Measure Theory and Integration
MATH 34000

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Prerequisites: MATH 20200 Metric Spaces 2

Unit aims: The aim of the unit is to introduce measure theory and the Lebesgue integral.

Relation to Other Units: This unit is an element of a sequence of analysis courses at Levels I/5, H/6 and M/7. However it also relates to the unit ergodic theory and dynamical systems and all units in probability.

Teaching Methods: A course of 30 lectures and additional problem classes.

Lecturer: Ivor McGillivray

Assessment Methods: The final assessment mark for the unit is calculated from a standard 2.5-hour written closed-book examination in January 2014. No calculators are permitted.

Texts:

- R. G. Bartle, The Elements of Integration and Lebesque Measure, Wiley, New York 1995. (this is the closest text to this unit).
- G. de Barra, Introduction to Measure Theory, Van Nostrand Reinhold Company, New York 1974.
- R. G. Bartle, D. R. Sherbert, Introduction to Real Analysis, Wiley, New York 1992.

1 Motivation and Background

1.1 The Riemann Integral

In further topics in analysis you saw the Riemann integral.

Definition 1.1. *Let $f : [a, b] \to \mathbb{R}$, $a < b$ and let $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$ with $a = x_0 < x_1 < \cdots < x_n = b$. Set*

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} \inf\{f(x) : x_{i-1} \leq x < x_i\}(x_i - x_{i-1}),$$

and likewise

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} \sup\{f(x) : x_{i-1} \leq x < x_i\}(x_i - x_{i-1}).$$

*Then f is **Riemann integrable** if*

$$\sup_{\mathcal{P}} L(\mathcal{P}, f) = \inf_{\mathcal{P}} U(\mathcal{P}, f),$$

*where the supremum and infimum are taken over all possible partitions of $[a, b]$. The **Riemann integral** $\int_a^b f(x)dx$ is defined to be this common value.*

However there are some features of this definition which are not ideal.

Remark. *Not all functions are Riemann integrable, and in particular the pointwise limit of a sequence of Riemann integrable functions need not be Riemann integrable.*

Example. *Define $f : [0, 1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1] \\ 0, & \textit{else} \end{cases}.$$

Let $q_1, q_2, \cdots$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Then f is the pointwise limit of the sequence of Riemann integrable functions $f_n : [0, 1] \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 1, & x \in \{q_1, \cdots, q_n\} \\ 0, & \textit{else} \end{cases},$$

but f is not Riemann integrable.

Definition 1.2. *Let X be a set and let $E \subset X$. We define the characteristic function of E, $\chi_E : X \to \mathbb{R}$ by*

$$\chi_E(x) = \begin{cases} 1 & \textit{if} \quad x \in E \\ 0 & \textit{if} \quad x \notin E \end{cases}$$

A step function on $\mathbb{R}$ is defined to be a finite linear combination of characteristic functions for intervals. One way of thinking about Riemann integration is that you first approximate the function from above and below by step functions. The approach in this course will be to instead approximate the function by linear combinations of more general characteristic functions (called simple functions). This will allow more functions to be integrable, such as the example above, and better behaviour when looking at limits at functions.

1.2 The Extended Real Number System

Definition 1.3. $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 1.4. ***Arithmetic on*** $\bar{\mathbb{R}}$***:***
Let $x \in \mathbb{R}$*. Then*

$$(\pm\infty) + (\pm\infty) = (\pm\infty) = (\pm\infty) + x,$$

$$(\pm\infty)(\pm\infty) = (+\infty),$$

$$(\pm\infty)(\mp\infty) = (-\infty),$$

and

$$x(\pm\infty) = \begin{cases} \pm\infty, & x > 0 \\ \mp\infty, & x < 0 \\ 0, & x = 0 \end{cases} .$$

Definition 1.5. *Let* (x_n) *be a sequence in* $\bar{\mathbb{R}}$*. Then*

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{r\geq n} x_r),$$

and

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{r\geq n} x_r).$$

Note that these always exist for sequences in $\bar{\mathbb{R}}$*.*

Theorem 1.6. *If* (x_n) *is a sequence in* $\bar{\mathbb{R}}$ *and* $\Lambda \in \mathbb{R}$ *then*

(i) $\Lambda = \limsup_{n\to\infty} x_n \iff \forall \epsilon > 0$

(a) $x_n < \Lambda + \epsilon$ *for all sufficiently large* n*, and*

(b) $x_n > \Lambda - \epsilon$ *for infinitely many* n*.*

Likewise

(ii) $\lambda = \liminf_{n\to\infty} x_n \iff \forall \epsilon > 0$

(a) $x_n > \lambda - \epsilon$ *for all sufficiently large* n*, and*

(b) $x_n < \lambda + \epsilon$ *for infinitely many* n*.*

Corollary 1.7. *Let (x_n) be a sequence in $\bar{\mathbb{R}}$. Then*

$$\lim_{n\to\infty} x_n = l \quad \iff \quad \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l.$$

2 Measurable Sets

Definition 2.1. *Let $X \neq \emptyset$. A family $\mathbb{X}$ of subsets of X is a **σ-algebra** if*

$$\begin{aligned} &(i) && \emptyset \in \mathbb{X},\ X \in \mathbb{X}, \\ &(ii) && A \in \mathbb{X} \implies A^c \in \mathbb{X},\ \text{and} \\ &(iii) && A_1, A_2, \dots \in \mathbb{X} \implies \cup_{n=1}^{\infty} A_n \in \mathbb{X}. \end{aligned}$$

*$(X, \mathbb{X})$ is called a **measurable space**, and the sets in $\mathbb{X}$ are called **measurable sets**.*

Remark. *Let $(X, \mathbb{X})$ be a measurable space, and let $A_1, A_2, \dots \in \mathbb{X}$. Then $\cap_{n=1}^{\infty} A_n \in \mathbb{X}$.*

Example. *Let X be a set then $\{\emptyset, X\}$ is always a sigma algebra. Similarly the power set $\mathcal{P}(X)$ consisting of all subsets of X is an example of a sigma algebra.*

If we take $X = \{a, b, c\}$ and $\mathbb{X} = \{X, \{a, b\}, \{c\}, \emptyset\}$ then we can see this is a sigma algebra (it can be thought of as the sigma algebra which does not distinguish between a and b). However $\mathbb{Y} = \{X, \{a\}, \{b\}, \{c\}, \emptyset\}$ is not a sigma algebra.

Definition 2.2. *Let $X \neq \emptyset$, and let $\mathbb{A}$ be a non-empty collection of subsets of X. Let $\mathbb{Y}$ be the collection of all σ-algebras containing $\mathbb{A}$. Then $\cap_{\mathbb{X}\in\mathbb{Y}}\mathbb{X}$ is the **σ-algebra generated by** $\mathbb{A}$. This is the smallest σ-algebra containing $\mathbb{A}$.*

Definition 2.3. *Let $X = \mathbb{R}$, and let $\mathbb{A} = \{(a, b) : a, b \in \bar{\mathbb{R}}, a < b\}$. The σ-algebra generated by $\mathbb{A}$ is the **Borel algebra** $\mathbb{B}$. This is the smallest σ-algebra containing all open sets. A set $B \in \mathbb{B}$ is a **Borel set**.*

The Borel σ-algebra is the one most commonly used throughout this unit.

Definition 2.4. *Let $(X, \mathbb{X})$ be a measurable space. Then $f : X \to \mathbb{R}$ is a $\mathbb{X}$-**measurable function** if for any $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) > \alpha\}$ is in $\mathbb{X}$.*

Example. *For any measurable space $(X, \mathbb{X})$ a constant function is measurable. If $\mathbb{X} = \mathcal{P}(X)$ then any function is measurable. If we let $X = \mathbb{N}$, $O = \{1, 3, 5, 7 \dots\}$ and $E = \{2, 4, 6, 8, \dots\}$ then $\mathbb{X} = \{\emptyset, O, E, \mathbb{N}\}$ is a sigma algebra. A function $f : \mathbb{N} \to \mathbb{R}$ is measurable if and only if $f(x) = f(y)$ for all $x, y \in E$ and for all $x, y \in O$.*

Lemma 2.5. *Let* $(X, \mathbb{X})$ *be a measurable space, and let* $f : X \to \mathbb{R}$. *Then the following are equivalent:*

$$\begin{aligned}
&(i) \quad \forall \alpha \in \mathbb{R},\ A_\alpha = \{x \in X : f(x) > \alpha\} \in \mathbb{X},\\
&(ii) \quad \forall \alpha \in \mathbb{R},\ B_\alpha = \{x \in X : f(x) \le \alpha\} \in \mathbb{X},\\
&(iii) \quad \forall \alpha \in \mathbb{R},\ C_\alpha = \{x \in X : f(x) \ge \alpha\} \in \mathbb{X},\\
&(iv) \quad \forall \alpha \in \mathbb{R},\ D_\alpha = \{x \in X : f(x) < \alpha\} \in \mathbb{X}.
\end{aligned}$$

Proof. The fact that (i) and (ii) are equivalent is immediate by taking complements (similarly (iii) and (iv)). So we need to show (i) and (iii) are equivalent. To do this let α_n be a sequence which converges monotonically to α from below. We can now show

$$\cap_n A_{\alpha_n} = C_\alpha.$$

Similarly if α_n is a sequence which converges monotonically to α from above then

$$\cup_n C_{\alpha_n} = A_\alpha.$$

The result now follows using the definition of a measurable function. $\square$

Lemma 2.6. *Let* $(X, \mathbb{X})$ *be a measurable space, and let* $f, g : X \to \mathbb{R}$ *be measurable. Let* $F : \mathbb{R}^2 \to \mathbb{R}$ *be continuous. Then* $h : X \to \mathbb{R}$ *given by* $h(x) = F(f(x), g(x))$ *is measurable.*

Proof. Fix $\alpha \in \mathbb{R}$ and note we need to show that the set

$$X_\alpha := \{x : F(f(x), g(x)) < \alpha\} = \{x : (f(x), g(x)) \in F^{-1}((-\infty, \alpha))\}$$

is measurable. We can let $A = F^{-1}((-\infty, \alpha))$ and note that it is an open set in $\mathbb{R}^2$ (F is continuous) and thus a countable union of open rectangles (sets of the form $(a, b) \times (c, d)$). So it suffices to show that for any set $(a, b) \times (c, d)$ we have that

$$\{x : (f(x), g(x)) \in (a, b) \times (c, d)\} = \{x : a < f(x) < b\} \cap \{x : c < g(x) < d\}$$

is measurable. We can now use the fact that f and g are measurable to complete the proof. $\square$

Corollary 2.7. *Let* f *and* g *be measurable and let* $c \in \mathbb{R}$. *Then* $f + g$, fg, $|f|$, $\frac{f}{g}$ *(for* $g \neq 0$*),* $\max\{f, g\}$, $\min\{f, g\}$ *and* cf *are all measurable.*

Definition 2.8. *Let* $(X, \mathbb{X})$ *be a measurable space. Then* $f : X \to \bar{\mathbb{R}}$ *is* $\mathbb{X}$-***measurable*** *if for any* $\alpha \in \mathbb{R}$, *the set* $\{x \in X : f(x) > \alpha\}$ *is in* $\mathbb{X}$. *The collection of all* $\bar{\mathbb{R}}$*-valued,* $\mathbb{X}$*-measurable functions on* X *is denoted by* $M(X, \mathbb{X})$.

Lemma 2.9. *Let $(X, \mathbb{X})$ be a measurable space. A function $f : X \to \bar{\mathbb{R}}$ is measurable if and only if*

(i) $A = \{x \in X : f(x) = +\infty\} \in \mathbb{X}$,
$B = \{x \in X : f(x) = -\infty\} \in \mathbb{X}$,

and

(ii) *the function* $f_1(x) = \begin{cases} f(x), & x \in (A \cup B)^c \\ 0, & x \in A \cup B \end{cases}$ *is measurable.*

Proof. Suppose f_1, A, B are measurable. For $\alpha \geq 0$ we have that

$$\{x : f(x) > \alpha\} = \{x : f_1(x) > \alpha\} \cup A$$

and so is measurable. Similar arguments work for $\alpha < 0$.

Now suppose f is measurable. We can write

$$A = \cap_{n \in \mathbb{N}} \{f(x) > n\} \text{ and } B = \cap_{n \in \mathbb{N}} \{f(x) \leq -n\}.$$

Thus both these sets are measurable. For $\alpha \geq 0$

$$\{x : f_1(x) > \alpha\} = \{x : f(x) > \alpha\} \backslash A$$

and so is measurable. For $\alpha < 0$

$$\{x : f_1(x) > \alpha\} = \{x : f(x) > \alpha\} \cup B$$

and is measurable. □

Corollary 2.10. *If $f \in M(X, \mathbb{X})$ and $c \in \mathbb{R}$, then f^2, cf, $|f|$, f^+, and f^- are all in $M(X, \mathbb{X})$.*

Lemma 2.11. *Let f_n be a sequence in $M(X, \mathbb{X})$, and define*

$$f(x) = \inf_n f_n(x), \quad F(x) = \sup_n f_n(x),$$
$$f^*(x) = \liminf_n f_n(x), \quad \text{and} \quad F^*(x) = \limsup_n f_n(x).$$

Then f, F, f^*, and F^* are in $M(X, \mathbb{X})$.

Proof. We prove this for f and f^* the proofs for F and F^* are analogous. Fix $\alpha \in \mathbb{R}$ and consider

$$A_\alpha = \{x \in X : f(x) > \alpha\} = \cup_{m \in \mathbb{N}} \cap_{n \in \mathbb{N}} \{x \in X : f_n(x) > \alpha + 1/m\}.$$

Since each $\{x \in X : f_n(x) > \alpha + 1/m\} \in \mathbb{X}$ it follows that $A_\alpha \in \mathbb{X}$ and so f is measurable.

For the second part note that $g_k = \inf_{n \geq k} f_n$ is measurable. Thus for any $\alpha \in \mathbb{R}$ we have that

$$\{x \in X : f^*(x) > \alpha\} = \cup_{k=1}^{\infty} \{x \in X : g_k(x) > \alpha\} \in \mathbb{X}.$$

□

Corollary 2.12. *Let f_n be a sequence in $M(X, \mathbb{X})$, and let $f_n \to g$ pointwise on X. Then $g \in M(X, \mathbb{X})$.*

Lemma 2.13. *If $f, g \in M(X, \mathbb{X})$, then $fg \in M(X, \mathbb{X})$.*

Proof. Truncate f for $k \in \mathbb{N}$ by

$$f_k(x) = \begin{cases} f(x) & \text{if} \quad -k < f(x) < k \\ k & \text{if} \quad f(x) \geq k \\ -k & \text{if} \quad f(x) \leq -k \end{cases}$$

It is an exercise to show this is measurable. Now do the same to g and use Lemma 2.11 and Corollary 2.7. $\square$

Definition 2.14. *A* ***simple function*** *is a finite linear combination of characteristic functions of measurable sets.*

Lemma 2.15. ***(Approximation by simple functions.)***
Let $f \in M(X, \mathbb{X})$, $f \geq 0$. Then there exists a sequence (ϕ_n) in $M(X, \mathbb{X})$ such that

$$\begin{aligned} &\text{(i)} \quad 0 \leq \phi_n(x) \leq \phi_{n+1}(x) \quad \forall x \in X,\ n \in \mathbb{N}, \\ &\text{(ii)} \quad \lim_{n \to \infty} \phi_n(x) = f(x), \\ &\text{(iii)} \quad \text{Each } \phi_n \text{ is a simple function.} \end{aligned}$$

Proof. We fix $n \in \mathbb{N}$ and for $k \in \{0, 1, \ldots, n2^n - 1\}$ define

$$E_{k,n} = \left\{ x : f(x) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) \right\} \text{ and } E_{n2^n,n} = \{x : f(x) \in [n, \infty)\}.$$

Note that $X = \cup_{k=0}^{2^n n} E_{k,n}$, the sets are disjoint and are all measurable. We then define

$$\phi_n(x) = \frac{k}{2^n} \text{ if } x \in E_{k,n}.$$

In other words $\phi_n = \sum_{k=0}^{2^n n} \frac{k}{2^n} \chi_{E_{k,n}}$ is a simple function. We can easily show $\phi_n(x) \leq \phi_{n+1}(x)$ and $\lim_{n \to \infty} \phi_n(x) = f(x)$. $\square$

Definition 2.16. *A function $f : X \to \mathbb{C}$ is measurable if Re(f) and Im(f) are measurable.*

Definition 2.17. *Let $(X, \mathbb{X})$ and $(Y, \mathbb{Y})$ be measurable spaces. A function $f : X \to Y$ is measurable if $A \in \mathbb{Y} \implies f^{-1}(A) \in \mathbb{X}$.*

3 Measures

Definition 3.1. *Let $(X, \mathbb{X})$ be a measurable space. Then $\mu : \mathbb{X} \to \bar{\mathbb{R}}$ is a* ***measure*** *if*

$$\begin{aligned} &(i) && \mu(\emptyset) = 0, \\ &(ii) && A \in \mathbb{X} \implies \mu(A) \geq 0, \text{ and} \\ &(iii) && \text{if } A_1, A_2, \cdots \text{ is a sequence of disjoint sets in } \mathbb{X}, \\ & && \text{then } \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n). \end{aligned}$$

$(X, \mathbb{X}, \mu)$ is called a ***measure space****.*

Definition 3.2. *A measure μ is called a* ***finite measure*** *if $\mu(X) < \infty$. It is called a* ***σ-finite measure*** *if there is a sequence (A_n) of sets in $\mathbb{X}$ with $X = \cup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for each n.*

Remark. *Let $(X, \mathbb{X})$ be a measure space and let $x \in X$. For $E \in \mathbb{X}$ we define*

$$\delta_x(E) = 1 \text{ if } x \in E \text{ and } \delta_x(E) = 0 \text{ if } x \notin E.$$

We can see that δ_x is a measure (often named after Dirac).

