

Jacobi triple product and the like via the exclusion process and the like

Joint with
Ross Bowen, Dan Fretwell, Jessica Jay

Márton Balázs

University of Bristol

4th Colloquium on Interacting Particle Systems
@Técnico, Lisbon
15 December, 2020.

Jacobi triple product

Theorem

Let $|x| < 1$ and $y \neq 0$ be complex numbers. Then

$$\prod_{i=1}^{\infty} (1 - x^{2i}) \left(1 + \frac{x^{2i-1}}{y^2}\right) (1 + x^{2i-1}y^2) = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

Mostly appears in **number theory** and **combinatorics of partitions**.

We'll prove it using interacting particles (for real x, y only).

Models

- Asymmetric simple exclusion
- Zero range

Blocking measures

State space

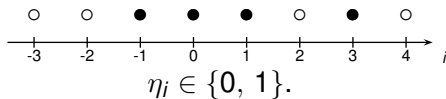
- No boundaries
- Boundaries

Lay down - stand up

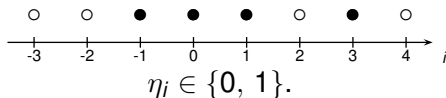
Jacobi triple product

More models

Asymmetric simple exclusion



Asymmetric simple exclusion



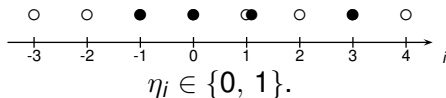
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



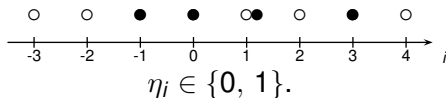
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



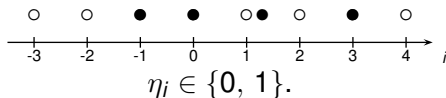
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



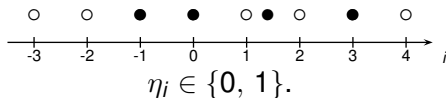
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



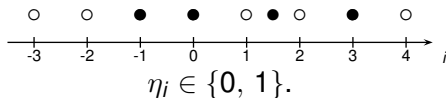
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



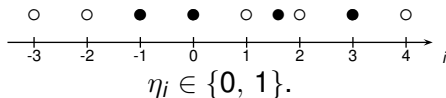
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



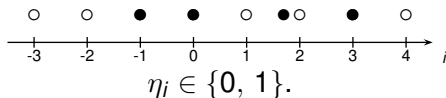
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



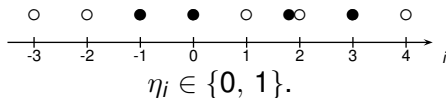
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



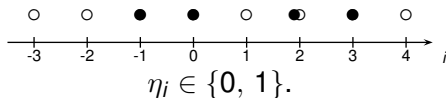
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



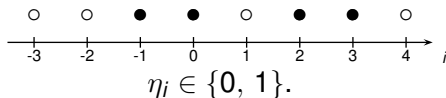
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



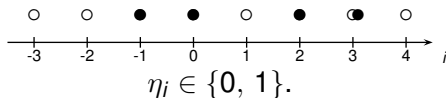
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



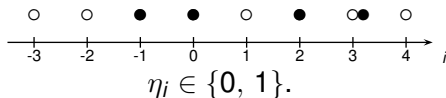
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



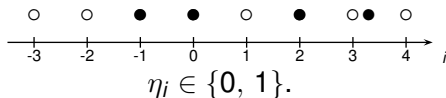
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



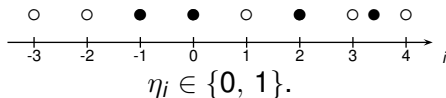
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



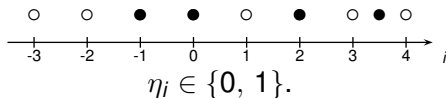
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



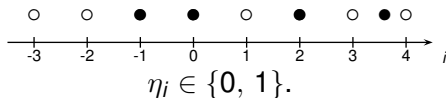
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



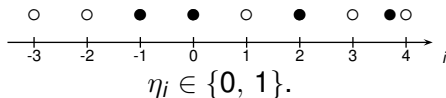
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



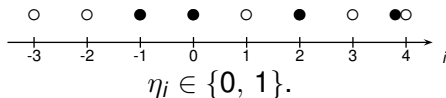
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



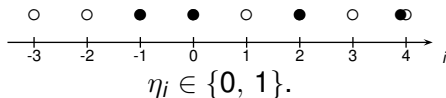
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



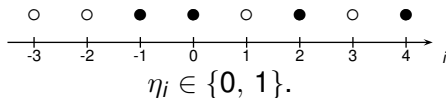
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



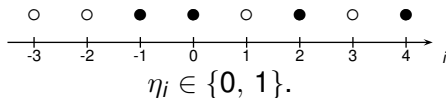
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



