The Poisson process

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This little write-up is part of important foundations of probability that were left out of the unit Probability 1 due to lack of time and prerequisites. Here we define the homogeneous Poisson process and prove some of its properties.

We will need the following notion of analysis:

Definition 1 *A function $g : \mathbb{R} \to \mathbb{R}$ is* little ordo *or* little o *of h^k, $g(h) = \mathfrak{o}(h^k)$ as $h \to 0$, if*

$$\lim_{h\to 0} \frac{g(h)}{h^k} = 0.$$

As an example, $h^2 = \mathfrak{o}(h)$, but $h \neq \mathfrak{o}(h)$, and $\sqrt{h} \neq \mathfrak{o}(h)$. A function is $\mathfrak{o}(1)$ as $h \to 0$ if and only if it tends to zero at the origin.

We consider now events (or marks) in time, and restrict ourselves to the simplest case of Poisson processes.

Definition 2 (homogeneous Poisson process) *Let $\lambda > 0$ be fixed, and consider random marks in time that satisfy the following.*

1. *The process is homogeneous in time: the probability of k marks in a given subset of the time line does not depend on where this subset actually is (we can shift it anywhere, probabilities won't change).*
2. *The probability of a mark in a given time interval of length h is $\lambda \cdot h + \mathfrak{o}(h)$.*
3. *The probability of two or more marks in a given time interval of length h is $\mathfrak{o}(h)$.*
4. *The number of marks in disjoint time intervals are independent random variables.*

Then we say that these marks form a homogeneous Poisson(λ) process.

This is a rather natural definition of events that occur independently with some density in time.

Proposition 3 *The number $N(t)$ of marks in a fixed interval of length t of a homogeneous Poisson(λ) process is of Poi($\lambda \cdot t$) distribution.*

Proof. There are various ways of proving this statement, here we do an elementary version. Divide the interval of length t into n pieces of length t/n each, and define, regarding these pieces,

$$\begin{aligned} p :&= \mathbf{P}\{\text{a given piece has precisely 1 mark}\},\\ M(t) :&= \#\{\text{pieces with precisely 1 mark}\}. \end{aligned}$$

Of course, both these quantities will also depend on the total number n of the little pieces. In fact,

$$p = \lambda \cdot \frac{t}{n} + \mathfrak{o}\Big(\frac{1}{n}\Big)$$

by the second property, which actually implies $\lim_{n\to\infty} n \cdot p = \lambda t$. By independence and the homogeneity property, we also know that $M(t) \sim \text{Binom}(n, p)$. The above limit allows us to apply the Poisson approximation of the Binomial distribution, and conclude that for any ℓ,

$$\lim_{n\to\infty} \mathbf{P}\{M(t) = \ell\} = \frac{(\lambda t)^\ell}{\ell!} \mathrm{e}^{-\lambda t}. \tag{1}$$

Now, to examine what this probability has to do with our original problem, write

$$\begin{aligned} \mathbf{P}\{M(t) = \ell\} = \mathbf{P}\{&\ell \text{ pieces have 1 mark, and } n-\ell \text{ pieces have 0 marks}\}\\ + \mathbf{P}\{&\ell \text{ pieces have 1 mark, and at least one piece has at least 2 marks}\}. \end{aligned} \tag{2}$$

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The reason we can do this is that the two lines have exclusive events (we can add the probabilities up), the union of which gives $\{M(t) = \ell\}$ (they add up to the left hand-side). Notice that for the second line we have

$$\begin{aligned}&\mathbf{P}\{\ell \text{ pieces have 1 mark, and at least one piece has at least 2 marks}\}\\ &\qquad\qquad\qquad\qquad\leq \mathbf{P}\{\text{at least one piece has at least 2 marks}\}.\end{aligned} \tag{3}$$

We then do a similar break-up for the quantity $N(t)$ (the total number of marks):

$$\begin{aligned}\mathbf{P}\{N(t) = \ell\} &= \mathbf{P}\{\ell \text{ pieces have 1 mark, and } n-\ell \text{ pieces have 0 marks}\}\\ &\quad + \mathbf{P}\{N(t) = \ell \text{ and at least one piece has at least 2 marks}\}.\end{aligned} \tag{4}$$