Take $(\mathbb{N}, P(\mathbb{N}))$ to be our measurable set for any $E \subset \mathbb{N}$ we define $\mu(E)$ to be the number of elements in E. μ is called the counting measure and is not finite but is sigma finite.

Remark. *Consider the measurable space $(\mathbb{R}, \mathbb{B})$. Then there is a unique measure λ supported on $\mathbb{B}$, which agrees with length for open intervals (a, b), i.e. $\lambda((a, b)) = b - a$. This measure is called the* ***Lebesgue measure****. We will justify this much later in the unit. If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function we can define $\lambda_f(a, b) = f(b) - f(a)$ and this also uniquely defines a measure.*

Lemma 3.3. ***(Monotonicity of measures.)***
Let μ be a measure on $(X, \mathbb{X})$. If $A, B \in \mathbb{X}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$. If in addition $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

Proof. We can write $B = (B \cap A) \cup (B \cap A^c)$ and since $A \subset B$ this gives that $B = A \cup B \cap A^c = A \cup (B \setminus A)$. Since this union is disjoint we can use properties (ii) and (iii) in the definition of a measure to prove both results. $\square$

Lemma 3.4. *Let μ be a measure on $(X, \mathbb{X})$.*
(i) If (A_n) is an increasing sequence $(A_1 \subset A_2 \subset \cdots)$ of measurable sets in $\mathbb{X}$, then $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.
(ii) If (B_n) is a decreasing sequence $(B_1 \supset B_2 \supset \cdots)$ of measurable sets in $\mathbb{X}$, and $\mu(B_1) < \infty$, then $\mu(\cap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n)$.

Proof. For part (i) let $A_0 = \emptyset$ and write $\cup_{n=1}^{\infty} A_n = \cup_{k=1}^{\infty} A_k \setminus A_{k-1}$. Now since this is a union of disjoint sets we can use Lemma 3.3 and property (iii) in the definition of a measure (as long as $\mu(A_n) < \infty$ for all n). This gives that

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{k=1}^{\infty} (\mu(A_k) - \mu(A_{k-1})) = \lim_{n\to\infty} \mu(A_n).$$

If $\mu(A_N) = \infty$ for some $N \in \mathbb{N}$ then $\mu(A_n) = \infty$ for all $n \geq N$ and the result follows.

For part (ii) use that $B_1 \setminus (\cap_{n=1}^{\infty} B_n) = \cup_{k=1}^{\infty} B_k \setminus B_{k+1}$. Since this union is disjoint (and $\mu(B_1) < \infty$)) we have that

$$\mu((B_1) \setminus (\cap_{n=1}^{\infty} B_n)) = \sum_{k=1}^{\infty} (\mu(B_k) - \mu(B_{k+1})) = \lim_{k\to\infty} \mu(B_1) - \mu(B_{k+1}).$$

Rearranging and using Lemma 3.3 gives the result. □

Definition 3.5. *Let $(X, \mathbb{X}, \mu)$ be a measure space. A proposition P holds **μ-almost everywhere** (μ-a.e.) if there is a set N in $\mathbb{X}$ such that $\mu(N) = 0$ and such that P holds on $X \setminus N$.*

Definition 3.6. *A measure space $(X, \mathbb{X}, \mu)$ is **complete** if $A \subset N \in \mathbb{X}$ and $\mu(N) = 0$ implies $A \in \mathbb{X}$.*

4 Integration of Non-Negative Functions

In this section we define the notion of an integral with respect to a measure for a non-negative measurable function. We do this first by defining the integral for simple functions and then by using the fact that non-negative measurable functions can be approximated by simple functions to extend the definition to non-negative measurable functions.

Definition 4.1. *Let $(X, \mathbb{X}, \mu)$ be a measure space. The collection of non-negative $\bar{\mathbb{R}}$-valued measurable functions on X is denoted by $M^+(X, \mathbb{X})$.*

Definition 4.2. *Recall that a simple function is a function of the form*

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i},$$

*where $c_i \in \mathbb{R}$ and $A_i \in \mathbb{X}$. It is in **standard representation** if $X = \cup_{i=1}^{n} A_i$, the sets A_i are pairwise disjoint, and the numbers c_i are distinct.*

Definition 4.3. *Let ϕ be a simple function with standard representation $\phi = \sum_{i=1}^{n} c_i \chi_{A_i}$. Then **the integral of ϕ with respect to μ** is*

$$\int \phi d\mu := \sum_{i=1}^{n} c_i \mu(A_i) \in \bar{\mathbb{R}}.$$

We now prove a few natural results about the integral for simple functions.

Lemma 4.4. *Let $(X, \mathbb{X}, \mu)$ be a measure space. Let ϕ, ψ be simple functions in $M^+(X, \mathbb{X})$ and let $c \geq 0$. Then*

$$\text{(i)} \quad \int c\phi d\mu = c \int \phi d\mu, \text{ and}$$

$$\text{(ii)} \quad \int (\phi + \psi) d\mu = \int \phi d\mu + \int \psi d\mu.$$

Proof. Part (i) is straightforward. To prove part (ii) we need to consider the intervals $A_1, \ldots, A_n$ which make up the standard representation for ϕ and the intervals $B_1, \ldots, B_m$ which make up the standard representation for ψ. We then take the intervals $A_i \cap B_j$ for $1 \leq i \leq n$ and $1 \leq m \leq m$. Note that these intervals will be disjoint and that $\phi + \psi$ will be constant on each interval. Now group these intervals into those on which $\psi + \phi$ take the same value and then we are in a standard form and the result will follow. □

Lemma 4.5. *Let $(X, \mathbb{X}, \mu)$ be a measure space, and let ϕ be a simple function in $M^+(X, \mathbb{X})$. Let $\nu : \mathbb{X} \to \bar{\mathbb{R}}$ be defined by $\nu(A) = \int \phi \chi_A d\mu$. Then ν is a measure.*

Proof. This is set as an exercise. □

We can now give the definition of an integral for a non-negative measurable function.

Definition 4.6. *Let $f \in M^+(X, \mathbb{X})$. Then **the integral of** f **with respect to** μ is*

$$\int f d\mu := \sup \left\{ \int \phi d\mu : \phi \leq f, \ \phi \textit{ is a simple measurable function} \right\} \in \bar{\mathbb{R}}.$$

Definition 4.7. *Let $f \in M^+(X, \mathbb{X})$, and let $A \in \mathbb{X}$. Then **the integral of** f **with respect to** μ **over** A is*

$$\int_A f d\mu := \int f \chi_A d\mu.$$

Lemma 4.8. *Let $(X, \mathbb{X}, \mu)$ be a measure space.*

(i) Let $f, g \in M^+(X, \mathbb{X})$, and let $f \leq g$ on X.
Then $\displaystyle\int f d\mu \leq \int g d\mu$.

(ii) Let $f \in M^+(X, \mathbb{X})$, and let $A, B \in \mathbb{X}$ with $A \subset B$.
Then $\displaystyle\int_A f d\mu \leq \int_B f d\mu$.

Proof. Part (i) follows since if $\phi \leq f$ where ϕ is a simple function then $\phi \leq g$. Part (ii) follows from part (i). $\square$

We now come to one of the most important Theorems for the unit.

Theorem 4.9. ***(Monotone Convergence Theorem)***
Let $(X, \mathbb{X}, \mu)$ be a measure space. Let (f_n) be a monotone increasing sequence of functions in $M^+(X, \mathbb{X})$, which converges point wise to f. Then $f \in M^+(X, \mathbb{X})$, and $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

Proof. Since $f_n \leq f$ for all n it follows Lemma 4.8 that $\lim_{n\to\infty} \int f_n \mathrm{d}\mu \leq \int f \mathrm{d}\mu$. To show the inequality in the other direction we fix a simple function $\phi \leq f$, $0 < \alpha < 1$ and let

$$A_n = \{x : \alpha\phi(x) \leq f_n(x)\}.$$

Note that for each n $A_n \in \mathbb{X}$ and $A_n \subseteq A_{n+1}$. We also have that $X = \cup_{n=1}^{\infty} A_n$. By Lemma 1 we can define a measure ν on $(X, \mathbb{X})$ by

$$\nu(A) = \int_A \phi \mathrm{d}\mu(x)$$

for all $A \in \mathbb{X}$. It follows from Lemma 3.4 that

$$\int \phi \mathrm{d}\mu = \nu(X) = \nu\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n).$$

However for all $n \in \mathbb{N}$

$$\int_X f_n \mathrm{d}\mu \geq \int_{A_n} f_n \mathrm{d}\mu \geq \alpha \int_{A_n} \phi \mathrm{d}\mu = \alpha\nu(A_n)$$

and thus $\alpha \int \phi \mathrm{d}\mu \leq \lim_{n\to\infty} \int f_n \mathrm{d}\mu$. Since this holds for all $0 < \alpha < 1$ and simple functions $\phi \leq f$ it follows that

$$\int f \mathrm{d}\mu \leq \lim_{n\to\infty} \int f_n \mathrm{d}\mu.$$

$\square$

This is a very important result on which much of the unit is based. We will now give several consequences of the Theorem.

Corollary 4.10.

(i) $f \in M^+(X, \mathbb{X}), c \geq 0 \implies cf \in M^+(X, \mathbb{X}), \int cf d\mu = c \int f d\mu,$ *and*

(ii) $f, g \in M^+(X, \mathbb{X}) \implies f + g \in M^+(X, \mathbb{X}), \int (f+g) d\mu = \int f d\mu + \int g d\mu.$

Remark. *The proof is based on approximation by simple functions (Lemma 2.14) and the Monotone Convergence Theorem. The proofs of the next three results are also based on the Monotone Convergence Theorem.*

Theorem 4.11. ***(Fatou's Lemma)***
Let $(X, \mathbb{X}, \mu)$ be a measure space. Let (f_n) be a sequence of functions in $M^+(X, \mathbb{X})$. Then

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Proof. We consider the functions defined $g_n(x) = \inf\{f_k(x) : k \geq n\}$ and use the Monotone Convergence Theorem. □

Theorem 4.12. *Let $(X, \mathbb{X}, \mu)$ be a measure space, and let (f_n) be a sequence in $M^+(X, \mathbb{X})$. Then*

$$\int \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

Proof. Note that $\sum_{n=1}^{k} f_n$ is a monotone increasing sequence and apply the Monotone Convergence Theorem. □

Theorem 4.13. *Let $(X, \mathbb{X}, \mu)$ be a measure space, and let $f \in M^+(X, \mathbb{X})$. Let $\nu : \mathbb{X} \to \bar{\mathbb{R}}$ be defined by $\nu(A) = \int_A f d\mu$. Then ν is a measure.*

Proof. By definition it follows that $\nu(\emptyset) = 0$ and $\nu(A) \geq 0$ for all $A \in \mathbb{X}$. Now let $A_1, A_2, ... \in \mathbb{X}$ be disjoint, $A = \cup_{n=1}^{\infty} A_n$ and note that $f\chi_A = \sum_{n=1}^{\infty} f\chi_{A_n}$. We have that

$$\nu(A) = \int f\chi_A \mathrm{d}\mu = \int \left(\sum_{n=1}^{\infty} f\chi_{A_n} \right) \mathrm{d}\mu = \sum_{n=1}^{\infty} \left(\int f\chi A_n \mathrm{d}\mu \right) = \sum_{n=1}^{\infty} \nu(A_i)$$

□

Theorem 4.14. *Let $(X, \mathbb{X}, \mu)$ be a measure space, and let $f \in M^+(X, \mathbb{X})$. Then*

$$f(x) = 0 \ \mu - a.e. \ on \ X \quad \Longleftrightarrow \quad \int f d\mu = 0.$$

Proof. Note that $f(x) = 0$ $\mu -$ a.e. on X implies that the set

$$E = \{x \in X : f(x) > 0\}$$

satisfies $\mu(E) = 0$. Let $f_n = n\chi_E$. We have that $\int f_n \mathrm{d}\mu = 0$ and $\liminf_{n\to\infty} f_n \geq f$. Thus by Fatou's Lemma

$$\int f \mathrm{d}\mu \leq \int \liminf_{n\to\infty} f_n \mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n \mathrm{d}\mu = 0.$$

On the other hand if $\int f \mathrm{d}\mu = 0$ let $A = \{x \in X : f(x) = 0\}$ and $B = A^c$. Let $B_n = \{x \in X : f(x) \geq \frac{1}{n}\}$ and note that

$$0 = \int f \mathrm{d}\mu \geq \int_{B_n} f \mathrm{d}\mu \geq \frac{\mu(B_n)}{n}.$$

Thus $\mu(B_n) = 0$ for all n and $\mu(B) = \mu(\cup_{n=1}^{\infty} B_n) = 0$. □

Remark. *Theorem 4.14 allows us to extend the Monotone Convergence Theorem to the case when we have convergence μ-a.e.*

Theorem 4.15. ***(Monotone Convergence Theorem for convergence μ-a.e.)***
Let $(X, \mathbb{X}, \mu)$ be a measure space. Let (f_n) be a monotone increasing sequence of functions in $M^+(X, \mathbb{X})$, which converges μ-a.e. on X to a function f. Then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

Proof. Let $A \in \mathbb{X}$ be such that if $x \in A$ then $\lim_{n\to\infty} f_n(x) = f(x)$ and $B := A^c$ satisfies $\mu(B) = 0$. If we consider the functions $f_n\chi_A$ then they are a monotonically increasing sequence of function which converge to $f\chi_A$. So by the Monotone Convergence Theorem we have that

$$\lim_{n\to\infty} \int f_n \chi_A \mathrm{d}\mu = \int f \chi_A \mathrm{d}\mu.$$

Furthermore

$$\int f \mathrm{d}\mu = \int f\chi_A \mathrm{d}\mu + \int f\chi_B \mathrm{d}\mu \text{ and } \int f_n \mathrm{d}\mu = \int f_n\chi_A \mathrm{d}\mu + \int f_n\chi_B \mathrm{d}\mu$$

However $\chi_B(x) = 0$ for μ almost every x so by our previous result $\int f\chi_B \mathrm{d}\mu = \int f_n\chi_B \mathrm{d}\mu = 0$ and the result follows. □

Definition 4.16. *Let μ and ν be measures on $(X, \mathbb{X})$. Then ν is **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if $A \in \mathbb{X}$ and $\mu(A) = 0$ implies $\nu(A) = 0$.*

Theorem 4.17. *Let $f \in M^+(X, \mathbb{X})$, and define $\nu : \mathbb{X} \to \bar{\mathbb{R}}$ by $\nu(A) = \int_A f d\mu$. Then ν is absolutely continuous with respect to μ.*

Proof. Let $A \in \mathbb{X}$ satisfy $\mu(A) = 0$. We then have that

$$\nu(A) = \int f\chi_A \mathrm{d}\mu.$$

However since $\mu(A) = 0$, $f\chi_A = 0$ for μ almost all x so $\nu(A) = 0$. □

Theorem 4.18. *Let $X = [a, b] \subset \mathbb{R}$, and let $\mathbb{X}$ be the collection of Borel sets in X. Let $f : [a, b] \to [0, \infty)$ be continuous and supported in [a,b]. The*

$$\int f d\lambda = \int_a^b f(x)dx,$$

where the LHS denotes the Lebesgue integral and the RHS denotes the Riemann integral.

Remark. *The idea of the proof is to show that the result holds for step functions, and then use approximation by these.*

5 Integrable functions

Throughout this section we will let $(X, \mathbb{X}, \mu)$ be an arbitrary measure space. We want to extend the definition of the integral with respect to μ to all measurable functions, f. To do this we will compute the integrals of f^+ and f^- and take the difference. However this can cause problems since we have not defined $\infty - \infty$ so for the definition to make sense we insist that both these integrals are finite.

Definition 5.1. *An $\mathbb{R}$-valued, $\mathbb{X}$-measurable function $f : X \to \mathbb{R}$ is **integrable** (or **summable**) if*

$$\int f^+ d\mu < +\infty \quad \text{and} \quad \int f^- d\mu < +\infty,$$

where $f^+ := \max\{f, 0\}$ and $f^- := -\min\{f, 0\}$.
We then define

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu.$$

For $A \in \mathbb{X}$ we define

$$\int_A f d\mu := \int_A f^+ d\mu - \int_A f^- d\mu.$$

The collection of all $\mathbb{R}$-valued, $\mathbb{X}$-measurable functions on X is denoted by $L(X, \mathbb{X}, \mu)$.