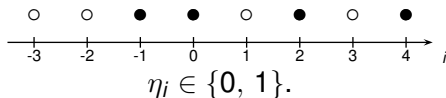
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



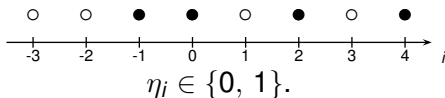
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



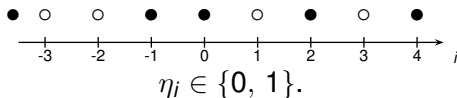
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



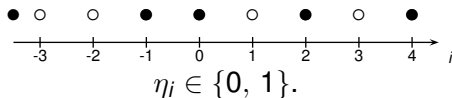
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



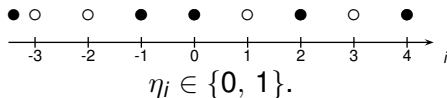
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



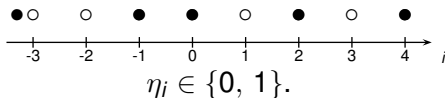
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



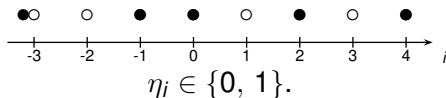
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



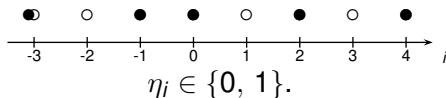
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



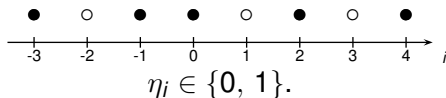
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



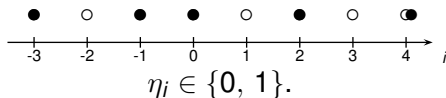
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



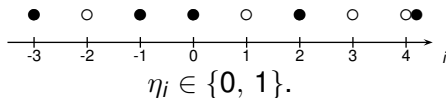
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



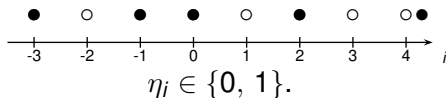
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



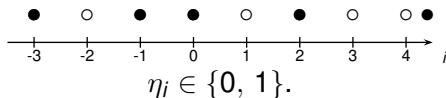
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



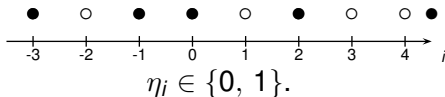
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



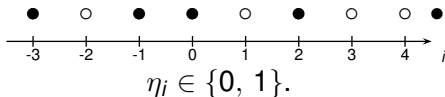
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



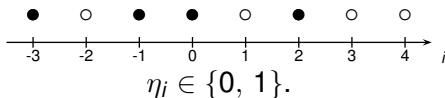
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



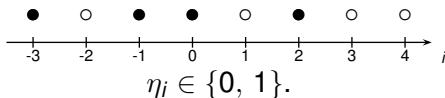
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



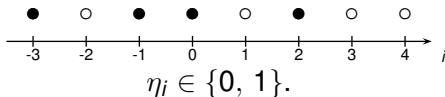
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



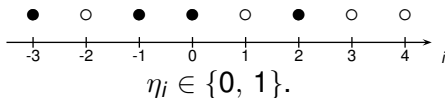
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



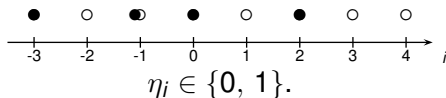
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



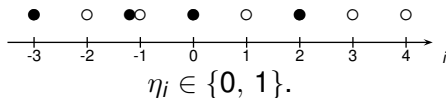
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



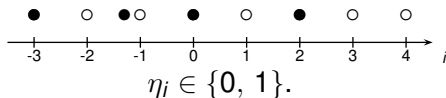
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



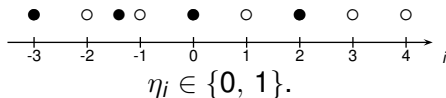
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



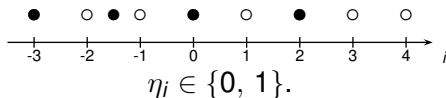
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



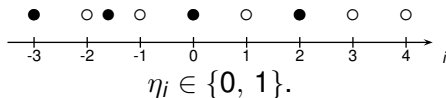
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



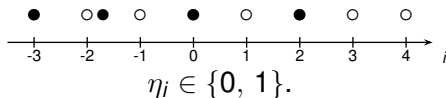
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



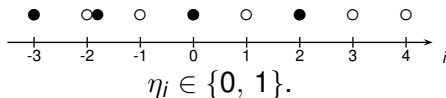
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



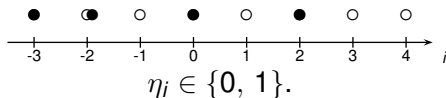
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