Again,

$$\mathbf{P}\{N(t) = \ell \text{ and at least one piece has at least 2 marks}\} \leq \mathbf{P}\{\text{at least one piece has at least 2 marks}\}. \tag{5}$$

Next we show that the common term occurring in both (3) and (5) is negligible. Defining

$$E_i = \{\text{the } i^{\text{th}} \text{ little piece contains at least 2 marks}\}, \qquad i = 1, 2, \ldots, n$$

that common term can be written as

$$\mathbf{P}\{\text{at least one piece has at least 2 marks}\} = \mathbf{P}\Big\{\bigcup_{i=1}^n E_i\Big\} \leq \sum_{i=1}^n \mathbf{P}\{E_i\} = \sum_{i=1}^n \mathfrak{o}\Big(\frac{t}{n}\Big) = \mathfrak{o}(t) \underset{n\to\infty}{\longrightarrow} 0$$

due to Boole's inequality, the third defining property of the Poisson process, and homogeneity. To finish the proof, take the difference of (2) and (4) and notice the cancellation of the first lines therein:

$$\begin{aligned}\mathbf{P}\{M(t) = \ell\} - \mathbf{P}\{N(t) = \ell\} &= \mathbf{P}\{\ell \text{ pieces have 1 mark, and at least one piece has at least 2 marks}\}\\ &\quad - \mathbf{P}\{N(t) = \ell \text{ and at least one piece has at least 2 marks}\}\\ &= \mathfrak{o}(t) - \mathfrak{o}(t) \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

which, together with (1) proves the statement by considering more and more of the little pieces. □

The Poisson process has other nice properties, which we do not detail here. It is of central importance in the theory of stochastic processes. There is also a natural extension to several dimensions.

Example 4 *The Poisson process serves as a model for*

- *earthquakes above a given magnitude in time;*
- *detection times of radioactive particles in a detector;*
- *raindrop impacts on a given area of the pavement;*
- *location of trees in the forest (planar Poisson process);*
- *location of visible stars on the sky.*

Example 5 *A city experiences on average 2 (moderate) earthquakes per week.*

(a) What is the probability that in the next two weeks there will be 3 or more earthquakes?

(b) What is the probability that we have more than time t before the next earthquake?

We use a Poisson process to model the earthquakes in time. According to the example our parameter is $\lambda = 2$ as $N(1) \sim \mathrm{Poi}(1 \cdot \lambda)$ implies $\mathbf{E}N(1) = \lambda = 2$.

(a) We know that in the next two weeks we have $N(2) \sim \mathrm{Poi}(2 \cdot 2)$ many earthquakes, therefore

$$\mathbf{P}\{N(2) \geq 3\} = 1 - \mathbf{P}\{N(2) = 0\} - \mathbf{P}\{N(2) = 1\} - \mathbf{P}\{N(2) = 2\} = 1 - \frac{4^0}{0!}\cdot \mathrm{e}^{-4} - \frac{4^1}{1!}\cdot \mathrm{e}^{-4} - \frac{4^2}{2!}\cdot \mathrm{e}^{-4} \simeq 0.76.$$

(b) Let T be the time left until the next earthquake. Then

$$\mathbf{P}\{T > t\} = \mathbf{P}\{N(t) = 0\} = \mathrm{e}^{-\lambda t} = \mathrm{e}^{-2t}.$$

In general, the waiting time T before the first event satisfies

$$\mathbf{P}\{T \leq t\} = 1 - \mathbf{P}\{T > t\} = 1 - \mathbf{P}\{N(t) = 0\} = 1 - \mathrm{e}^{-\lambda t},$$

in which we recognise the Exponential(λ) distribution. It is a fact that the first event, and then events thereafter follow each other in i.i.d. Exponential(λ) distributed times in a homogeneous Poisson(λ) process.