Lemma 5.2. *Let $f, g \in M^+(X, \mathbb{X})$ with $\int f d\mu < +\infty$ and $\int g d\mu < +\infty$. Then $f - g \in L(X, \mathbb{X}, \mu)$ and*

$$\int (f - g)d\mu = \int f d\mu - \int g d\mu.$$

Proof. To proof this we let $h = (f - g)$ and note that $h^+ \leq f$ and $h^- \leq g$. So h is integrable and

$$\int h\mathrm{d}\mu = \int h^+\mathrm{d}\mu - \int h^-\mathrm{d}\mu.$$

We can then write

$$h^+ + g = h^- + f$$

and since the integrals of all of h^+, h^-, f, g are finite we can use Corollary 4.10 to deduce the result. □

We now introduce the notion of a charge which is the same as the notion of measures except we allow negative values for the charge of sets.

Definition 5.3. *Let $(X, \mathbb{X})$ be a measurable space. Then $\nu : \mathbb{X} \to \mathbb{R}$ is a* ***charge*** *if*

(i) $\nu(\emptyset) = 0,$ *and*

(ii) *If $A_1, A_2, \cdots$ is a sequence of disjoint sets in $\mathbb{X}$,*

then $\nu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n).$

Note that the sum $\sum_{n=1}^{\infty} \nu(A_n)$ must converge unconditionally whenever $A_1, A_2, \ldots \in \mathbb{X}$ is a sequence of disjoint sets, as the union is independent of order, and ν is $\mathbb{R}$-valued.

Theorem 5.4. *Let $f \in L(X, \mathbb{X}, \mu)$, and define $\nu : \mathbb{X} \to \mathbb{R}$ by $\nu(A) = \int_A f d\mu$. Then ν is a charge.*

Proof. If we let $\nu^+(A) = \int_A f^+\mathrm{d}\mu$ and $\nu^-(A) = \int_A f^-\mathrm{d}\mu$ then $\nu = \nu^+ - \nu^-$. ν^+ and ν^- are measures and thus charges. It is straightforward to now show that ν is a charge. □

Theorem 5.5. *If (X, X, μ) is a measure space and $f \in M(X, \mathbb{X})$ then*

$$f \in L(X, \mathbb{X}, \mu) \iff |f| \in L(X, \mathbb{X}, \mu).$$

Proof. Write $|f| = f^+ + f^-$. Thus if $|f| \in L(X, \mathbb{X}, \mu)$ then $f^-, f^+ \in M^+(X, \mathbb{X})$ and $\max\{\int f^+\mathrm{d}\mu, \int f^-\mathrm{d}\mu\} < \infty$. This means that $f \in L(X, \mathbb{X}, \mu)$. On the other hand if $f \in L(X, \mathbb{X}, \mu)$ then by definition $f^-, f^+ \in M^+(X, \mathbb{X})$ and $\max\{\int f^+\mathrm{d}\mu, \int f^-\mathrm{d}\mu\} < \infty$. This means that $|f| \in L(X, \mathbb{X}, \mu)$. □

We make use of this with the following simple corollary.

Corollary 5.6. *Let f be measurable, let $g \in L(X, \mathbb{X}, \mu)$, and suppose $|f| \leq |g|$. Then $f \in L(X, \mathbb{X}, \mu)$.*

Proof. It immediately follows that $\int |f| \mathrm{d}\mu \leq \int |g| \mathrm{d}\mu < \infty$ and thus $|f| \in L(X, \mathbb{X}, \mu)$. We can now apply Theorem 5.5 to deduce the result. $\square$

Theorem 5.7. *$L(X, \mathbb{X}, \mu)$ is a vector space over $\mathbb{R}$.*

Proof. We need to show that for all $\alpha \in \mathbb{R}$ and $f, g \in L(X, \mathbb{X}, \mu)$ that $\alpha f, f+g \in L(X, \mathbb{X}, \mu)$. Firstly $0 \cdot f = 0 \in L(X, \mathbb{X}, \mu)$. Now let $\alpha \neq 0$ and $f = f^+ - f^-$. If $\alpha > 0$ then we can immediately see that $\alpha f \in L(X, \mathbb{X}, \mu)$. If $\alpha < 0$ then $(\alpha f)^+ = |\alpha| f^-$ and $(\alpha f)^- = |\alpha| f^+$ and again it follows that $\alpha f \in L(X, \mathbb{X}, \mu)$.

For the second part we use the triangle inequality and the previous two results. We have

$$|f+g| \leq |f| + |g|.$$

By Theorem 5.5 and corollary 4.10 it follows that $|f| + |g| \in L(X, \mathbb{X}, \mu)$. Thus $|f+g| \in L(X, \mathbb{X}, \mu)$ by Corollary 5.6. (NB to see that $\int (f+g) \mathrm{d}\mu = \int f \mathrm{d}\mu + \int g \mathrm{d}\mu$ write $f + g = f^+ + g^+ - (f^- + g^-)$ and use Lemma 5.2.) $\square$

Theorem 5.8. ***(Lebesgue's Dominated Convergence Theorem)***
Let (f_n) be a sequence of functions in $M(X, \mathbb{X})$, which converges μ-a.e. on X to a measurable function $f : X \to \mathbb{R}$. Suppose there exists a function $g \in L(X, \mathbb{X}, \mu)$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then $f_n \in L(X, \mathbb{X}, \mu)$ for each $n \in \mathbb{N}$, $f \in L(X, \mathbb{X}, \mu)$ and

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu.$$

Proof. We will first prove the result in the case when convergence is for all of X. By corollary 5.6 it follows that each function $f_n \in L(X, \mathbb{X}, \mu)$, and since $|f| \leq g$ it also follows that $f \in L(X, \mathbb{X}, \mu)$. We now prove the result by using Fatou's lemma twice. Note that $f_n + g \geq 0$ for all n so we can apply Fatou. This gives

$$\int f + g \mathrm{d}\mu = \int (\liminf_{n \to \infty} f_n + g) \mathrm{d}\mu \leq \liminf_{n \to \infty} \left(\int f_n \mathrm{d}\mu + \int g \mathrm{d}\mu \right)$$

and so $\int f \mathrm{d}\mu \leq \liminf_{n \to \infty} \int f_n \mathrm{d}\mu$. On the other hand $g - f_n > 0$ and we can apply Fatou again to see that

$$\int -f \mathrm{d}\mu \leq \liminf_{n \to \infty} \int -f_n \mathrm{d}\mu.$$

Multiplying through by -1 gives $\int f \mathrm{d}\mu \geq \limsup_{n \to \infty} \int f \mathrm{d}\mu$. So putting this together gives $\int f \mathrm{d}\mu = \lim_{n \to \infty} \int f_n \mathrm{d}\mu$.

We now need to consider the case when the convergence is only for μ-a.e $x \in X$. In this case we know there exists $A \in \mathbb{X}$ where for all $x \in A$ we have that $\lim_{n\to\infty} f_n(x) = f(x)$ and where $\mu(A^c) = 0$. So

$$\int f \mathrm{d}\mu = \int f\chi_A \mathrm{d}\mu + \int f\chi_{A^c} \mathrm{d}\mu$$

and

$$\int f_n \mathrm{d}\mu = \int f_n\chi_A \mathrm{d}\mu + \int f_n\chi_{A^c} \mathrm{d}\mu.$$

We can deduce from above that $\lim_{n\to\infty} \int f_n\chi_A \mathrm{d}\mu = \int f\chi_A \mathrm{d}\mu$ and use Theorem 4.14 to show that the integrals over A^c are 0. The result follows. $\square$

Example. *Take* $(\mathbb{R}^+, \mathbb{B}, \lambda)$ *as our measure space Let* $f_n(x) = \frac{e^{-nx}}{\sqrt{x}}$ *and* $I_n = \int f_n d\lambda$. *We have that* $\lim_{n\to\infty} f_n(x) = 0$ *and* $|f_n(x)| \leq g(x)$ *where*

$$g(x) = \begin{cases} x^{-1/2} & \textit{if} \quad 0 < x < 1 \\ e^{-x} & \textit{if} \quad x \geq 1 \end{cases}$$

We can use that for non-negative continuous functions on closed bounded intervals that the Riemann integral and the Lebesgue integral agree to show that

$$\int g(x) d\lambda < \infty.$$

So we can apply Lebesgue's dominated convergence theorem to show that $\lim_{n\to\infty} I_n = 0$ *(without actually having to integrate* f_n*). We can easily see that the convergence is not uniform (for each* n, $\lim_{x\to 0} f_n(x) = \infty$*), so we could not show this so easily working just with the Riemann integral.*

Remark. *Lebesgue's Dominated Convergence Theorem allows us to deduce several results about the behaviour of limits inside integrals The following results follow from Lebesgue's Dominated Convergence Theorem. Throughout the rest of this section let* $f : X \times [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a function such that* $f(., t)$ *is* $\mathbb{X}$*-measurable for* $a \leq t \leq b$.

Theorem 5.9. ***(Convergence for a family of functions parametrized by t)***
Suppose $f(x, t_0) = \lim_{t\to t_0} f(x, t)$ *for all* $x \in X$, *and suppose there exists* $g \in L(X, \mathbb{X}, \mu)$ *such that* $|f(x, t)| \leq g(x)$ *for all* $t \in [a, b]$. *Then*

$$\int f(x, t_0) d\mu(x) = \lim_{t\to t_0} \int f(x, t) d\mu(x).$$

Proof. Take a sequence t_n, with elements in $[a, b]$, such that $\lim_{n\to\infty} t_n = t_0$. Let $f_n(x) = f(x, t_n)$ and apply the Lebesgue dominated convergence theorem. $\square$

Corollary 5.10. *Suppose $t \longrightarrow f(x,t)$ is continuous on $[a,b]$ for each $x \in X$, and suppose there exists $g \in L(X, \mathbb{X}, \mu)$ such that $|f(x,t)| \leq g(x)$ for all $t \in [a,b]$. Then F defined by*

$$F(t) = \int f(x,t) d\mu(x)$$

is continuous on $[a,b]$.

Proof. This is an immediate corollary of Theorem 5.9. □

Corollary 5.11. ***(Interchanging differentiation and integration)*** *Suppose the function $x \longrightarrow f(x,t_0)$ is integrable on X for some $t_0 \in [a,b] \subset \mathbb{R}$. Also suppose that $\frac{\partial f}{\partial t}$ exists on $X \times [a,b]$, and that there exists $g \in L(X, \mathbb{X}, \mu)$ such that $|\frac{\partial f}{\partial t}(x,t)| \leq g(x)$ for all $t \in [a,b]$. Then*

$$\frac{\partial F}{\partial t}(t) := \frac{\partial}{\partial t} \int f(x,t) d\mu(x) = \int \frac{\partial f}{\partial t}(x,t) d\mu(x).$$

Proof. First of all we show that $f(x,t)$ is integrable for all $t \in [a,b]$. Fix $x \in X$, $t \in [a,b]$ and let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence in [a,b] which satisfies $\lim_{n \to \infty} t_n = t$ then

$$\frac{\partial f}{\partial t}(x,t) = \lim_{n \to \infty} \frac{f(x,t) - f(x,t_n)}{t - t_n}$$

which is measurable. Moreover we can write

$$|f(x,t)| = |f(x,t_0) + f(x,t) - f(x,t_0)| \leq |f(x,t_0)| + |f(x,t) - f(x,t_0)|$$

and by the mean value theorem we have that there exists $y \in (a,b)$ such that

$$|f(x,t) - f(x,t_0)| = \left| \frac{\partial df}{\partial dt}(x,y) \right| |t - t_0|. \leq |g(x)|(b-a).$$

So $|f(x,t)| \leq |f(x,t_0)| + |g(x)|(b-a)$ and thus $f(x,t) \in L(X, \mathbb{X}, \mu)$. So $f(x,t)$ is integrable for all $t \in [a,b]$.

We now consider $f_n(x) = \frac{f(x,t_n) - f(x,t)}{t_n - t}$. We know that by the mean value theorem there exists $z \in [a,b]$ such that $f_n(x) = \frac{\partial F}{\partial t}(x,z)$ and so $|f_n(x)| \leq g(x)$ for all n. So we can apply the Lebesgue dominated convergence theorem to get

$$\int \frac{\partial f}{\partial t}(x,t) d\mu(x) = \lim_{n \to \infty} \int f_n \mathrm{d}\mu.$$

We then have that since $f(x,t)$ is integrable for all t that

$$\lim_{n \to \infty} \int f_n(x) \mathrm{d}\mu(x) = \lim_{n \to \infty} \frac{\int f(x,t_n) \mathrm{d}\mu - \int f(x,t) \mathrm{d}\mu}{t_n - t}.$$

This holds for any sequence $t_n \to t$, with elements, in $[a,b]$ and we can deduce that $\frac{\partial F}{\partial t}(t)$ exists and

$$\frac{\mathrm{d}F}{\partial t}(t) = \frac{\partial}{\partial t}\int f(x,t)d\mu(x) = \int \frac{\partial f}{\partial t}(x,t)d\mu(x).$$

□

Corollary 5.12. ***(Interchanging Lebesgue integration with Riemann integration)***
Suppose $t \longrightarrow f(x,t)$ is continuous on $[a,b]$ for each $x \in X$, and suppose there exists $g \in L(X, \mathbb{X}, \mu)$ such that $|f(x,t)| \leq g(x)$ for all $t \in [a,b]$. Then

$$\int_a^b F(t)dt := \int_a^b \int f(x,t)d\mu(x)dt = \int \int_a^b f(x,t)dtd\mu(x),$$

where the integral with respect to t is the Riemann integral and $F(t) = \int f(x,t)d\mu(x)$.

Proof. We'll make use of the fundamental Theorem of calculus to relate the Riemann integral to derivatives. First of all let $h(x,t) = \int_a^t f(x,s)\mathrm{d}s$. We know that $\frac{\partial h}{\partial t}(x,t) = f(x,t)$ and that for a fixed t the function $x \to h(x,t)$ is measurable. Moreover $|h(x,t)| \leq (b-a)g(x)$ so $h(x,t)$ is integrable with respect to μ and we can define $H(t) = \int h(x,t)\mathrm{d}\mu$. We now use Corollary 5.11 to see that

$$\frac{dH}{dt}(t) = \int \frac{\partial h}{\partial t}(x,t)\mathrm{d}\mu = \int f(x,t)\mathrm{d}\mu(x) = F(t).$$

So

$$\int_a^b F(t)\mathrm{dt} = H(b) - H(a) = \int h(x,b) - h(x,a)\mathrm{d}\mu(x).$$

We know that $h(x,a) = 0$ and so

$$\int_a^b F(t)\mathrm{dt} = \int h(x,b)\mathrm{d}\mu(x) = \int \int_a^b f(x,t)\mathrm{dtd}\mu(x).$$

□

6 L_p spaces

Throughout this section we will again let $(X, \mathbb{X}, \mu)$ be an arbitrary measure space.

Definition 6.1. *Let V be a vector space over $\mathbb{R}$. Then $N : V \to \mathbb{R}$ is a **norm** if for all $u, v \in V$ and for all $\alpha \in \mathbb{R}$*

$$\begin{aligned} &\textit{(i)} && 0 \leq N(v) < \infty, \\ &\textit{(ii)} && N(v) = 0 \iff v = 0, \\ &\textit{(iii)} && N(\alpha v) = |\alpha| N(v), \text{ and} \\ &\textit{(iv)} && N(u+v) \leq N(u) + N(v). \end{aligned}$$

Remark. *Normed spaces are metric spaces with the usual metric being $d(u, v) = N(u - v)$.*

Example. *Take the vector space over $\mathbb{R}$,*

$$C([0,1]) = \{f : [0,1] \to \mathbb{R} : \ f \textit{ is continuous}\}$$

and $N(f) = \int_0^1 |f(x)|dx$. This is a normed space. However consider

$$f_n(x) = \begin{cases} 0 & \textit{if} \quad x \leq \frac{1}{2} - \frac{1}{2n} \\ n\left(x - \frac{1}{2} + \frac{1}{2n}\right) & \textit{if} \quad x \in \left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\right) \\ 1 & \textit{if} \quad x > \frac{1}{2} + \frac{1}{2n} \end{cases}$$

This is a Cauchy sequence, however f_n can not converge to a continuous function in the metric induced by this norm. This means the norm is not complete.

Definition 6.2. *$N_0 : V \to \mathbb{R}$ is a **seminorm** if it satisfies (i), (iii) and (iv).*

Remark. *$N_\mu(f) = \int |f| d\mu$ is a seminorm on $L(X, \mathbb{X}, \mu)$. However it is not necessarily a norm since $\int |f - g| d\mu = 0$ does not always imply $f = g$. For example take $(\mathbb{R}, B, \lambda)$, $f(x) = 0$ for all $x \in \mathbb{R}$.*

$$f(x) = \begin{cases} 1 & \textit{if} \quad x \in \mathbb{Q} \\ 0 & \textit{if} \quad x \notin \mathbb{Q} \end{cases}$$

We have that $f \neq g$ but $\int |f - g| d\lambda = 0$.

Definition 6.3. *Two functions $f, g \in L(X, \mathbb{X}, \mu)$ are **μ-equivalent** if $f = g$ μ-almost everywhere.*

Remark. *The relation μ-equivalent is an equivalence relation.*

Definition 6.4. *The equivalence class containing an element $f \in L(X, \mathbb{X}, \mu)$ is denoted by $[f]$. The **Lebesgue space** $L_1(X, \mathbb{X}, \mu)$ consists of all μ-equivalence classes of functions in $L(X, \mathbb{X}, \mu)$. For $[f] \in L_1(X, \mathbb{X}, \mu)$ we define*

$$\|[f]\|_1 := \int |f| d\mu,$$

where f is any representative of $[f]$.