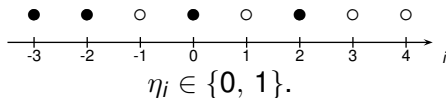
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



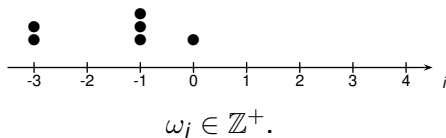
Particles try to jump

to the right with rate p ,

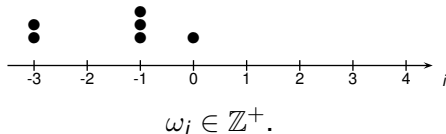
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

The asymmetric zero range process

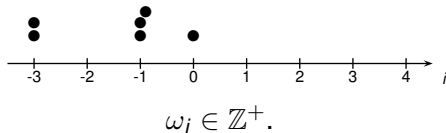


The asymmetric zero range process



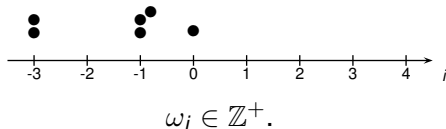
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



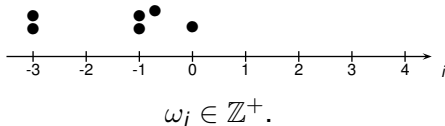
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



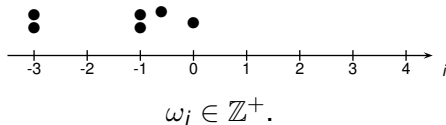
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



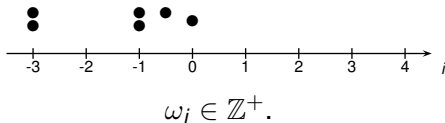
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



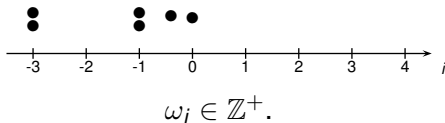
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



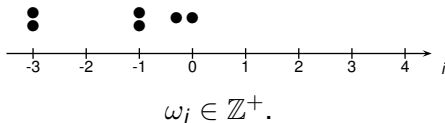
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



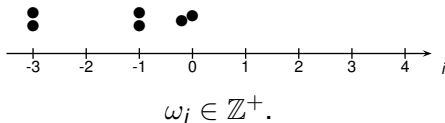
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



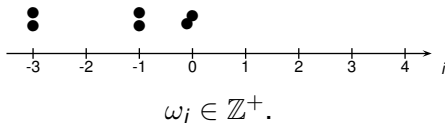
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



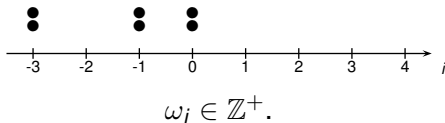
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



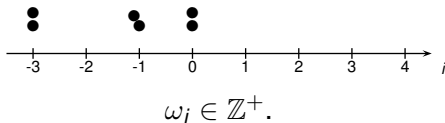
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



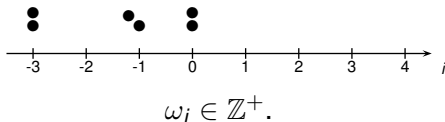
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



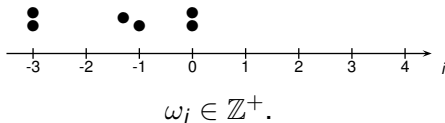
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



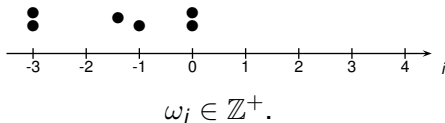
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



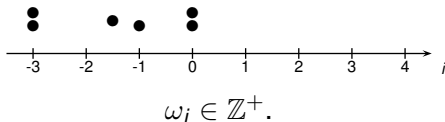
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



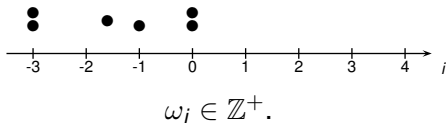
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



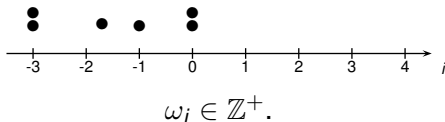
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



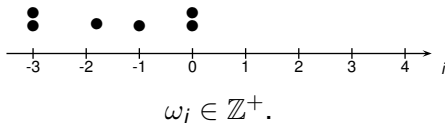
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



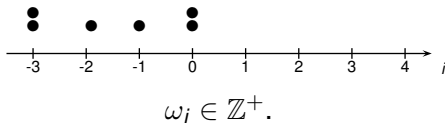
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



