}$ is not a sigma-algebra of subsets of X.

   Now let $(X, \mathbb{X})$ be a general measurable space.

   (c) **(3 marks)**
   What does it mean for a function $f : X \to \mathbb{R}$ to be measurable.

   (d) **(3 marks)**
   Show, directly from your definition, that if $f : X \to \mathbb{R}$ is measurable then $|f| : X \to \mathbb{R}$ is also measurable.

   (e) **(5 marks)**
   Let $(A, \mathbb{A})$ be the measurable space where $A = \{1, 2, 3\}$ and $\mathbb{A} = \{A, \emptyset, \{2, 3\}, \{1\}\}$. Show that $f : A \to \mathbb{R}$ is measurable if and only if $f(2) = f(3)$.

   (f) **(4 marks)**
   Is the statement '$|f| : X \to \mathbb{R}$ being measurable implies that $f : X \to \mathbb{R}$ is measurable' true? (Justify your answer).

2. Let $(X, \mathbb{X})$ be a measurable space.

   (a) **(5 marks)**
   Explain what is meant by "$\mu : \mathbb{X} \to \mathbb{R}$ is a measure".

   Now let $(X, \mathbb{X}, \mu)$ be a measure space.

   (b) **(5 marks)**
   Let $A_n \in \mathbb{X}$ for all $n \in \mathbb{N}$. Prove, directly from the definition, that

   $$\mu\left(\cup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

   (c) **(4 marks)**
   Suppose that (F_n) is a sequence of sets such that $\mu(F_1) < \infty$ and $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{N}$. Show that

   $$\mu\left(\cap_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n).$$

   (d) **(6 marks)**
   Let $A_n \in \mathbb{X}$ be a sequence of sets such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ and

   $$A = \{x : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

   Show that $\mu(A) = 0$.

   (e) **(5 marks)**
   Now take the measure space $(\mathbb{R}, \mathbb{B}, \lambda)$ (where $\mathbb{B}$ is the Borel sigma algebra and λ is Lebesgue measure), $\delta > 0$ and

   $$V = \left\{ x \in \mathbb{R} : \text{ there exist infinitely many } p \in \mathbb{Z}, q \in \mathbb{N} \text{ where } \left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\delta}} \right\}.$$

   Show that $\lambda(V) = 0$.

Continued...

3. Let $(X, \mathbb{X}, \mu)$ be a measure space.

   (a) **(2 marks)**
   State Fatou's lemma.

   (b) **(7 marks)**
   State Lebesgue's dominated convergence theorem and prove it using Fatou's lemma.

   Now let $X = (0,1)$, let $\mathbb{B}$ denote the Borel subsets of $(0,1)$, and let λ be Lebesgue measure.

   (c) **(5 marks)** Show that for a non-negative measurable function $f : (0,1) \to \mathbb{R}$ we have that $\lim_{\delta \to 0} \int_{(\delta,1)} f \mathrm{d}\lambda = \int_{(0,1)} f \mathrm{d}\lambda$. (You may use the monotone convergence theorem without proof as long as it is clearly stated).

   (d) **(6 marks)**
   Let $0 < \alpha < 1$ and $f_n : (0,1) \to \mathbb{R}$ be given by

   $$f_n(x) = \frac{n x^\alpha}{1 + n^2 x^2}.$$

   Find $\lim_{n\to\infty} \int_{(0,1)} f_n \mathrm{d}\lambda(x)$, and justify your answer.

   (e) **(5 marks)**
   Let $f_n(x) = \frac{n}{1+n^2x^2}$, show that $\lim_{n\to\infty} \int_{(0,1)} f_n d\lambda \neq 0$. Does there exist $g \in L_1$ such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and $x \in (0,1)$?

4. Let $(X, \mathbb{X}, \mu)$ be a measure space, and let $1 < p < \infty$.

   (a) **(6 marks)**
   What is meant by "$f = g$ μ - almost everywhere"? Prove that $f = g$ μ almost everywhere if and only if $\int |f - g| \mathrm{d}\mu = 0$.

   (b) **(6 marks)**
   State and prove Hölder's inequality. (you may assume Young's inequality that if $1 \leq p < \infty$, $A, B \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$.)

   (c) **(3 marks)**
   State Minkowski's inequality.

   (d) **(5 marks)**
   Let $1 \leq p \leq q < \infty$ show that if $f \in L_p$ and $f \in L_q$ then $f \in L_r$ for all $p \leq r \leq q$.

   (e) **(5 marks)**
   Let $1 \leq p < \infty$ and suppose that $f \in L_p \cap L_\infty$ show that $f \in L_r$ for all $r \geq p$.

Continued...

5. Let $(X, \mathbb{X})$ be a measure space and let μ, ν be measures on $\mathbb{X}$.

   (a) **(3 marks)**
   What is meant by "μ is absolutely continuous with respect to ν"?

   (b) **(5 marks)**
   Let $f \in M^+(X)$ be a non-negative measurable function and suppose that $\lambda : \mathbb{X} \to \overline{\mathbb{R}}$ is the measure defined by
   $$\lambda(A) = \int_A f \mathrm{d}\mu.$$
   Show that λ is absolutely continuous with respect to μ and that if $f(x) > 0$ for μ almost every $x \in X$ then μ is absolutely continuous with respect to λ.

   (c) **(4 marks)** Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions and $f : X \to \mathbb{R}$ be a measurable function. Is it always true that if μ is absolutely continuous with respect to ν and $\lim_{n\to\infty} f_n(x) = f(x)$ for μ almost all $x \in X$ then $\lim_{n\to\infty} f_n(x) = f(x)$ for ν almost all $x \in X$? (Justify your answer)

   (d) **(5 marks)** Suppose that μ is absolutely continuous with respect to ν and $A_n \in \mathbb{X}$ are a sequence of sets such that $\lim \nu(A_n) = 0$. Is it always true that $\lim_{n\to\infty} \mu(A_n) = 0$? (Justify your answer)

   (e) **(3 marks)**
   State the Radon-Nikodým Theorem.

   (f) **(5 marks)** Let μ and ν be σ-finite measures on $\mathbb{X}$ where μ is absolutely continuous with respect to ν and such that the Radon-Nikodým derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ is in L_∞. Suppose that $f_n : X \to \mathbb{R}$ are a sequence of measurable functions and $f : X \to \mathbb{R}$ is a measurable function such that f_n converges in measure to f with respect to ν. Show that f_n also converges in measure to f with respect to μ.

End of examination.