Theorem 6.5. *$L_1(X, \mathbb{X}, \mu)$ with $\|\|_1$ is a normed vector space over $\mathbb{R}$.*

Proof. The quickest way to see this is a vector space is to consider the subspace

$$N = \{f \in L(X, \mathbb{X}, \mu) : f(x) = 0 \text{ for } \mu \text{ a.e. } x\}$$

and observe that $L_1(X, \mathbb{X}, \mu) = L(X, \mathbb{X}, \mu)/N$, i.e. that $L_1(X, \mathbb{X}, \mu)$ is a quotient space. If you are not familiar with quotient spaces then just check that you get a vector space with the operations $[f] + [g] = [f + g]$ and $\alpha[f] = [\alpha f]$.

To see that $\|f\|$ is indeed a norm we first have to check it is well defined. Let $f, g \in L(X, \mathbb{X}, \mu)$ with $[g] = [f]$ then $f = g$ for μ almost every x and so $\|[f]\| = \int |f| \mathrm{d}\mu = \int |g| \mathrm{d}\mu = \|[g]\|$. We can then see that for $[f], [g] \in L_1(X, \mathbb{X}, \mu)$

1. $\|[f]\| = \int |f| \mathrm{d}\mu \geq 0$.
2. $\|[0]\| = 0$ and if $0 = \|[f]\| = \int |f| \mathrm{d}\mu$ then $f(x) = 0$ for μ almost every x and $[f] = [0]$.
3. For $\alpha \in \mathbb{R}$, $\|[\alpha f]\| = \int |\alpha f| \mathrm{d}\mu = |\alpha| \int |f| \mathrm{d}\mu = \alpha \|[f]\|$.
4. $\|[f + g]\| = \int |f + g| \mathrm{d}\mu \leq \int |f| \mathrm{d}\mu + \int |g| \mathrm{d}\mu = \|[f]\| + \|[g]\|$.

□

Remark. *The elements of $L_1(X, \mathbb{X}, \mu)$ are equivalence classes of functions in $L(X, \mathbb{X}, \mu)$, but these classes are represented by functions which are equal μ-a.e.. We will usually write f rather than $[f]$.*

Definition 6.6. *For $1 \leq p < \infty$, $L_p(X, \mathbb{X}, \mu)$ consists of all μ-equivalence classes of $\mathbb{X}$-measurable, $\mathbb{R}$-valued functions f for which*

$$\int |f|^p d\mu < \infty.$$

We define

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{\frac{1}{p}}.$$

When there is no risk of confusion we will write $L_p(X)$ or L_p instead of $L_p(X, \mathbb{X}, \mu)$.

We'll now prove several inequalities, the final one of which will show the triangle inequality for L_p.

Lemma 6.7. *[Youngs' inequality] If $A, B, p, q \in \mathbb{R}^+$ where $p, q \geq 1$ and $1 = \frac{1}{p} + \frac{1}{q}$ then*

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Proof. Fix A, B as positive real numbers and $p, q \geq 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. We also let $a = A^p, b = B^q$ and $0 < \alpha = \frac{1}{p} < 1$. We define a function $\phi : (0, \infty) \to \mathbb{R}$ by

$$\phi(t) = \alpha t - t^\alpha.$$

We can see that $\phi'(t) < 0$ for $t < 1$, $\phi'(1) = 0$ and $\phi'(t) > 0$ for $t \geq 1$. Thus by the mean value theorem ϕ has a unique minimum value at $t = 1$. Thus for all positive t we have that

$$\alpha t - t^\alpha \geq \alpha - 1.$$

We now substitute in $t = \frac{a}{b}$ which gives that

$$\frac{a^\alpha}{b^\alpha} \leq (1 - \alpha) + \alpha \frac{a}{b}$$

and multiplying through by b gives

$$a^\alpha b^{1-\alpha} \leq (1 - \alpha)b + \alpha a.$$

Recall that $\frac{1}{q} = 1 - \frac{1}{p} = 1 - \alpha$ and substitute this in to get

$$a^{1/p} b^{1/q} \leq \frac{b}{q} + \frac{a}{p}.$$

We can now use that $a = A^p$ and $b = B^q$ to get

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

□

Theorem 6.8. *[Hölder's Inequality]*
Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and suppose $f \in L_p(X)$ and $g \in L_q(X)$. Then $fg \in L_1(X)$, and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. Let $f \in L_p$ and $g \in L_q$. If either $|f|_p = 0$ or $|g|_q = 0$ then $fg = 0$ μ almost everywhere and the inequality follows. So we assume $|f|_p \neq 0$ and $|g|_q \neq 0$. We now let $x \in \mathbb{X}$, $A = \frac{|f(x)|}{\|f\|_p}$ and $B = \frac{|g(x)|}{\|g\|_q}$. We can now apply Young's inequality for a and b to get that

$$\frac{|f(x)||g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|q\|_q^q}.$$

Since f^p and g^q are integrable it follows that fg is integrable and so $fg \in L^1(X)$. If we integrate both sides with respect to μ we obtain that

$$\frac{1}{\|f\|_p \|g\|_q} \int |f(x)g(x)| \mathrm{d}\mu(x) \leq \frac{1}{p} + \frac{1}{q} = 1$$

and we can rearrange to get

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

$\square$

Corollary 6.9. ***(Cauchy-Schwartz Inequality)***
Suppose $f, g \in L_2(X)$. Then $fg \in L_1(X)$, and

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2.$$

Proof. This is Hölder's inequality with $p = q = 2$. $\square$

Corollary 6.10. *Let $p, q, r > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and suppose $f \in L_p(X)$ and $g \in L_q(X)$. Then $fg \in L_r(X)$, and*

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Proof. We use that $|f|^r \in L_{p/r}$, $|g|^r \in L_{q/r}$ and that $\frac{r}{p} + \frac{r}{q} = 1$ to allow us to apply Hölder's inequality. $\square$

Corollary 6.11. *Let $1 \leq p < r < q$, and suppose $f \in L_p(X) \cap L_q(X)$. Then $f \in L_r(X)$ with*

$$\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha},$$

where $\alpha \in (0, 1)$ is such that $\frac{\alpha}{p} + \frac{1-\alpha}{q} = \frac{1}{r}$.

Proof. We use that $|f|^\alpha \in L_{p/\alpha}$ and $|f|^{1-\alpha} \in L_{q/(1-\alpha)}$ to apply the previous inequality. $\square$

Remark. *The next result is the triangle inequality for the L_p-norms.*

Theorem 6.12. ***(Minkowski's Inequality)***
Suppose $f, g \in L_p(X)$ for some $p \geq 1$. Then $f + g \in L_p(X)$, and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. We have already proved this for $p = 1$ so we suppose $p > 1$ and let q satisfy that $\frac{1}{p} + \frac{1}{q} = 1$ (so $p = (p-1)q$). To avoid trivialities we assume $\|f + g\|_p > 0$ and we also know straight away that $f + g$ is measurable. We can write for all x that

$$|f + g(x)|^p \leq (2 \max\{|f(x)|, |g(x)|\})^p \leq 2^p(|f(x)|^p + |g(x)|^p)$$

and we can conclude that by Corollary 5.6 and Theorem 5.7 that $|f + g|^p$ is integrable and so $f + g \in L_p$.

We now write

$$|(f+g)(x)|^p = |(f+g)(x)||(f+g)(x)|^{p-1}| \leq |f(x)||(f+g)(x)|^{p-1} + |g(x)||(f+g)(x)|^{p-1}.$$

Since $p = (p-1)q$ we know that $(f+g)^{p-1} \in L_q$ and so we can apply Holder's inequality to get

$$\|f|f+g|^{p-1}\|_1 \leq \|f\|_p \|(f+g)^{p-1}\|_q = \|f\|_p \|f+g\|_p^{p/q}.$$

and

$$\|g|f+g|^{p-1}\|_1 \leq \|g\|_p \|f+g\|_p^{p/q}.$$

So

$$\begin{aligned} \|f+g\|_p^p &\leq \|f\|_p \|f+g\|_p^{p/q} + \|g\|_p \|f+g\|_p^{p/q} \\ &\leq (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/q}. \end{aligned}$$

Dividing through by $\|f+g\|_p^{p/q} > 0$ and using that $p - p/q = p(1-1/q) = 1$ gives that

$$\|f+g\|_p^{p-p/q} = \|f+g\|_p \leq (\|f\|_p + \|g\|_p).$$

□

Definition 6.13. *A sequence (f_n) in L_p is a* ***Cauchy sequence in*** L_p *if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } m,n \geq N \implies \|f_n - f_m\|_p < \epsilon.$$

A sequence (f_n) in L_p ***converges to*** f ***in*** L_p *if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies \|f_n - f\|_p < \epsilon.$$

A normed space is ***complete*** *if every Cauchy sequence converges in the metric induced by the norm.*
A ***Banach space*** *is a complete normed space.*

Theorem 6.14. *Let $1 \leq p < \infty$. Then L_p is a Banach space.*

Proof. Using Minkowski's inequality we can see that L_p is a normed space so we just need to show the metric induced by the norm is complete. Let $\{f_n\}$ be a Cauchy sequence in L_p. Thus for all $\epsilon > 0$ there exists N such that for all $n, m \geq N$ $\|f_n - f_m\|_p < \epsilon$. We can pick a subsequence g_k of f_n such that for all k we have that $\|g_{k+1} - g_k\|_p \leq 2^{-k}$. Let

$$g(x) = |g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$$

which is measurable. Thus by Fatou's Lemma

$$\int g^p \mathrm{d}\mu \leq \liminf_{n\to\infty} \int \left(|g_1(x)| + \sum_{k=1}^{n} |g_{k+1}(x) - g_k(x)| \right)^p \mathrm{d}\mu.$$

However by Minkowski's inequality

$$\left(\int\left(|g_1(x)|+\sum_{k=1}^{n}|g_{k+1}(x)-g_k(x)|\right)^p \mathrm{d}\mu\right)^{1/p} \leq \|g_1\|_p+\sum_{k=1}^{n}\|g_{k+1}(x)-g_k(x)\|_p \leq \|g_1\|_p+1$$

which means that $\int g^p \mathrm{d}\mu < \infty$.

Thus if we let

$$E = \{x : g(x) < \infty\}$$

then $\mu(E^c) = 0$ and $g\chi_E \in L_p$. If we let

$$f(x) := \begin{cases} g_1(x) + \sum_{k=1}^{\infty}(g_{k+1}(x) - g_k(x)) & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \notin E \end{cases}$$

then since $|f(x)|^p \leq g(x)^p$ we have that $f \in L_p$. Furthermore we have that for μ almost all x $f(x) = \lim_{k\to\infty} g_k(x)$ and

$$|f - g_k(x)| \leq \begin{cases} g(x) + \left|g_1(x) + \sum_{n=1}^{k-1}(g_{n+1}(x) - g_n(x))\right| \leq 2g(x) & \text{if} \quad x \in E \\ \left|g_1(x) + \sum_{n=1}^{k-1} g_{n+1}(x) - g_n(x)\right| \leq g(x) & \text{if} \quad x \notin E. \end{cases}$$

Thus by the Lebesgue Dominated Convergence Theorem $\lim_{k\to\infty}\int|f - g_k(x)|^p = 0$ and so $\lim_{k\to\infty}\|f - g_k\|_p = 0$.

So we have shown convergence of f_n to f in L_p along a subsequence. Now let $\epsilon > 0$ choose M such that for $n, m \geq N$ that $\|f_n - f_m\|_p \leq \epsilon/2$ and choose k such that $\|f - g_k\|_p \leq \epsilon/2$ and where k is sufficiently large (i.e $g_k = f_l$ where $l \geq N$). Now for $n \geq N$ we can use Minkowski to get that

$$\|f_n - f\|_p = \|f_n - g_k + g_k - f\|_p \leq \|f_n - g_k\|_p + \|g_k - f\|_p \leq \epsilon.$$

□

Definition 6.15. *$L_\infty(X, \mathbb{X}, \mu)$ is the set of all μ-equivalence classes of $\mathbb{X}$-measurable, $\mathbb{R}$-valued functions which are bounded μ-almost everywhere. We define*

$$\|f\|_\infty := \inf\{M \in \mathbb{R} : |f| \leq M \ \mu - a.e.\}.$$

This is called the essential supremum. When there is no risk of confusion we will write $L_\infty(X)$ or L_∞ instead of $L_\infty(X, \mathbb{X}, \mu)$.

Theorem 6.16. *$\|\|_\infty$ is a norm on $L_\infty(X, \mathbb{X}, \mu)$.*

Proof. Exercise □

Theorem 6.17. *L_∞ is a Banach space.*

Proof. Again we let f_n be a Cauchy sequence in L_∞. Let $E \in X$ satisfy that $\mu(E^c) = 0$ and $x \in E$ if $|f_n(x) \leq \|f_n\|_\infty$ and $|f_n - f_m| \leq \|f_n - f_m\|_\infty$ for all $n, m \in \mathbb{N}$. For all $x \in E$ we have that $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$ so $\lim_{n\to\infty} f_n(x)$ exists. Set

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \notin E \end{cases}$$

For any $\epsilon > 0$ we can find N such that for all $n, m > N$ we have that $\|f_n - f_m\|_\infty < \epsilon$. Moreover for any $x \in E$ there exists $k > N$ such that $|f(x) - f_k(x)| < \epsilon$. Thus for $x \in E$ and $n \geq N$

$$|f(x) - f_n(x)| \leq |f(x) - f_k(x)| + |f_k(x) - f(x)| \leq 2\epsilon.$$

So since $\mu(E^c) = 0$ we have that $f \in L_\infty$ and $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$. □

7 Modes of Convergence

As usual we let $(X, \mathbb{X}, \mu)$ be any measure space.

Remark. *We are already familiar with uniform convergence, pointwise convergence, convergence μ-a.e. and convergence in L_p $(1 \leq p \leq \infty)$. Now we will introduce several new modes of convergence, and study the links between the different types of convergence.*

Theorem 7.1. *Suppose $\mu(X) < \infty$. Let (f_n) be a sequence in L_p, $1 \leq p < \infty$, which converges uniformly to a function f. Then $f \in L_p$ and $f_n \to f$ in L_p.*

Proof. Let $\epsilon > 0$ and choose N such that for all $n \geq N$ we have that $\sup_{x \in X} |f_n(x) - f(x)| < \epsilon$. We then have that

$$\int |f_n - f|^p \mathrm{d}\mu \leq \int \epsilon^p \mathrm{d}\mu = \epsilon^p \mu(X)$$

and the result follows. □

Example. *Uniform convergence does not in general imply convergence in L_p. For example take $(\mathbb{R}, \mathbb{B}, \lambda)$ as our measure space and define $f_n(x) = n^{-1/p}\chi_{[0,n]}$. We have that*

$$\sup_{x \in \mathbb{R}} |f_n(x) - 0| = n^{-1/p}$$

and thus f_n converges uniformly to 0. However for all n

$$\int |f_n|^p \, d\lambda = n^{-1} n = 1$$

so f_n does not converge to 0 in L_p.

Remark. *Let (f_n) be a sequence in L_∞, which converges uniformly to a function f. Then $f \in L_\infty$, and $f_n \to f$ in L_∞.*

Theorem 7.2. *Let (f_n) be a sequence in L_p, $1 \leq p < \infty$, which converges μ-a.e. to a measurable function f. Suppose there exists $g \in L_p$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then $f \in L_p$, and $f_n \to f$ in L_p.*

Proof. This is an application of the Lebesgue Dominated Convergence Theorem. See exercise sheet 5 question 8. □

Corollary 7.3. *Suppose $\mu(X) < \infty$. Let (f_n) be a sequence in L_p, $1 \leq p < \infty$, which converges μ-a.e. to a function f. Suppose there exists $K \in \mathbb{R}$ such that $|f_n| \leq K$ for all $n \in \mathbb{N}$. Then $f \in L_p$ and $f_n \to f$ in L_p.*

Remark. *Convergence in L_p does not imply pointwise convergence. For example for $(\mathbb{R}, \mathbb{B}, \lambda)$ take $f_n(x) = 1$ if $x \in \mathbb{Q}$ and $f_n(x) = \frac{1}{n}$ if $x \notin \mathbb{Q}$. We then have $\int |f_n|^p d\lambda = \frac{1}{n^p}$ and so $\lim_{n\to\infty} ||f_n|||_p = 0$ but f_n does not converge to f everywhere.*

Definition 7.4. *A sequence (f_n) of measurable functions* ***converges to f in measure*** *if for any $\alpha > 0$*

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) = 0.$$

A sequence (f_n) of measurable functions is ***Cauchy in measure*** *if for any $\alpha, \epsilon > 0$, there exists $M \in \mathbb{R}$ such that*

$$m, n \geq M \implies \mu(\{x \in X : |f_n(x) - f_m(x)| \geq \alpha\}) < \epsilon.$$

Remark. *Uniform convergence implies convergence in measure.*

Remark. *Convergence in L_p implies convergence in measure. To see this fix $f_n \in L_p$ such that f_n converges to f in L_p. Let $a > 0$ and let $A_n = \{x : |f_n(x) - f(x)| > a\}$. Thus*

$$\int |f_n - f|^p \geq \mu(A_n) a^p$$

and since $\lim_{n\to\infty} \int |f_n - f|^p d\mu = 0$ we must have that $\lim_{n\to\infty} \mu(A_n) = 0$.