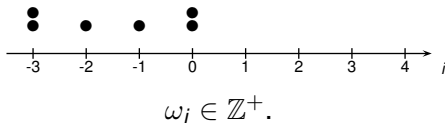
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



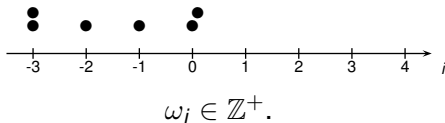
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



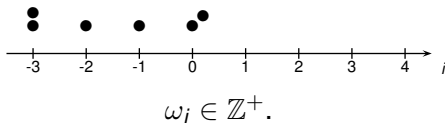
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



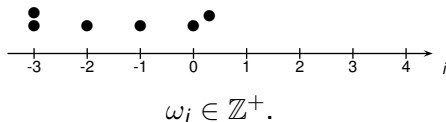
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



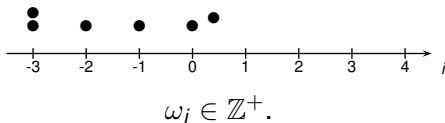
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



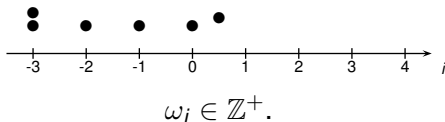
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



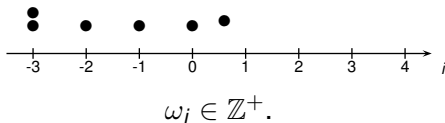
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



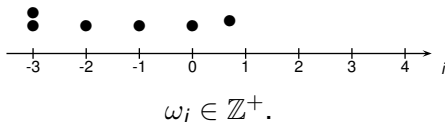
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



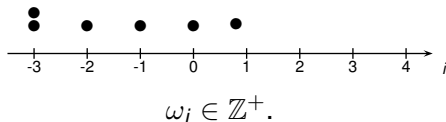
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



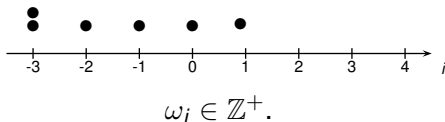
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



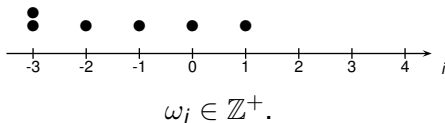
Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process

We need r **non-decreasing** and assume, as before,
 $q = 1 - p < p$.

Examples:

- ▶ 'Classical' ZRP: $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$.
- ▶ Independent walkers: $r(\omega_i) = \omega_i$.

Product blocking measures

Can we have a reversible stationary distribution in product form:

$$\underline{\mu}(\underline{\omega}) = \bigotimes_i \mu_i(\omega_i);$$

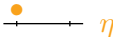
$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \rightsquigarrow i+1}) = \underline{\mu}(\underline{\omega}^{i \rightsquigarrow i+1}) \cdot \text{rate}(\underline{\omega}^{i \rightsquigarrow i+1} \rightarrow \underline{\omega}) \quad ?$$

Here

$$\underline{\omega}^{i \rightsquigarrow i+1} = \underline{\omega} - \underline{\delta}_i + \underline{\delta}_{i+1}.$$

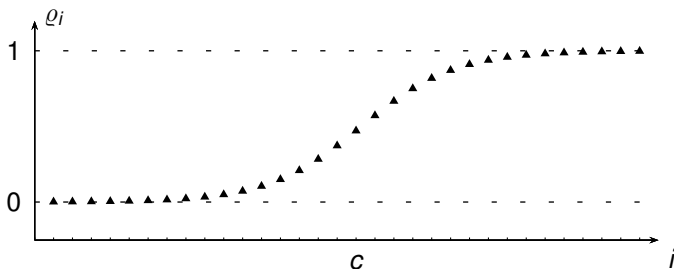
Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}(\underline{\eta} \rightarrow \underline{\eta}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\eta}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\eta}^{i \curvearrowright i+1} \rightarrow \underline{\eta})$$

ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$;  η

$$\varrho_i(1 - \varrho_{i+1}) \cdot p = (1 - \varrho_i)\varrho_{i+1} \cdot q$$

Solution:
$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$



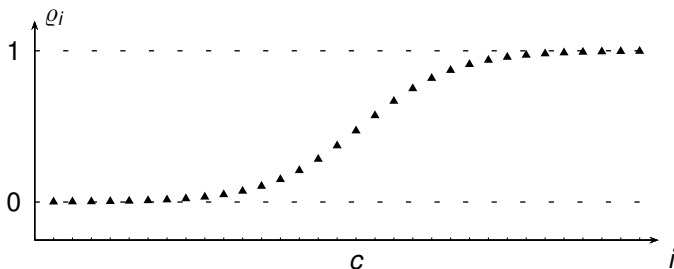
Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}(\underline{\eta} \rightarrow \underline{\eta}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\eta}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\eta}^{i \curvearrowright i+1} \rightarrow \underline{\eta})$$

ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$; $\text{---} \overset{\bullet}{\text{---}} \underline{\eta}^{i \curvearrowright i+1}$

$$\varrho_i(1 - \varrho_{i+1}) \cdot p = (1 - \varrho_i)\varrho_{i+1} \cdot q$$

Solution:
$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$



Asymmetric zero range process

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\omega}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\omega}^{i \curvearrowright i+1} \rightarrow \underline{\omega}) \quad ?$$

AZRP:

$$\mu_i(\omega_i) \mu_{i+1}(\omega_{i+1}) \cdot p \mathbf{1}\{\omega_i > 0\} = \mu_i(\omega_i - 1) \mu_{i+1}(\omega_{i+1} + 1) \cdot q$$

Solution: $\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i - \text{const}}\right).$

State space: ASEP

Notice:

$$\mathbf{P}\{\eta_i = 0\} = 1 - \varrho_i = \frac{1}{1 + (\frac{p}{q})^{i-c}} \quad \text{as } i \rightarrow \infty$$

$$\mathbf{P}\{\eta_i = 1\} = \varrho_i = \frac{1}{(\frac{q}{p})^{i-c} + 1} \quad \text{as } i \rightarrow -\infty$$

are both summable. Hence by Borel-Cantelli there is $\underline{\mu}$ -a.s. a rightmost hole and a leftmost particle,

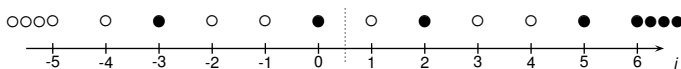
$$N := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

is $\underline{\mu}$ -a.s. finite.

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

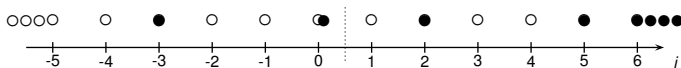
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

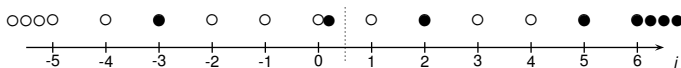
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

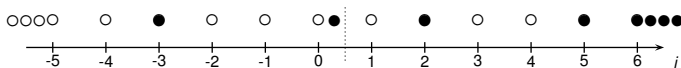
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

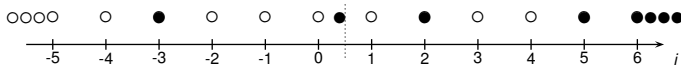
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

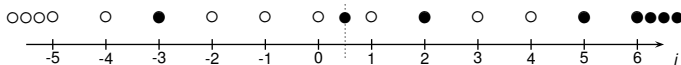
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

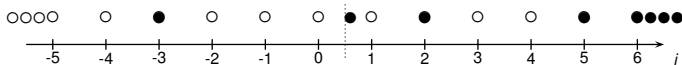
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

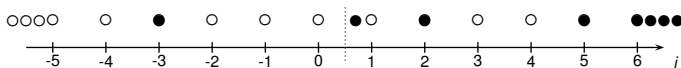
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

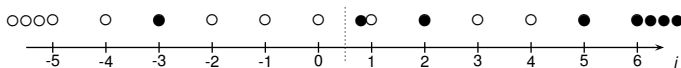
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

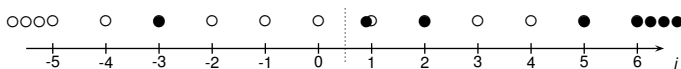
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

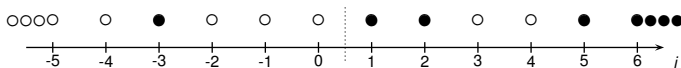
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

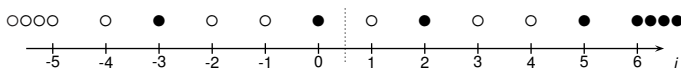
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

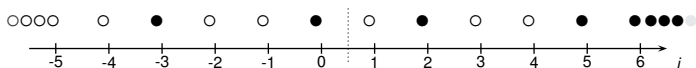
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

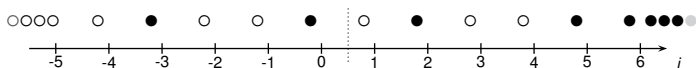
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

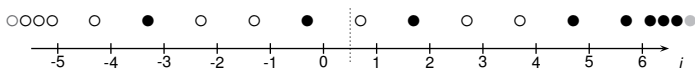
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

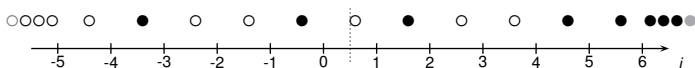
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

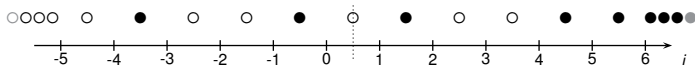
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

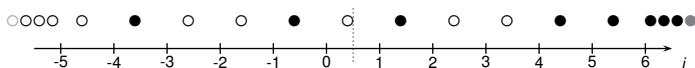
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

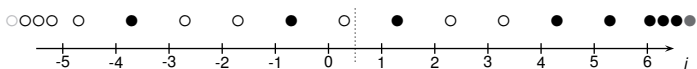
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

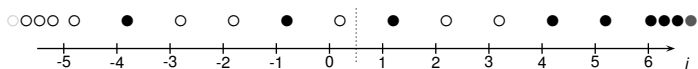
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

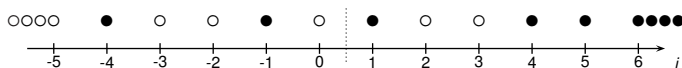
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

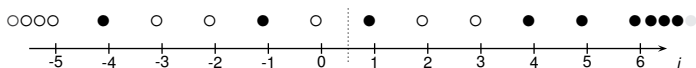
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

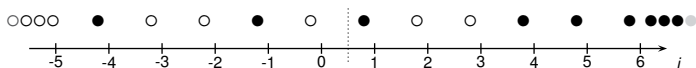
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

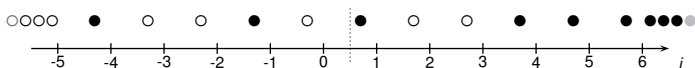
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

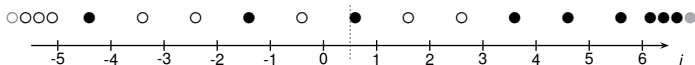
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

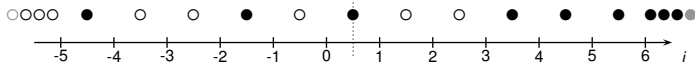
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

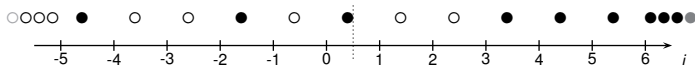
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

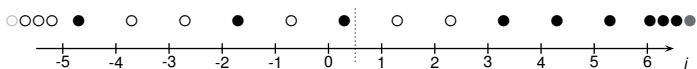
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

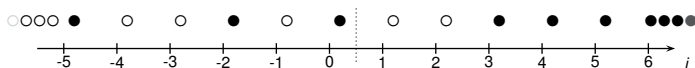
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

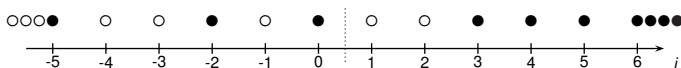
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$

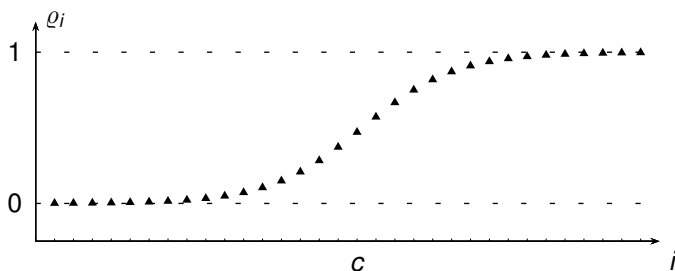


$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Recall

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}},$$

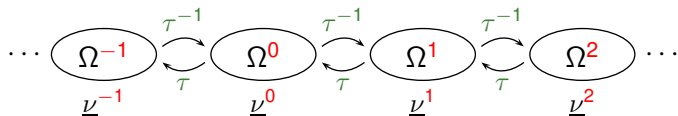


but

$$\underline{\nu}^n(\cdot) := \underline{\mu}(\cdot \mid N(\cdot) = n)$$

doesn't depend on c anymore. **Stationary measure on Ω^n .**

State space: ASEP



$$\underline{\mu}(\cdot) = \sum_{n=-\infty}^{\infty} \underline{\mu}(\cdot | N(\cdot) = n) \underline{\mu}(N(\cdot) = n) = \sum_{n=-\infty}^{\infty} \underline{\nu}^n(\cdot) \underline{\mu}(N(\cdot) = n).$$

Ergodic decomposition of $\underline{\mu}$.

Let's find the coefficients $\underline{\mu}(N(\cdot) = n)$!