Remark. *Pointwise convergence need not imply convergence in measure. Again take $(\mathbb{R}, \mathbb{B}, \lambda)$ and let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n = \chi_{[n,\infty]}$. We then have that $\lim_{n\to\infty} f_n(x) = 0$ but*

$$\lambda\{x : |f_n(x) \geq 1\} = \infty \text{ for all } n \in \mathbb{N}.$$

Theorem 7.5. *Let (f_n) be a sequence of $\mathbb{R}$-valued, measurable functions, which is Cauchy in measure. Then there is a subsequence of f_n, which converges μ-a.e. and in measure to a $\mathbb{R}$-valued, measurable function f.*

Proof. Since f_n is Cauchy in measure if we take $k \in \mathbb{N}$ then there exists $N(k)$ such that for all $n, m \geq N(k)$ we have that

$$\mu(\{x : |f_n(x) - f_m(x)| \geq 2^{-k}\} \leq 2^{-k}$$

and we can take $N(1) < N(2) < \cdots$. Thus we can choose a subsequence $f_{n_k} = g_k$ such that $n_k \geq N(k)$ and thus

$$\mu(\{x : |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\} \leq 2^{-k}.$$

Let $E_k = \{x : |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\}$, $F_k = \cup_{j=k}^{\infty} E_k$ and note that $\mu(F_k) \leq 2^{-k+1}$. If $x \notin F_k$ and $i > j > m > k$ then we have that

$$|g_i(x) - g_j(x)| = \left| \sum_{z=i}^{j-1} g_z(x) - g_{z+1}(x) \right| \leq \sum_{z=m}^{\infty} |g_z(x) - g_{z+1}(x)| \leq 2^{-m+1}.$$

Thus $g_n(x)$ is a Cauchy sequence in $\mathbb{R}$ and so convergent. Now let $F = \cap_{k=1}^{\infty} F_k$ and note that $\mu(F) = \lim_{k \to \infty} \mu(F_k) = 0$ and that for $x \in F^c$ we will have that $g_n(x)$ is convergent.

Let $f(x) = \lim_{n \to \infty} g_n(x)$ if $x \notin F$ and set $f(x) = 0$ if $x \in F$. We have that g_n converges μ a.e. to f so the first part is done. For the second part let $\epsilon > 0$ and let $a > 0$. Fix k such that $2^{-k+1} < \min\{a, \epsilon\}$. Thus

$$\{x : |g_k(x) - f(x)| \geq a\} \subset \{x : |g_k(x) - f(x)| \geq 2^{-k+1}\} \subset F_k$$

So

$$\mu(\{x : |g_k(x) - f(x)| \geq a\}) \leq \mu(F_k) \leq 2^{-k+1} < \epsilon.$$

□

Corollary 7.6. *Let (f_n) be a sequence of $\mathbb{R}$-valued, measurable functions, which is Cauchy in measure. Then (f_n) converges in measure to a $\mathbb{R}$-valued, measurable function f, which is uniquely determined μ-a.e.*

Proof. We know from Theorem 7.5 that there exists a real measurable function f and a subsequence g_k of f_n such that g_k converges in measure to f. Let $a, \epsilon > 0$ we can choose N such that for all $n, m > N$

$$\mu(\{x : |f_n(x) - f_m(x)| \geq a/2\}) \leq \epsilon/2$$

and k such that $g_k = f_l$ with $l > N$ and

$$\mu(\{x : |g_k(x) - f(x)| \geq a/2\}) \leq \epsilon/2.$$

Thus by considering the union of these two sets and using the triangle inequality we have that for all $n > N$

$$\mu(\{x : |f(x) - f_n(x)| \geq a\}) \leq \epsilon.$$

Thus f_n converges to f in measure. We now need to show f is uniquely determined μ a.e.. So suppose f_n converges to some real valued function g in measure. Thus for any $a, \epsilon > 0$ we can find N such that for any $n \geq N$ the sets

$$E_1 = \{x : |f - f_n(x)| \leq a/2\} \text{ and } E_2 = \{x : |g - f_n(x)| \leq a/2\}$$

satisfy $\mu((E_1 \cap E_2)^c) \leq \epsilon$. By the triangle inequality it follows that for any $x \in (E_1 \cap E_2)$ we have $|f(x) - g(x)| \leq a$. Thus for any $n \in N$ we have that $\mu(\{x : |f(x) - g(x)| > 1/n\} = 0$ and so $\mu(\{x : f(x) \neq g(x)\}) = 0$. □

Remark. *Convergence in measure does not imply convergence in L_p.*

Theorem 7.7. *Let (f_n) be a sequence of functions in L_p, which converges to f in measure. Suppose there exists $g \in L_p$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then $f \in L_p$, and $f_n \to f$ in L_p.*

Proof. We prove this by contradiction and by applying Theorem 7.2. Let $\epsilon > 0$ and suppose there exists a subsequence g_k of f_n such that $\|g_k - f\|_p > \epsilon$ for all $k \in \mathbb{N}$. We know that g_k converges to f in measure and is thus Cauchy in measure. So by Theorem 7.5 and Corollary 7.5 it follows that there exists a subsequence of h_l which converges almost everywhere to f. Thus by Theorem 7.2 it follows that $f \in L_p$ and $f_n \to f$ in L_p. □

Definition 7.8. *A sequence (f_n) of measurable functions is **almost uniformly convergent to** $f \in M$ if for any $\delta > 0$, there exists $E_\delta \in \mathbb{X}$ with $\mu(E_\delta) < \delta$, such that (f_n) is uniformly convergent to f on $X \setminus E_\delta$.*
*A sequence (f_n) of measurable functions is **almost uniformly Cauchy** if for any $\delta > 0$, there exists $E_\delta \in \mathbb{X}$ with $\mu(E_\delta) < \delta$, such that (f_n) is uniformly convergent on $X \setminus E_\delta$.*

Remark. *Almost uniform convergence implies convergence almost everywhere.*

Theorem 7.9. *Let (f_n) be almost uniformly Cauchy. Then (f_n) is almost uniformly convergent to a measurable function f.*

Proof. For any $k \in \mathbb{N}$ we can find a set $E_k \in \mathbb{X}$ for which f_n converges uniformly to some function g on E_k^c and $\mu(E_k) < \frac{1}{2^k}$. We let $F_k = \cup_{n=k}^{\infty} E_n$ and note that since $F_k^c \subset E_k^c$ we know that f_n will converge uniformly to g on F_k^c and $\mu(F_k) < \frac{1}{2^{k-1}}$. Thus we can define a measurable function by

$$g_k(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if} \quad x \in F_k^c \\ 0 & \text{if} \quad x \in F_k \end{cases}$$

Moreover if we let $F = \cap_{k=1}^{\infty} F_k$ then $\mu(F) = 0$. Thus if $x \notin F$ then for k sufficiently large $g_k(x) = \lim_{n\to\infty} f_n(x)$. So $\lim_{k\to\infty} g_k(x) = \lim_{n\to\infty} f_n(x)$

if $x \in F^c$ and $\lim_{k\to\infty} g_k(x) = 0$ if $x \in F$. So $g = \lim_{k\to\infty} g_k$ is a measurable function for which $f_n(x)$ converges to g for μ a.e. x. We need to show this convergence is almost uniform.

Fix $\delta > 0$ and choose k such that $\frac{1}{2^{k-1}} < \delta$. We take the set F_k and note that $\mu(F_k) < \epsilon$ and $f_n(x)$ converges uniformly to $g_k(x) = g(x)$ on F_k^c. The proof is complete.

□

Theorem 7.10. *Let (f_n) be almost uniformly convergent to f. Then (f_n) converges to f in measure.*

Proof. Fix $a, \epsilon > 0$. We can find a set $A_\epsilon \in \mathbb{X}$ such that f_n is uniformly convergent to f on A_ϵ^c and $\mu(A_\epsilon) < \epsilon$. Thus we can choose N such that for $n \geq N$

$$\{x \in X : |f_n(x) - f(x)| \geq a\} \subset A_\epsilon$$

and thus

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq a\}) < \epsilon.$$

□

Theorem 7.11. *Let (f_n) converge to f in measure. Then there is a subsequence that converges almost uniformly to f.*

Proof. This can be deduced as an exercise from the proof of Theorem 7.5(Note that converging in measure implies Cauchy in measure). □

Corollary 7.12. *Let (f_n) converge to f in L_p. Then there is a subsequence that converges almost uniformly to f.*

Proof. We've seen that L_p convergence implies convergence in measure and so this follows from Theorem 7.11. □

Remark. *Convergence almost everywhere can be replaced by convergence in measure in the statements of the Monotone Convergence Theorem and Lebesgue's Dominated Convergence Theorem. Similarly, a version of Fatou's Lemma holds for convergence in measure. (See Exercise Sheet 6 for details.)*

Remark. *In general convergence almost everywhere does not imply almost uniform convergence. The next result shows that this holds if $\mu(X) < \infty$.*

Theorem 7.13. ***(Egorov's Theorem)***
Suppose $\mu(X) < \infty$. Let (f_n) be a sequence of measurable, $\mathbb{R}$-valued functions which converges almost everywhere to a measurable, $\mathbb{R}$-valued function f. Then (f_n) converges almost uniformly to f.

Proof. Fix $m, k \in \mathbb{N}$ and let

$$E_{m,k} = \{x \in X : |f_k(x) - f(x)| \geq m^{-1}\}.$$

Let $F_{m,k} = \cup_{n=k}^{\infty} E_{m,n}$ and $F_m = \cap_{k=1}^{\infty} F_{m,k}$. We know that $\mu(F_m) = 0$ and since $\mu(X) < \infty$ we have that $\lim_{k\to\infty} \mu(F_{m,k}) = 0$. Now let $\delta > 0$ and choose k_m such that $\mu(F_{m,k_m}) < \delta/2^k$. Let $F = \cup_{m=1}^{\infty} F_{m,k_m}$ and observe that $\mu(F) < \delta$. Moreover if we let $\epsilon > 0$ and choose $m > \frac{1}{\epsilon}$ then for all $n \geq k_m$ and $x \in F^c$ we know that $|f_n(x) - f(x)| \leq m^{-1} < \epsilon$. Thus f_n converges uniformly to f on F^c. □

8 Radon-Nikodým Theorem

Throughout this section we will let $(X, \mathbb{X})$ be a measurable space.
Recall that $\nu : \mathbb{X} \to \mathbb{R}$ is a **charge** on $(X, \mathbb{X})$ if

(i) $\nu(\emptyset) = 0$, and

(ii) If $A_1, A_2, \cdots$ is a sequence of disjoint sets in $\mathbb{X}$,

then $\nu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$.

Definition 8.1. *Let $\nu : \mathbb{X} \to \mathbb{R}$ be a* ***charge****.*

- *A set $P \in \mathbb{X}$ is* ***positive*** *with respect to ν if $\nu(A \cap P) \geq 0$, $\forall A \in \mathbb{X}$.*
- *A set $N \in \mathbb{X}$ is* ***negative*** *with respect to ν if $\nu(A \cap N) \leq 0$, $\forall A \in \mathbb{X}$.*
- *A set $M \in \mathbb{X}$ is* ***a null set*** *with respect to ν if $\nu(A \cap M) = 0$, $\forall A \in \mathbb{X}$.*

Lemma 8.2. *(i) A measurable subset of a positive set is positive.*
(ii) The union of two positive sets is positive.

Theorem 8.3. ***(The Hahn Decomposition Theorem)***
Let $\nu : \mathbb{X} \to \mathbb{R}$ be a ***charge****. Then there exists positive $P \in \mathbb{X}$ and negative $N \in \mathbb{X}$ such that $P \cup N = X$ and $P \cap N = \emptyset$.*

Proof. We let $\mathcal{P} \subset \mathbb{X}$ denote the set of all positive measurable subsets of X. Since $\emptyset \in \mathcal{P}$ we know that $\mathcal{P}$ is nonempty and we can define

$$\alpha = \sup\{\nu(A) : A \in \mathcal{P}\}.$$

Thus we can find a sequence of sets $A_n \in \mathcal{P}$ where $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \nu(A_n) = \alpha$. We now let $P = \cup_{n=1}^{\infty} A_n$ and note that $\nu(P) = \lim_{n\to\infty} \nu(A_n) = \alpha$ which means that $\alpha < \infty$. To see that P is positive we let $E \in \mathbb{X}$ and calculate

$$\nu(E \cap P) = \nu(E \cap \cup_{n=1}^{\infty} A_n) = \nu(\cup_{n=1}^{\infty} E \cap A_n) = \lim_{n\to\infty} \nu(E \cap A_n) \geq 0.$$

We now let $N = X\backslash P$ and show that this is a negative set by contradiction. Suppose N is not negative which means we can find a set $E \in \mathbb{X}$ such that $E \subset N$ and $\nu(E) > 0$. E cannot be a positive set since $\nu(E \cup P) = \nu(E) + \nu(P) > \alpha$. Therefore we can let

$$n_1 = \inf\{n \in \mathbb{N} : \text{ there exists } A \subset E \text{ with } \nu(A) \leq -n^{-1}\}$$

and choose $E_1 \subset E$ such that $\nu(E_1) \leq -n_1^{-1}$. Thus $\nu(E\backslash E_1) > \nu(E)+n_1^{-1} > 0$. We now proceed inductively by letting

$$n_k = \inf\{n \in \mathbb{N} : \text{ there exists } A \subset E\backslash \left(\cup_{m=1}^{k-1} E_m\right) \text{ with } \nu(A) \leq -n^{-1}\}$$

and then choosing $E_k \subset E\backslash \left(\cup_{m=1}^{k-1} E_m\right)$ such that $\nu(E_1) \leq -n_k^{-1}$.

We will now let $F = \cup_{k=1}^{\infty} E_k$ and note that since the sets E_k are disjoint

$$\nu(F) = \nu\left(\cup_{k=1}^{\infty} E_k\right) \leq -\sum_{k=1}^{\infty} n_k^{-1} < 0.$$

Since $\nu(F) > -\infty$ this implies that $\lim_{k\to\infty} n_k^{-1} = 0$. We know that $\nu(E\backslash F) = \nu(E) - \nu(F) > 0$ and since $\nu(E\backslash F) \cup P) = \nu(E\backslash F) + \nu(P) > \alpha$ $E\backslash F$ cannot be positive. Therefore we can find A such that $A \subset E\backslash F$ and $\nu(A) < 0$. Let k be such that $\nu(A) \leq -(n_k - 1)^{-1}$ and note that $A \subset E\backslash \left(\cup_{m=1}^{k-1} E_m\right)$ with $\nu(A) \leq -(n_k - 1)^{-1}$ which contradicts the definition of n_k. Thus N is negative and we have that $X = P \cup N$ where P is positive, N is negative and $P \cap N = \emptyset$. □

We now show that the Hahn decomposition is essentially unique.

Lemma 8.4. *Let P_1, N_1 and P_2, N_2 be Hahn Decompositions for ν. Then for any $A \in \mathbb{X}$*

$$\nu(A \cap P_1) = \nu(A \cap P_2), \text{ and}$$
$$\nu(A \cap N_1) = \nu(A \cap N_2).$$

Proof. We have that

$$\nu(A \cap (P_1 \cup P_2)) = \nu(A \cap P_1) + \nu(A \cap (P_2\backslash P_1)) = \nu(A \cap P_2) + \nu(A \cap (P_1\backslash P_2)).$$

However $P_2\backslash P_1 \subset N_1$ and $P_1\backslash P_2 \subset N_2$ and so $\nu(A \cap (P_1\backslash P_2)) = \nu(A \cap (P_2\backslash P_1)) = 0$. Therefore

$$\nu(A \cap P_1) = \nu(A \cap (P_1 \cup P_2)) = \nu(A \cap P_2).$$

□

Definition 8.5. *Let $\nu : \mathbb{X} \to \mathbb{R}$ be a charge and let P, N be the Hahn Decomposition for ν. The **positive variation** of ν is the finite measure ν^+ on $\mathbb{X}$ defined by $\nu^+(A) = \nu(A \cap P)$. The **negative variation** of ν is the finite measure ν^- defined by $\nu^-(A) = -\nu(A \cap N)$. The **total variation** of ν is the finite measure $|\nu| = \nu^+ + \nu^-$.*

Remark. *Note that $\nu = \nu^+ - \nu^-$.*

Theorem 8.6. ***(The Jordan Decomposition Theorem)***
Let $\nu : \mathbb{X} \to \mathbb{R}$ be a charge. If $\nu = \nu_1 - \nu_2$, where ν_1, ν_2 are finite measures, then $\nu_1(A) \geq \nu^+(A)$ and $\nu_2(A) \geq \nu^-(A)$ for any $A \in \mathbb{X}$.