State space: ASEP

Recall:

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$\underline{\mu}(\underline{\eta}) = \prod_{i \leq 0} \frac{\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{1 + \left(\frac{p}{q}\right)^{i-c}} \cdot \prod_{i > 0} \frac{\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

Recall:

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$\underline{\mu}(\underline{\eta}) = \prod_{i \leq 0} \frac{\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{1 + \left(\frac{p}{q}\right)^{i-c}} \cdot \prod_{i > 0} \frac{\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

$$\underline{\mu}(\underline{\tau\eta}) = \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}
 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau}\underline{\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{-c}
 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{-c} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}
 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{-c} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c} \\
 &= \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}.
 \end{aligned}$$

State space: ASEP

So,

$$\underline{\mu}(N = n - 1) = \sum_{\underline{\eta}: N(\underline{\eta}) = n - 1} \underline{\mu}(\underline{\eta})$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\tau}\underline{\eta})=n-1} \underline{\mu}(\underline{\tau}\underline{\eta})\end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\tau}\underline{\eta})=n-1} \underline{\mu}(\underline{\tau}\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c}\end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c}\end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c} \\ &= \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.\end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}
 \underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c} \\
 &= \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.
 \end{aligned}$$

Solution:

$$\underline{\mu}(N = n) = \frac{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}$$

discrete Gaussian.

State space: ASEP

and, if $N(\underline{\eta}) = n$,

$$\begin{aligned} \underline{\nu}^n(\underline{\eta}) &= \frac{\underline{\mu}(\underline{\eta} \mid N(\underline{\eta}) = n)}{\mu(N(\underline{\eta}) = n)} \\ &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}. \end{aligned}$$

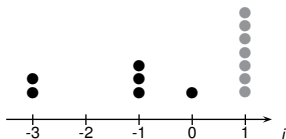
This is the unique stationary distribution on Ω^n .