Proof. We let $X = N \cup P$ be the Hahn decomposition. We then have for any $A \in \mathbb{X}$

$$\nu_1(A) \geq \nu_1(A \cap P) \geq \nu(A \cap P) = \nu^+(A)$$

and

$$\nu_2(A) \geq \nu_2(A \cap N) \geq -\nu(A \cap N) = \nu^-(A).$$

□

Theorem 8.7. *Let $f \in L(X, \mathbb{X}, \mu)$, and consider the charge $\nu : \mathbb{X} \to \mathbb{R}$ given by $\nu(A) = \int_A f d\mu$. Then ν^+ and ν^- are given by $\nu^+(A) = \int_A f^+ d\mu$ and $\nu^-(A) = \int_A f^- d\mu$.*

Proof. See exercise sheet. □

Remark. *Let $f \in M^+(X, \mathbb{X})$. Recall that $\lambda : \mathbb{X} \to \bar{\mathbb{R}}$ defined by $\lambda(A) = \int_A f d\mu$ is a measure, and is absolutely continuous with respect to μ. A partial converse to this is given by the Radon-Nikodym Theorem. To see that in general there is no converse let μ be counting measure on $(\mathbb{R}, \mathbb{B})$ and λ Lebesgue measure. We know that if $\mu(A) = 0$ then $A = \emptyset$ so $\lambda(A) = 0$ which means that λ is absolutely continuous with respect to μ. In this case we can easily see that no such f exists, since if such an f exists then for any x, $0 = \lambda(\{x\}) = \int_{\{0\}} f d\mu$ and thus $f(x) = 0$. However this would then mean that every set has Lebesgue measure 0 which is a contradiction. The problem here is that counting measure on $\mathbb{R}$ is not sigma finite.*

Theorem 8.8. ***(Radon-Nikodým Theorem)***
Let λ, μ be σ-finite measures with $\lambda \ll \mu$. Then there exists $f \in M^+(X, \mathbb{X})$ such that $\lambda(A) = \int_A f d\mu$ for all $A \in \mathbb{X}$. This f is uniquely determined $\mu-$almost everywhere.

Proof. We first prove this in the case when μ and λ are finite and then extend to case when they are sigma-finite. So we suppose μ and λ are finite measures where λ is absolutely continuous with respect to μ. Our aim is to consider sets A on which for all subsets $B \subset A$ $\lambda(A)/\mu(A)$ is close to some common value. to do this we use the Hahn decomposition.

For $c > 0$ consider the charge $\lambda - c\mu$ and let $P(c)$ and $N(c)$ be the positive and negative parts of the Hahn decomposition for this charge. We now let $A(1) = N(c)$ and define inductively $A_{k+1} = N(c(k+1))\backslash(\cup_{i=1}^{k} A_i)$. We can see that the sets A_k are disjoint and

$$A_{k+1} = N(c(k+1))\backslash(\cup_{j=1}^{k} N(jc)) = N(c(k+1)) \cap (\cap_{j=1}^{k} P(cj)).$$

We now let $B := (\cup_{k=1}^{\infty} A_k)^c = \cap_{j=1}^{\infty} P(jc)$. Thus we have that for all k

$$0 \leq ck\mu(B) \leq \lambda(B) < \infty$$

which means that $\mu(B) = 0$ and thus $\lambda(B) = 0$ since λ is absolutely continuous with respect to μ.

For any measurable $E \subset A_k$ we have that

$$\lambda(E) - ck\mu(E) \leq 0 \leq \lambda(E) - c(k-1)\mu(E).$$

which means if we define $f_c : X \to \mathbb{R}$ by $f_c(x) = c(k-1)$ if $x \in A_k$ and $f_c(x) = 0$ when $x \in B$ then for any $A \in \mathbb{X}$

$$\int_A f_c \mathrm{d}\mu \leq \lambda(A) \leq \int_A (f_c + c)\mathrm{d}\mu = \int_A f_c \mathrm{d}\mu + c\mu(A).$$

We now use this construction with $c = 2^{-k}$ to create a sequence of functions $f_k \in M^+$ such that

$$\int f_k \mathrm{d}\mu \leq \lambda(A) \leq \int (f_k + 2^{-k})\mathrm{d}\mu = \int_A f_c \mathrm{d}\mu + 2^{-k}\mu(A). \tag{1}$$

Thus for positive integers $m \geq n$ we have that for any $A \in \mathbb{X}$

$$\left| \int_A f_n \mathrm{d}\mu - \int_A f_m \mathrm{d}\mu \right| \leq 2^{-n}\mu(A).$$

By taking sets $\{x : f_n(x) \geq f_m(x)\}$ and $\{x : f_m(x) \geq f_n(x)\}$ respectively we can see that

$$\int |f_n - f_m| \mathrm{d}\mu \leq 2^{-n+1}\mu(X).$$

Note that each function $f_n \in L_1(\mu)$ and so f_n converges in $L_1(\mu)$ to some function $f \in L_1(\mu)$ and thus f_n also converges in measure to f. Since each $f_n \in M^+$ and by Theorem 7.5 f_n must converge to f almost everywhere along a subsequence we can deduce that $f \in M^+$. Furthermore for any $E \in \mathbb{X}$ we have that

$$\left| \int_E f_n \mathrm{d}\mu - \int_E f \mathrm{d}\mu \right| \leq \int_E |f_n - f| \mathrm{d}\mu \leq \int |f_n - f| \mathrm{d}\mu \to 0 \text{ as } n \to \infty.$$

Thus by equation (1) we can conclude that $\lambda(E) = \lim_{k\to\infty} \int_E f_k \mathrm{d}\mu = \int_E f \mathrm{d}\mu$. So we've shown the existence of f in the case when μ and λ are finite.

To show that f is uniquely defined a.e. we suppose there exists g such that for all $E \in \mathbb{X}$

$$\lambda(E) = \int_E g \mathrm{d}\mu.$$

If we let $A = \{x : g(x) > f(x)\}$

$$\int |f - g| \mathrm{d}\mu = \int_{A^c} (f - g) \mathrm{d}\mu + \int_A (g - f) \mathrm{d}\mu = 0$$

and therefore $f = g$ μ a.e..

Finally we turn to the case where μ and λ are sigma finite. Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of sets such that for each n, $\lambda(A_n) < \infty$, $\mu(A_n) < \infty$ and $A_n \subset A_{n+1}$. For each n we can use the finite case result to find $f_n \in M^+$ such that for all $E \subset A_n$, $\lambda(E) = \int_E f_n \mathrm{d}\mu$ and $f(x) = 0$ for $x \in A_n^c$. So f_n is a monotone increasing sequence of functions so we can let $f(x) = \lim_{n\to\infty} f_n(x)$. By the monotone convergence theorem for all $A \in X$

$$\lambda(A) = \lambda(\cup_{n=1}^{\infty} A \cap A_n) = \lim_{n\to\infty} \lambda(A \cap A_n) = \lim_{n\to\infty} \int_A f_n \mathrm{d}\mu.$$

□

Remark. *f is called the Radon-Nikodym derivative of λ with respect to μ and is denoted by $\frac{d\lambda}{d\mu}$.*

Definition 8.9. *Two measures λ and μ on $\mathbb{X}$ are mutually singular if $\exists A, B \in \mathbb{X}$ with $A \cup B = X$, $A \cap B = \emptyset$ and $\lambda(A) = \mu(B) = 0$. We write $\lambda \perp \mu$.*

Theorem 8.10. ***(Lebesgue Decomposition Theorem)***
Let λ, μ be σ-finite measures on $\mathbb{X}$. Then there exists measures λ_1, λ_2 such that $\lambda_1 \perp \mu$, $\lambda_2 \ll \mu$ and $\lambda = \lambda_1 + \lambda_2$. The measures λ_1, λ_2 are unique.

Proof. Let $\nu = \lambda + \mu$. Thus ν is sigma finite and both λ and μ are absolutely continuous with respect to ν. We can now use the Radon Nikodym Theorem to see there exists $f, g \in M^+$ such that for all $A \in \mathbb{X}$

$$\mu(A) = \int_A f \mathrm{d}\nu \text{ and } \lambda(A) = \int_A g \mathrm{d}\nu.$$

Take the set $B = \{x : f(x) = 0\}$ and note that $\mu(B) = 0$. We define λ_1 and λ_2 to be the measures such that

$$\lambda_1(A) = \lambda(A \cap B) \text{ and } \lambda_2(A) = \lambda(A \cap B^c).$$

$\lambda_1(X \backslash B) = 0$ so we can see it is singular with respect to μ. To show λ_2 is absolutely continuous with respect to μ we let $A \in \mathbb{X}$ satisfy $\mu(A) = 0$. We then have that $f(x) = 0$ for ν almost all $x \in A$ and thus for λ almost all x. Therefore $\lambda_1(A) = \lambda(A \cap B) = 0$ and so λ_1 is absolutely continuous with respect to μ.

□

8.1 Linear Functionals

Definition 8.11. *A **linear functional** on $L_p(X, \mathbb{X}, \mu)$ is a map $G : L_p(X, \mathbb{X}, \mu) \to \mathbb{R}$ such that $\forall a, b \in \mathbb{R}$, $\forall f, g \in L_p$, $G(af + bg) = aG(f) + bG(g)$.*
*A linear functional G is **bounded** if $\exists M \in \mathbb{R}$ such that $\forall f \in L_p$, $|G(f)| \le M\|f\|_p$.*
*The **norm** of G is the real number $\|G\| := \sup\{|G(f)| : f \in L_p, \|f\|_p \le 1\}$.*

Theorem 8.12. *(1) Let $q \ge 1$ and let $g \in L_q(X, \mathbb{X}, \mu)$.*
Define $G : L_p(X, \mathbb{X}, \mu) \to \mathbb{R}$ by $G(f) = \int fg d\mu$ for $p = \frac{q}{q-1} > 1$. Then G is a bounded linear functional, and $\|G\| = \|g\|_q$.
(2) Suppose μ is σ-finite and let $q = \infty$, $p = 1$.
Define $G : L_1(X, \mathbb{X}, \mu) \to \mathbb{R}$ by $G(f) = \int fg d\mu$. Then G is a bounded linear functional, and $\|G\| = \|g\|_\infty$.

Proof. (1) It is straightforward to see that G is linear. Furthermore for any $f \in L_p$ we have that by Hölder's inequality

$$|G(f)| = \left| \int fg \mathrm{d}\mu \right| \le \|f\|_p \|g\|_q.$$

So G is bounded and $\|G\| \le \|g\|_q$. Now let $h(x) = \operatorname{sgn}(g(x))|g(x)|^{q-1}$ and observe that since $p(q-1) = q$, $h \in L_p$, $\|h\|_p = \|g\|_q^{q/p} = \|g\|_q^{q-1}$ and

$$G(h) = \int hg \mathrm{d}\mu = \int |g|^q \mathrm{d}\mu = \|g\|_q^q = \|g\|_q \|h\|_p.$$

Thus $\|G\| = \|g\|_q$.

(2) Again it is straightforward to see that G is a bounded linear functional and $\|G\| \le \|g\|_\infty$. We now fix $\epsilon > 0$ and choose $X_\epsilon = \{x : g(x) \ge \|g\|_\infty - \epsilon\}$. We now $\mu(X_\epsilon) > 0$ and since μ is sigma finite we can find $Y_\epsilon \subset X_\epsilon$ such that $0 < \mu(Y_\epsilon) < \infty$. We now let $h = \chi_{Y_\epsilon}\operatorname{sgn}(g(x))$ which is in L_1 and $\|h\|_1 = \mu(Y_\epsilon)$. We then have

$$G(h) = \int hg \mathrm{d}\mu \ge (\|g\|_\infty - \epsilon)\mu(Y_\epsilon) = (\|g\|_\infty - \epsilon)\|h\|_1.$$

□

Definition 8.13. *A linear functional G is **positive** if $f \ge 0 \implies G(f) \ge 0$.*

Lemma 8.14. *Let G be a bounded linear functional on $L_p(X, \mathbb{X}, \mu)$. Then there exists bounded, positive linear functionals G^+ and G^- on $L_p(X, \mathbb{X}, \mu)$ such that $G(f) = G^+(f) + G^-(f)$, $\forall f \in L_p(X, \mathbb{X}, \mu)$.*

Theorem 8.15. ***(Riesz Representation Theorem for** L_1**)***
Let $(X, \mathbb{X}, \mu)$ *be a* σ*-finite measure space, and let* G *be a bounded linear functional on* $L_1(X, \mathbb{X}, \mu)$*. Then there exists* $g \in L_\infty(X, \mathbb{X}, \mu)$ *such that* $G(f) = \int fg d\mu$, $\forall f \in L_1(X, \mathbb{X}, \mu)$*.*
We have $\|G\| = \|g\|_\infty$*. Moreover, if* G *is positive, then* $g \geq 0$*.*

Proof. 'Outline the details will be filled in via a problem sheet'. We start with the finite case, assume that G is a positive bounded linear functional and define $\lambda : \mathbb{X} \to \mathbb{R}$ by $\lambda(E) = G(\chi_E)$. We then show that λ is a finite measure and is absolutely continuous. We can then apply the Radon-Nikodym derivative to find $g \in M^+$ such that $G(\chi_E) = \lambda(E) = \int g \mathrm{d}\mu$.

Thus it follows for all simple functions $\phi \in L_1$ that $G(\phi) = \int \phi g \mathrm{d}\mu$. We then use the Monotone convergence theorem to extend this to all non-negative functions in L_1 and finally use linearity to extend it to all functions in L_1.

To extend this to σ measure spaces we work on sets A_n where $\cup_{n=1}^\infty A_n = X$, $\mu(A_n) < \infty$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$. This can be readily seen to yield the result for non-negative functions (Monotone Convergence Theorem) and positive functionals. Finally for general bounded linear functionals we use Lemma 8.14. □

For the case when $1 < p < \infty$ we can relax the condition that the measure space is sigma finite.

Theorem 8.16. ***(Riesz Representation Theorem for** L_p**)***
Let $(X, \mathbb{X}, \mu)$ *be any measure space, and let* G *be a bounded linear functional on* $L_p(X, \mathbb{X}, \mu)$, $1 < p < \infty$*. Then there exists* $g \in L_q(X, \mathbb{X}, \mu)$, $q = \frac{p}{p-1}$, *such that* $G(f) = \int fg d\mu$, $\forall f \in L_p(X, \mathbb{X}, \mu)$*.*
We have $\|G\| = \|g\|_q$*.*

Proof. For a sigma-finite measure the proof for L_1 can be adapted without difficulty. To see how it may be extended to the non-sigma finite case please consult Bartle (NB: However nearly all measure theory is done with the assumption of the measure space being sigma finite and in my opinion it is the sigma finite case which is important to understand.) □

9 Constructing measures and unmeasurable sets

In this section we show how Lebesgue measure can properly be defined and in doing this give a method which can be used to generate many different measures. We'll also show at the end how to construct sets which are not Lebesgue measurable.

Definition 9.1. *Let* X *be a space we define an algebra to be a set* $\mathbf{A}$ *of subsets of* X *which satisfies that*

1. $\emptyset, X \in \mathbf{A}$

2. If $Y \in \mathbf{A}$ *then* $Y^c \in \mathbf{A}$.

3. If $A_1, \ldots, A_n \in \mathbf{A}$ *then* $\cup_{k=1}^{n} A_k \in \mathbf{A}$

Definition 9.2. *A premeasure on an algebra of subsets of* X *is a function* $\mu : X \to \overline{\mathbb{R}}$ *such that*

1. $\mu(\emptyset) = 0$

2. $\mu(A) \geq 0$ *for all* $A \in \mathbf{A}$

3. If $A_1, A_2, \ldots \in \boldsymbol{A}$*, the sets* A_i *are pairwise disjoint and* $\cup_{i=1}^{\infty} A_i \in \boldsymbol{A}$ *then*

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Lemma 9.3. *The collection* $\mathbb{F}$ *of all finite unions of sets of the form* $(a, b]$*,*$(-\infty, b)$*,* $[b, \infty)$ *and* $(-\infty, \infty)$ *is an algebra on* $\mathbb{R}$ *and the length gives a premeasure on these sets.*

Proof. To show that these sets form an algebra and that length satisfies the first two properties to be a premeasure is straight forward. So we need to show that length is countably subadditive on $\mathbb{F}$. To prove (iii) we just need to show that if one of the sets of the form $(a, b]$, (a, ∞), $(-\infty, b]$, $(-\infty, \infty)$ is a disjoint countable union of other sets of this form then length adds up countably. We do this for $(a, b]$ and note that other sets can be dealt with similarly.

Suppose we have a sequence of disjoint intervals $(a_1, b_1], (a_2, b_2], \ldots$ such that $(a, b] = \cup_{n=1}^{\infty} (a_n, b_n]$ and $a_n < b_n < a_{n+1}$ for all $n \in \mathbb{N}$. For any $k \in \mathbb{N}$ we have that $\cup_{n=1}^{k} (a_n, b_n] \subset (a, b]$ and thus

$$l\left(\cup_{n=1}^{k} (a_n, b_n]\right) = \sum_{n=1}^{k} b_n - a_n \leq b - a.$$

This holds for any k so we must have

$$l\left(\cup_{n=1}^{\infty} (a_n, b_n]\right) < b - a.$$

On the other hand let $\epsilon > 0$ and $\epsilon_j > 0$ satisfy that $\sum_{j=1}^{\infty} \epsilon_j < \epsilon/2$. We then have that

$$[a, b] \subset \cup_{j=1}^{\infty} (a_j - \epsilon/2, b_j + \epsilon/2)$$

and since $[a, b]$ is compact we can do this with finitely many intervals. So we can find k such that

$$[a, b) \subset \cup_{j=1}^{k} (a_j - \epsilon_j/, b_j + \epsilon_j)$$

and thus

$$b - a \leq \sum_{j=1}^{k}(b_j - a_j) + \epsilon$$

and the result follows. $\square$

We now want to show that we can extend premeasures on an algebra to give a measure on a suitable sigma algebra.

Definition 9.4. *If μ is a premeasure defined on an algebra $\boldsymbol{A}$ we can define $\mu^* : P(X) \to \mathbb{R}$ by*

$$\mu^*(B) = \inf\left\{\sum_{j=1}^{\infty}\mu(A_j) : A_j \in \boldsymbol{A} \text{ and } B \subseteq \cup_{j=1}^{\infty}A_j\right\}.$$

It can be shown that μ^* has several properties similar to that of a measure and it is called an outer measure, however it is not in general a measure.

Lemma 9.5. ***(Properties of the outer measure:)***

(a) $\mu^*(\emptyset) = 0,$

(b) $\mu^*(B) \geq 0, \quad \forall B \subseteq X,$

(c) $A \subseteq B \implies \mu^*(A) \leq \mu^*(B),$

(d) $B \in \mathbb{A} \implies \mu^*(B) = \mu(B),$ *and*

(e) *If* $B_1, B_2, \cdots$ *is a sequence of subsets of* X*, then* $\mu^*(\cup_{k=1}^{\infty}B_k) \leq \sum_{k=1}^{\infty}\mu^*(B_k).$

Proof. Parts (a), (b) and (c) all follow immediately from the definition. For part d for $B \in \mathbb{A}$ write $B = B \cup \emptyset \cup \emptyset \cup \cdots$ to see that $\mu^*(B) \leq \mu(B)$. Now let $A_j \in \mathbb{A}$ be a family of sets such that $B \subseteq \cup_{j=1}^{\infty}A_j$. We know that $\sum_{j=1}^{\infty}\mu(A_j) \geq \mu(B)$ and so $\mu^*(B) \geq \mu(B)$.

Finally to establish (e), we let $\epsilon > 0$ and note that for each $n, k \in \mathbb{N}$ we can find $A_{n,k} \in \mathbb{A}$ such that $B_n \subseteq \cup_{k=1}^{\infty}A_{n,k}$ and $\sum_{k=1}^{\infty}\mu^*(A_{n,k}) \leq \mu^*(B_n) + \epsilon/2^n$. So $\cup_{n=1}^{\infty}B_n \subseteq \cup_{n,k\in\mathbb{N}}A_{n,k}$ and

$$\sum_{n,k\in\mathbb{N}}\mu^*(A_{n,k}) \leq \sum_{n=1}^{\infty}\mu^*(B_n) + \epsilon.$$

Therefore it follows that $\mu^*(\cup_{k=1}^{\infty}B_k) \leq \sum_{k=1}^{\infty}\mu^*(B_k)$.

$\square$

Definition 9.6. *For an outer measure μ^* we say that a set E is μ^* measurable if for all $A \subset X$ we have that*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

We call the set of all μ^* measurable sets A^*.

Theorem 9.7. ***(Caratheodory's Extension Theorem)***
(i) The collection $\mathbb{A}^*$ of all μ^*-measurable sets is a $\sigma-$algebra containing $\mathbb{A}$.
(ii) If $A_1, A_2, \cdots$ is a sequence of disjoint sets in $\mathbb{A}^*$, then $\mu^*(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$.

Proof. 'Outline'. **Step 1** We first need to show that $\mathbb{A}^*$ is an algebra and that μ^* is additive on $\mathbb{A}^*$. It is easy to show $X, \emptyset \in A^*$ and to show that A^* is closed under taking complements. We then show it is closed under taking finite intersections which is enough to show it is an algebra. Finally taking $E, F \in \mathbb{A}^*$ with $E \cap F = \emptyset$ it can be shown that $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$.

Step 2 We show that $\mathbb{A}^*$ is in fact a sigma algebra and μ^* is a measure on $\mathbb{A}^*$. Let $\{E_n\}_{n\in\mathbb{N}}$ be a disjoint collection of sets in $\mathbb{A}^*$. If we let $F_n = \cup_{k=1}^{n} E_k$ then it follows from step 1. that $F_n \in \mathbb{A}^*$. Now fix $A \subset X$ and let $E = \cup_{k=1}^{\infty} E_k$. By parts (c) and (e) of Lemma 9.5 it follows that for any $n \in \mathbb{N}$

$$\mu^*(A\cap E)+\mu^*(A\cap E^c) \leq \sum_{k=1}^{\infty} \mu^*(A\cap E_k)+\mu^*(A\cap F_n^c) = \mu^*(A)+ \sum_{k=n+1}^{\infty} \mu^*(A\cap E_k) \tag{2}$$

where we have used that $\mu^*(A) = \mu^*(A\cap F_n)+\mu^*(A\cap F_n^c)$. Thus if $\mu^*(A) < \infty$ then by letting $n \to \infty$ we can see that $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$ and if $\mu * (A) = \infty$ then the result is trivial. On the other hand we have that by part (e) of Lemma 9.5

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Thus we have shown that $E \in \mathbb{A}^*$. To see that μ^* is countably additive on A^* take $A = E$ in (2) to get

$$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k) \leq \mu^*(E) + \sum_{k=n+1}^{\infty} \mu^*(E_k)$$

and letting $n \to \infty$ yields the result.

Step 3 We show that $A \subset A^*$. Let $A \in \mathbb{A}$ (we need to show that $E \in A^*$) and $E \subset X$. By part (e) of Lemma 9.5 we have that

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

On the other hand let $\epsilon > 0$ and $F_n \in \mathbb{A}$ be a sequence such that $E \subset \cup_{n=1}^{\infty} F_n$ and $\sum_{n=1}^{\infty} \mu^*(F_n) \leq \mu^*(E) + \epsilon$. Since each $F_n \in \mathbb{A}$ we have that

$$\mu^*(F_n) = \mu^*(F_n \cap A) + \mu^*(F_n \cap A^c).$$

So

$$\mu^*(E) \geq \sum_{n=1}^{\infty} \mu^*(F_n)+\epsilon = \sum_{n=1}^{\infty} \mu^*(F_n\cap A)+\mu^*(F_n\cap A^c)+\epsilon \geq \mu^*(E\cap A)+\mu^*(E\cap A^c)+\epsilon.$$

Thus letting $\epsilon \to 0$ yields that

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

and the proof is complete. $\square$

Remark. *$(X, \mathbb{A}^*, \mu^*)$ is a complete measure space. Recall that this means for any $A \in \mathbb{A}^*$ with $\mu^*(A) = 0$ all subsets of A are in $\mathbb{A}^*$.*

Theorem 9.8. ***(Hahn's Uniqueness Theorem)***
Let μ be a $\sigma-$finite measure on $\mathbb{A}$. Then the extension of μ to a measure on $\mathbb{A}^*$ is unique.

Proof. We first suppose ν is finite a measure on $\mathbb{A}^*$ such that $\nu(A) = \mu(A)$ for all $A \in \mathbb{A}^*$. We use the definition of μ^* to show for all $E \subset A^*$ that $\nu(E) = \mu^*(E)$. We then extend this to the sigma finite case. In the case where μ is finite, let $E \subset A^*$, for $\epsilon > 0$ we can find sets $E_n \in A$ such that $E \subseteq \cup_{n=1}^{\infty} E_n$ and $\mu^*(E) \geq \sum_{n=1}^{\infty} \mu^*(E_n) - \epsilon$. Thus

$$\nu(E) \leq \sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} \mu^*(E_n) \leq \mu^*(E) + \epsilon.$$

This is using that ν and μ^* must agree on elements of $\mathbb{A}$ and since it holds for any $\epsilon > 0$ it follows that $\nu(E) \leq \mu^*(E)$. However this argument works equally well for E^c and we have since $X \in \mathbb{A}$

$$\mu^*(X) = \mu^*(E) + \mu^*(E^C) \geq \nu(E) + \nu(E^c) = \nu(X).$$

Since $\mu(X) < \infty$ this must mean $\nu(E) = \mu^*(E)$.

We can extend this to the sigma finite case in the usual way. $\square$

Definition 9.9. *$\mathbb{F}^*$ is the $\sigma-$algebra of **Lebesgue measurable sets**, and l^* is the **Lebesgue measure**.*

Remark. *Recall that $\mathbb{B}$ is the smallest $\sigma-$algebra containing $\mathbb{F}$, and hence $\mathbb{B} \subseteq \mathbb{F}^*$. The restriction of l^* to $\mathbb{B}$ is the **Borel measure** (also called Lebesgue measure.) It is possible to find a set which is not Lebesgue measurable.*

Remark. *(i) $\mathbb{B}$ is a proper subset of $\mathbb{F}^*$.*
(ii) Not every subset of $\mathbb{R}$ is Lebesgue measurable.

The following procedure constructs a subset of $\mathbb{R}$ which is not Lebesgue measurable (It is due to Giuseppe Vitali and was done in 1905). For $x, y \in \mathbb{R}$ we say $x \sim y$ if and only if $x - y \in \mathbb{Q}$. This gives an equivalence relation. We then consider the equivalence classes $x + \mathbb{Q}$ and take $V \subset (0, 1)$ to be a set which contains exactly one element in each equivalence class $x + \mathbb{Q}$ (we can do this by the axiom of choice, if you haven't heard about this before look it up).

We can then take r_n to be an enumeration of the rationals in $(-1, 1)$ and consider the sets $V + r_n$ for each $n \in \mathbb{N}$. since Lebesgue measure is invariant under translation if V was measurable we would have that $\lambda(V + r_n) = \lambda(V)$. However we can see that $(0, 1) \subset \cup_{n \in \mathbb{N}}(V + r_n) \subset (-1, 2)$ which would contradict either $\lambda(V) = 0$ or $\lambda(V) > 0$ so V cannot be measurable.

Exercise sheet 1

1. Show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $A \subset \mathbb{R}$ is an open set then $f^{-1}(A) = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in A\}$ is an open set in $\mathbb{R}^2$ (take the usual Euclidean metric on $\mathbb{R}^2$).

2. Show that

   (*i*) $[a, b] = \bigcap_{n=1}^{\infty} (a - 1/n,\, b + 1/n)$;

   (*ii*) $(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n,\, b - 1/n]$.

   Conclude that any σ-algebra of subsets of $\mathbb{R}$ which contains all open (resp. closed) intervals also contains all closed (resp. open) intervals.

3. Show that every open set $A \subset \mathbb{R}$ can be written as a countable union of open intervals.

4. Show that the Borel σ-algebra $\mathbb{B}$ is generated by the collection of half-open intervals $(a, b]$. Show that it is also generated by the collection of half-rays (a, ∞), $a \in \mathbb{R}$.

5. Let $f : (X, \mathbb{X}) \to \mathbb{R}$ (or $\overline{\mathbb{R}}$). Show that the following are equivalent.

   (*i*) f is $\mathbb{X}$-measurable;

   (*ii*) $\{f > q\} \in \mathbb{X}$ for each $q \in \mathbb{Q}$;

   (*iii*) $\{f \leq q\} \in \mathbb{X}$ for each $q \in \mathbb{Q}$.

6. Give an example of an $\mathbb{R}$-valued function f on some measurable space $(X, \mathbb{X})$ which is not $\mathbb{X}$-measurable, but is such that f^2 is $\mathbb{X}$-measurable.

7. Consider the Borel σ-algebra $(\mathbb{R}, \mathbb{B})$. Show that any monotone function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

8. Let $x \in (0, 1]$ have the expansion $x = 0.x_1x_2x_3\ldots$ in base 2, the non-terminating expansion being used in cases of ambiguity. Show that $f_n(x) = x_n$ is a Borel measurable function of x for each n.

9. Let $f : X \to \overline{\mathbb{R}}$ be a measurable function on $(X, \mathbb{X})$ and $A > 0$. The **truncation** f_A of f is defined by

$$f_A(\omega) = \begin{cases} f(\omega), & \text{if } |f(\omega)| \leq A, \\ A, & \text{if } f(\omega) > A, \\ -A, & \text{if } f(\omega) < -A. \end{cases}$$

Show that f_A is measurable.

10. Let f be a non-negative measurable function on $(X, \mathbb{X})$ which is bounded (so $0 \leq f(\omega) \leq K$ for all $\omega \in X$). Show that the sequence of simple measurable functions φ_n constructed in the lecture converges to f uniformly on X.

11. Let $(X, \mathbb{X})$ be a measurable space, and let $f : X \to \mathbb{R}$ be a function. Show that f is measurable if and only if $f^{-1}(G)$ is measurable for each open set $G \subset \mathbb{R}$.

C1 Show that the set of all $x \in [0, 1]$ which are normal base 2 is Borel measurable (if you don't know what normal base 2 means look it up).

Exercise sheet 2

1. Consider the measure space $(\mathbb{R}, \mathbb{B}, \lambda)$ where λ is Lebesgue measure. For any $x \in \mathbb{R}$ show that $\lambda(\{x\}) = 0$. Thus show that any countable set $X \subset \mathbb{R}$ satisfies $\lambda(X) = 0$.

2. Again let λ denote Lebesgue measure. Show the following.

   a) If E is a non-empty open subset of $\mathbb{R}$ then $\lambda(E) > 0$.

   a) If K is a compact subset of $\mathbb{R}$ then $\lambda(K) < +\infty$. (Use the Heine-Borel theorem)

3. Find a set with positive Lebesgue measure but which contains no non-void open interval.

4. Let $(X, \mathbb{X}, \mu)$ be a measure space. Show that if for any $A \in \mathbb{X}$ we define $\mu_A : \mathbb{X} \to \overline{\mathbb{R}}$ by $\mu_A(B) = \mu(A \cap B)$ then μ_A is a measure.

5. Let $(X, \mathbb{X})$ be a measurable space, $\mu_1, \mu_2, \ldots, \mu_n$ be measures on $(X, \mathbb{X})$ and $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that $\nu = \sum_{k=1}^{n} a_k \mu_k$ is a measure on $(X, \mathbb{X})$. Thus prove Lemma 4.5 from the lectures. That is that for any measure μ on $(X, \mathbb{X})$ and simple function $\phi \in M^+(X, \mathbb{X})$ we can define a measure by $\nu(A) = \int_A \phi \mathrm{d}\mu$ for all $A \in \mathbb{X}$.

6. Let $X = \mathbb{N}$ and $\mathbb{X}$ be the family of all subsets of X. If E is finite, let $\mu(E) = 0$; if E is infinite, let $\mu(E) = +\infty$. Is μ a measure on $\mathbb{X}$?

7. Let (A_n) be a sequence of sets in X. Put

$$A = \bigcap_{m=1}^{\infty} [\bigcup_{n=m}^{\infty} A_n], \qquad B = \bigcup_{m=1}^{\infty} [\bigcap_{n=m}^{\infty} A_n].$$

The set A is called the **limit superior** of the sets (A_n) and denoted by lim sup A_n. The set B is called the **limit inferior** of the sets (A_n) and denoted by lim inf A_n. A consists of all those $x \in X$ that belong to infinitely many of the sets A_n while B consists of all those $x \in X$ that belong to all but finitely many of the sets A_n.

Let $(X, \mathbb{X}, \mu)$ be a measure space and let (A_n) be a sequence in $\mathbb{X}$. Show that

a) $\mu(\text{lim inf } A_n) \leq \text{lim inf } \mu(A_n)$;
and, if $\mu(\cup A_n) < +\infty$,

b) $\text{lim sup } \mu(A_n) \leq \mu(\text{lim sup } A_n)$.

Show that the inequality in b) may fail if $\mu(\cup A_n) = +\infty$.

C2 Let $(X, \mathbb{X}, \mu)$ be a measure space where $\mu(X) = 1$. Show that

a If $A_1, A_2, \ldots \in \mathbb{X}$ and $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then

$$\mu(\{x \in X : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\} = 0.$$

b If $A_1, A_2, \ldots \in X$, $\mu(\cap_{i \in I} A_i) = \prod_{i \in I} \mu(A_i)$ for all finite subsets $I \subset \mathbb{N}$ and $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ then

$$\mu(\{x \in X : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\} = 1.$$

It may help to use the inequality that for $x \geq 0$, $1 - x \leq e^{-x}$.

C3 Show that the set of numbers which are weakly normal base 2 in $[0, 1]$ (i.e the numbers for which the digit 0 occurs with frequency $1/2$ in the binary expansion) has Lebesgue measure 1. It'll help to use question C2. and Stirling's formula (if you have not seen this before look it up).

Exercise sheet 3

Notation: $(X, \mathbb{X}, \mu)$ will denote a measure space and $(\mathbb{R}, \mathbb{B}, \lambda)$ will be the specific measure space of the real numbers with the Borel sigma algebra and Lebesgue measure.

1. Working on $(\mathbb{R}, \mathbb{B}, \lambda)$ let $f_n = (1/n)\chi_{[0,n]}$ and $f = 0$. Show that the sequence (f_n) converges uniformly to f, but that

$$\int f \, d\lambda \neq \lim \int f_n \, d\lambda.$$

Why does this not contradict the Monotone Convergence Theorem? Does Fatou's Lemma apply?

2. Show that if $(X, \mathbb{X}, \mu)$ is a finite measure space (i.e. $\mu(X) < \infty$), and if (f_n) is a real-valued sequence in $M^+(X, \mathbb{X})$ which converges uniformly to a function f, then f belongs to $M^+(X, \mathbb{X})$ and

$$\int f \, d\mu = \lim \int f_n \, d\mu.$$

3. Let $A_n \in \mathbb{X}$ and $B_n = A_n^c$. suppose that for all n $A_n \subseteq A_{n+1}$ and $\cup_{n=1}^{\infty} A_n = X$. If $f \in M^+(X, \mathbb{X})$ satisfies that $\int f \mathrm{d}\mu < \infty$ then show that $\lim_{n\to\infty} \int_{B_n} f \mathrm{d}\mu = 0$ (Hint: Use Theorem 4.13). Is this true if we don't assume that $\int f \mathrm{d}\mu < \infty$.

4. Let X be a finite closed interval $[a, b]$ in $\mathbb{R}$, $\mathbb{B}$ the collection of Borel sets in X and λ be Lebesgue measure on X. Let f be a non-negative continuous function on X. Show that

$$\int f \mathrm{d}\lambda = \int_a^b f(x) \mathrm{d}x$$

where the right side denotes the Riemann integral.

5. If $f \in M^+$ and $\int f \, d\mu < +\infty$ then $\mu(\{f = +\infty\}) = 0$.

6. Let $f_n \in M^+$, $\lim f_n = f$ and $f_n \leq f$. Show that $\int f \, d\mu = \lim \int f_n \, d\mu$.

7. Let (f_n) be a sequence of non-negative $\mathbb{R}$-valued measurable functions such that $f_n \downarrow f$.
(a) Show that if $\int f_k \, d\mu < +\infty$ for some k, then $\lim \int f_n \, d\mu = \int f \, d\mu$.
(b) Show that $\int f_k \, d\mu = +\infty$ for all k does not imply $\int f \, d\mu = \infty$.

c4 Suppose that $(f_n) \subseteq M^+$, that (f_n) converges to f, and that

$$\int f \, d\mu = \lim \int f_n \, d\mu < +\infty.$$

Prove that

$$\int_E f \, d\mu = \lim \int_E f_n \, d\mu$$

for each $E \in \mathbb{X}$. *Hint.* It may help to consider $A_n := \{f_n \geq \alpha f\}$ (for some $1 < \alpha < \infty$) and $B_n := \bigcap_{m \geq n} A_n^c$.

Exercise sheet 4

NB: Throughout this sheet $(X, \mathbb{X}, \mu)$ refers to a measure space.

1. Let $f \in L(X, \mathbb{X}, \mu)$ and $a > 0$. Show that $\{x : |f(x)| \geq a\}$ has finite measure. Show that $\{x : f(x) \neq 0\}$ has σ-finite measure (i.e for each $n \in \mathbb{N}$ there exists $A_n \in \mathbb{X}$ such that $\cup_{n\in\mathbb{N}} A_n = \{x \in X : f(x) \neq 0\}$ and $\mu(A_n) < \infty$.)

2. Let $f \in L(X, \mathbb{X}, \mu)$ and $\varepsilon > 0$. Show that there exists an $\mathbb{X}$-measurable simple function φ such that

$$\int |f - \varphi| \, d\mu < \varepsilon.$$

3. Let $f_n \in L$ such that $f_n \to f \in L$. Suppose that

$$\lim_{n \to \infty} \int |f_n - f| \, d\mu = 0.$$

Show that

$$\int |f| \, d\mu = \lim \int |f_n| \, d\mu.$$

4. Let $f \in L(X, \mathbb{X}, \mu)$ and $g : X \to \mathbb{R}$ is a measurable function such that $f = g$ for μ-almost all x. Show that $g \in L(X, \mathbb{X}, \mu)$ and that $\int g \mathrm{d}\mu = \int f \mathrm{d}\mu$.

5. Show that $\lim_{n \to \infty} \int_{[0,1]} f_n \, d\lambda = 0$ where $f_n(x) := \frac{n\sqrt{x}}{1+n^2x^2}$.

6. Show that for the function $f(x) := \frac{1}{1+x^2}$ you have $\int_{\mathbb{R}} f \, d\lambda = \pi$.

7. Let $n \geq 2$, $x \geq 0$. Prove that $(1 + x/n)^n \geq (1/4)x^2$. and deduce that $x \to (1 + x/n)^{-n}$ is Lebesgue-integrable over $[1, \infty)$ and that

$$\lim_{n \to \infty} \int_{[1,\infty)} (1 + \frac{x}{n})^{-n} \, d\lambda = e^{-1}.$$

c5 Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that $\lim_{y \to \infty} \int_0^y f(x) \mathrm{d}x$ exists and is finite. Show that it is possible that $f \notin L((0, \infty), \mathbb{B}, \lambda)$.

Exercise sheet 5

1. Let $(X, \mathbb{X}, \mu)$ be a finite measure space. If f is $\mathbb{X}$-measurable, let $E_n = \{x : (n-1) \leq |f(x)| < n\}$. Show that $f \in L^1$ if and only if

$$\sum_{n=1}^{\infty} n\mu(E_n) < +\infty.$$

2. Let $X = \mathbb{N}$ and let $\mathbb{X}$ be the collection of all subsets of $\mathbb{N}$. Let λ be defined by

$$\lambda(E) = \sum_{n \in E} (1/n^2), \; E \in \mathbb{X}.$$

   (a) Show that $f : X \to \mathbb{R}$, $f(n) = \sqrt{n}$ satisfies $f \in L^p$ if and only if $1 \leq p < 2$.

(b) Find a function f such that $f \in L^p$ if and only if $1 \leq p \leq p_0$.

3. Let $(X, \mathbb{X}, \mu)$ be a finite measure space and let $f \in L^p$. Show that $f \in L^r$ for any $1 \leq r < p$ with

$$\|f\|_r \leq \mu(X)^{\frac{1}{r}-\frac{1}{p}}\|f\|_p.$$

4. Suppose that $X = \mathbb{N}$ and μ is the counting measure on X. Show that $f \in L^p$ implies $f \in L^s$ for $1 \leq p \leq s < \infty$.

5. Let $(X, \mathbb{X}, \mu)$ be a measure space and suppose $f \in L^{p_1}$ and $f \in L^{p_2}$ with $1 \leq p_1 < p_2 < \infty$. Prove that $f \in L^p$ for any p with $p_1 \leq p \leq p_2$. (Hint: Prove Corollary 6.11 from the lecture notes)

6. Let $f \in L^p$, $1 \leq p \leq \infty$ and $g \in L^\infty$. Show that $fg \in L^p$ and $\|fg\|_p \leq \|f\|_p\|g\|_\infty$.

7. (a) Prove that the space $L^\infty(X, \mathbb{X}, \mu)$ is contained in $L^1(X, \mathbb{X}, \mu)$ if and only if $\mu(X) < \infty$.

   (b) Show that if $\mu(X) = 1$ and $f \in L^\infty$, then

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

8. Let $1 \leq p < \infty$, $f : X \to \mathbb{R}$ be a measurable function and $f_n : X \to \mathbb{R}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$ and there exists $g \in L_p$ such that $|f_n(x)| \leq g(x)$ for all $N \in \mathbb{N}$ and $x \in X$. Show that $f \in L_P$ and $\lim_{n\to\infty}\|f_n - f\|_p = 0$ (i.e f_n tends to f in L_p.)

9 This is an additional question on the Dominated Convergence Theorem. In the measure space $((0, \infty), \mathbb{B}, \lambda)$ show that

$$\lim_{n\to\infty} \int \frac{\cos(x^5)}{1 + nx^2} \mathrm{d}\lambda = 0.$$

c6 A function $\phi : \mathbb{R} \to \mathbb{R}$ is called convex if for every $x_1, x_2 \in \mathbb{R}$ and $t \in [0, 1]$ we have that $\phi(tx_1 + (1-t)x_2) \leq t\phi(x_1) + (1-t)\phi(x_2)$. Prove

(a) If ϕ is convex then for all $x \in \mathbb{R}$

$$\phi(x) = \sup\{\psi(x) : \psi \text{ is affine and } \psi(y) \leq \phi(y) \text{ for all } y \in \mathbb{R}\}.$$

(b) Jensen's inequality that if $(X, \mathbb{X}, \mu)$ is a probability space, $f \in L(X, \mathbb{X}, \mu)$ and $\phi : \mathbb{R} \to \mathbb{R}$ is convex then

$$\phi\left(\int f \mathrm{d}\mu\right) \leq \int \phi(f)\mathrm{d}\mu.$$

(c) Hölders inequality using Jensen's inequality. For $g \in L_q$ it may help to assume that $g > 0$, consider the measure ν defined by $\nu(A) = \frac{1}{\|g\|_q^q} \int_A g^q \mathrm{d}\mu$ and $h = |f|/g^{q-1}$.

Exercise sheet 6

1. Let $(X, \mathbb{X}, \mathbb{P})$ be a **probability space** (i.e. $\mathbb{P}(X) = 1$). Let $Y : X \to \mathbb{R}$ be an $\mathbb{X}$-measurable **random variable**. Assume that $Y \in L_2(X, \mathbb{X}, \mathbb{P})$. Define

$$\mu = \mathbb{E}[Y] = \int Y \, d\mathbb{P} \qquad (\textbf{expectation} \text{ of } Y),$$

$$\sigma^2 = \mathbb{E}[(Y - \mu)^2] \qquad (\textbf{variance} \text{ of } Y)$$

Prove **Chebychev's inequality**,

$$\mathbb{P}(\{x \in X : |Y(x) - \mu| \geq \alpha\}) \leq (\sigma/\alpha)^2$$

for any $\alpha > 0$.

2. Consider the measure space $([0, 1], \mathbb{B}, \lambda)$. Give an example of a sequence of functions (f_n) in L_1 and a function $f \in L_1$ where f_n converges to f in L_1 but for which f_n is not almost everywhere convergent to f.

3. Let (f_n), (g_n) be $\mathbb{R}$-valued $\mathbb{X}$-measurable functions which converge in measure to $\mathbb{X}$-measurable functions f and g respectively. Prove that

   a) $(a\, f_n + b\, g_n)$ converges in measure to $a\, f + b\, g$ for any $a, b \in \mathbb{R}$;

   b) $(|f_n|)$ converges to $|f|$ in measure.

4. Prove the following versions of Fatou's Lemma. Assume f_n, $f \in M_+$ $(n \in \mathbb{N})$.

   i) If $f_n \to f$ μ-a.e. then $\int f \, d\mu \leq \liminf \int f_n \, d\mu$.

   ii) If $f_n \to f$ in measure then $\int f \, d\mu \leq \liminf \int f_n \, d\mu$.

5. Show that Lebesgue's Dominated Convergence Theorem holds if almost everywhere convergence is replaced by convergence in measure.

6. Let $(X, \mathbb{X}, \mu)$ be a finite measure space. If f is an $\mathbb{X}$-measurable function, put

$$r(f) = \int \frac{|f|}{1 + |f|} \, d\mu.$$

Show that a sequence (f_n) of $\mathbb{X}$-measurable $\mathbb{R}$-valued functions converges in measure to f if and only if $r(f_n - f) \to 0$.

7. Let (f_n) be a sequences of $\mathbb{R}$-valued $\mathbb{X}$-measurable functions which converge in measure to a $\mathbb{X}$-measurable function f. Show that (f_n) is Cauchy in measure.

8. Give an example to show that convergence a.e does not always imply almost uniform convergence (i.e. show that the assumption $\mu(X) < \infty$ is necessary in Egorov's Theorem).

c7 Prove Vitali's Convergence Theorem. This states that for $1 \leq p < \infty$ a sequence $\{f_n\}_{n\in\mathbb{N}}$ in L_p is convergent if and only if the following three criteria are satisfied.

   (a) f_n is Cauchy in measure.

   (b) For each $\epsilon > 0$ there exists $E_\epsilon \in \mathbb{X}$ such that $\mu(E_\epsilon) < \infty$ such that if $F \in \mathbb{X}$ and $F \cap E_\epsilon = \emptyset$ then
$$\int_F |f_n|^p \mathrm{d}\mu < \epsilon^p \text{ for all } n \in \mathbb{N}.$$

   (c) For each $\epsilon > 0$ there is a $\delta > 0$ such that if $E \in \mathbb{X}$ and $\mu(E) < \delta(\epsilon)$ then
$$\int_E |f_n|^p \mathrm{d}\mu < \epsilon^p \text{ for all } n \in \mathbb{N}.$$

Exercise sheet 7

1. Let ν be a charge on $(X, \mathbb{X})$.

   a) Let $\mathbb{X} \ni E_n \uparrow E$ (this means for all n, $E_n \subset E_{n+1}$). Show that $\nu(E) = \lim \nu(E_n)$.

   b) Let $\mathbb{X} \ni F_n \downarrow F$ (this means for all n, $F_n \supset F_{n+1}$). Show that $\nu(F) = \lim \nu(F_n)$.

2. Let ν be a charge on $(X, \mathbb{X})$. Show that

   a) $\nu^+(E) = \sup\{\nu(F) : \mathbb{X} \ni F \subseteq E\}$,

   b) $\nu^-(E) = -\inf\{\nu(F) : \mathbb{X} \ni F \subseteq E\}$.

3. Let $(X, \mathbb{X}, \mu)$ be a measure space and $f \in L(X, \mathbb{X}, \mu)$. Let $\nu : \mathbb{X} \to \mathbb{R}$ be the charge given by $\nu(A) = \int_A f \mathrm{d}\mu$. Show that a set E is null with respect to ν if and only if $\mu(E \cap \{x \in X : f(x) \neq 0\}) = 0$.

4. Prove Theorem 8.7 from the lecture notes. That is if $f \in L(X, \mathbb{X}, \mu)$ and $\nu : \mathbb{X} \to \mathbb{R}$ is the charge given by $\nu(A) = \int_A f \mathrm{d}\mu$ then show that the positive and negative variations of ν, are given by $\nu^+(A) = \int_A f^+ \mathrm{d}\mu$ and $\nu^- = \int_A f^- \mathrm{d}\mu$ respectively.

5. Let $\nu(E) = \int_E xe^{-x^2}\mathrm{d}\lambda$ ($E \in \mathbb{B}$, λ is Lebesgue measure). Give a Hahn decomposition of $\mathbb{R}$ with respect to ν.

6. Let ν, μ be σ-finite measures on $(X, \mathbb{X})$ with $\nu << \mu$. Let $f = \frac{d\nu}{d\mu} \in M^+$. Show that for any $g \in M^+$,

$$\int g\, d\nu = \int gf\, d\mu.$$

Hint: Apply Monotone Convergence Theorem to simple functions.

7. Let ν, λ, μ be σ-finite measures on $(X, \mathbb{X})$ with $\nu << \lambda$ and $\lambda << \mu$. Show that $\nu << \mu$ and

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu} \qquad \mu\text{-a.e.}$$

8. Let

$$f(x) = \begin{cases} \sqrt{1-x}, & x \leq 1 \\ 0, & x > 1 \end{cases}$$

and

$$g(x) = \begin{cases} x^2, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

Let

$$\nu(E) = \int_E f\, d\lambda \quad \text{and} \quad \mu(E) = \int_E g\, d\lambda \quad (E \in \mathbb{B})$$

Find the Lebesgue decomposition of ν with respect to μ.

9. Let $(X, \mathbb{X}_0, \mu)$ be a probability space, $f : X \to \mathbb{R}$ an integrable function and $\mathbb{X} \subset \mathbb{X}_0$ a sigma algebra. Show that there exists a function g which is integrable with respect to the measurable space $(X, \mathbb{X})$ and for which any $A \in \mathbb{X}$ satisfies $\int_A f\mathrm{d}\mu = \int_A g\mathrm{d}\mu$. (This function (random variable) is known as the conditional expectation).

Exercise sheet 8

This sheet goes through the proof of the Riesz representation theorem for $p = 1$. Suppose that $(X, \mathbb{X}, \mu)$ is a measure space where μ is sigma finite. For questions 1,2 and 3 suppose that μ is finite. For questions 1 to 4 $G : L_1 \to \mathbb{R}$ is a positive bounded linear functional.

1. Show that if we define $\lambda : \mathbb{X} \to \mathbb{R}$ by

$$\lambda(A) = G(\chi_A)$$

then λ is a measure which is absolutely continuous with respect to μ. (Hint: The hardest part is to show that for disjoint sets $A_n \in \mathbb{X}$ we have $\lambda(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \lambda(A_n)$ use Corollary 7.3 and the functions $\chi_{\cup_{n=1}^{k} A_n}$).

2. Use the Radon-Nikodým Theorem to show that for all measurable simple functions $\phi : X \to \mathbb{R}$ there exists a nonnegative measurable function $g : X \to \mathbb{R}$ such that

$$G(\phi) = \int \phi g \mathrm{d}\mu.$$

3. Let $f \in L_1$ be nonnegative and ϕ_n a monotone increasing sequences of simple functions which converge almost everywhere and in L_1 to f. Show that

$$G(f) = \int f g \mathrm{d}\mu$$

and that this can be extended to all L_1 functions by the linearity of G.

4. Show that the previous result is still true if we suppose that μ is a sigma finite measure.

5. Use Lemma 8.14 and the previous results to extend these results to show that for all bounded linear functionals $G : L_1 \to \mathbb{R}$ there exists $g \in L_\infty$ with $G(f) = \int g f \mathrm{d}\mu$ for all $f \in L_1$ and $\|G\| = \|g\|_\infty$.