State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

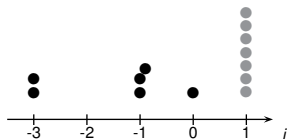


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

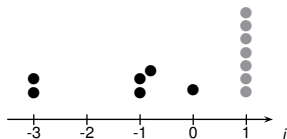


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

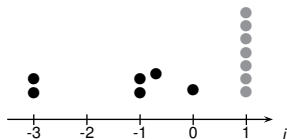


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

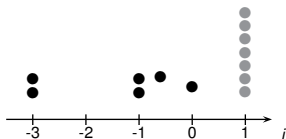


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

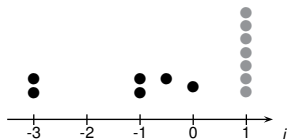


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

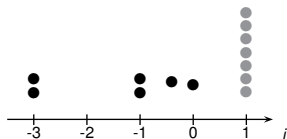


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

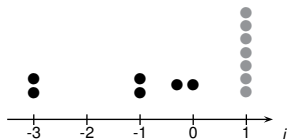


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

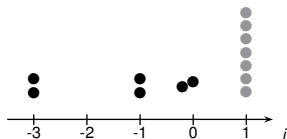


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

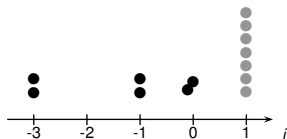


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

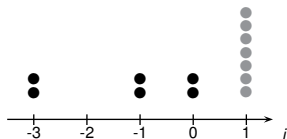


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

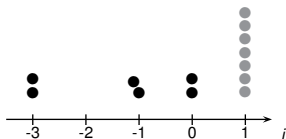


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

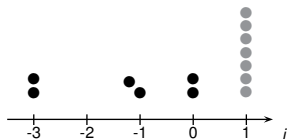


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

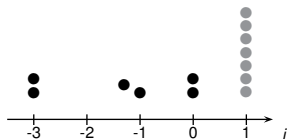


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

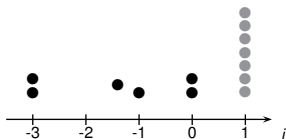


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

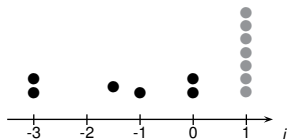


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

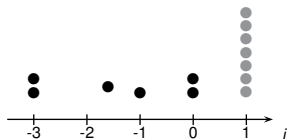


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

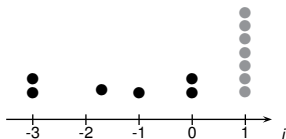


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

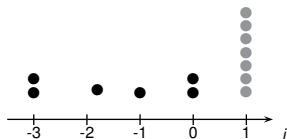


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

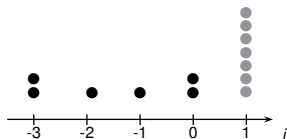


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

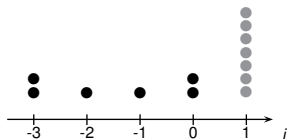


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

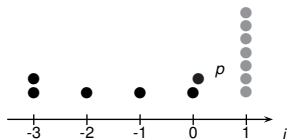


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

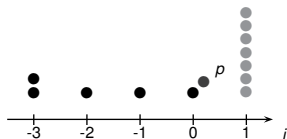


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

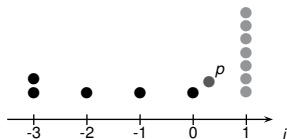


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

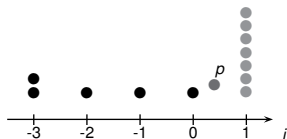


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

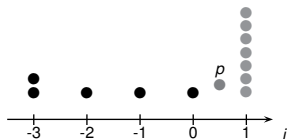


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

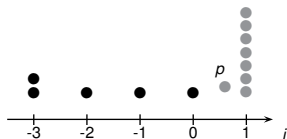


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

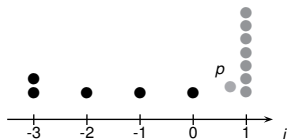


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

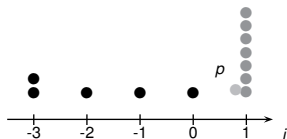


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

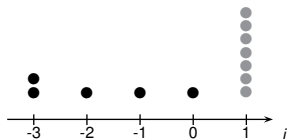


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

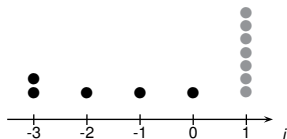


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

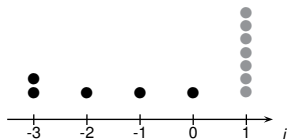


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

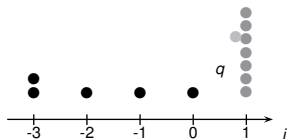


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

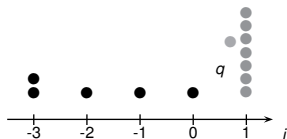


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

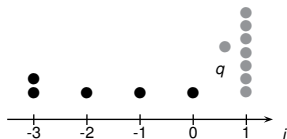


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

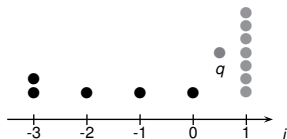


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

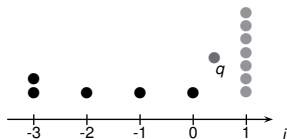


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

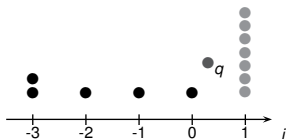


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

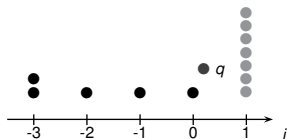


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

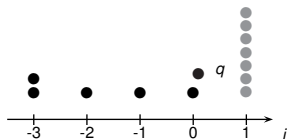


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

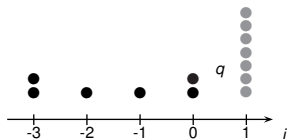


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

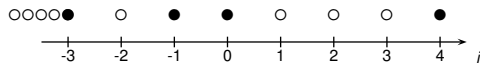
\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.



\rightsquigarrow The product measure stays stationary on the half-line.

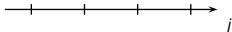
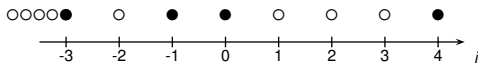
Lay down / stand up

ASEP



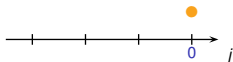
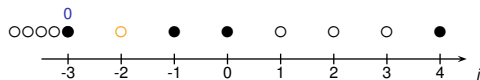
Lay down / stand up

ASEP



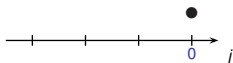
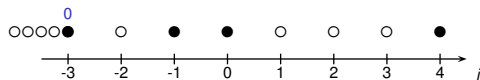
Lay down / stand up

ASEP



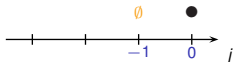
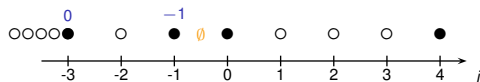
Lay down / stand up

ASEP



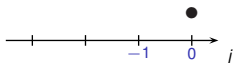
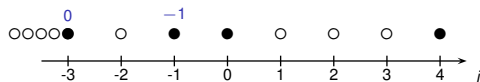
Lay down / stand up

ASEP



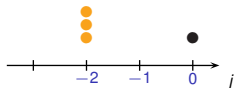
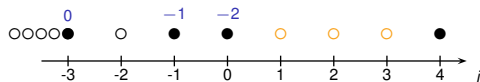
Lay down / stand up

ASEP



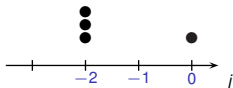
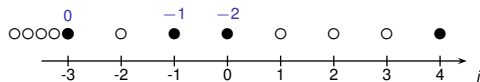
Lay down / stand up

ASEP



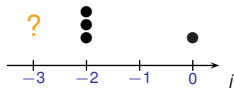
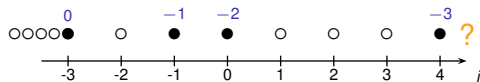
Lay down / stand up

ASEP



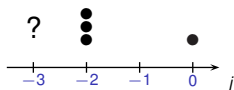
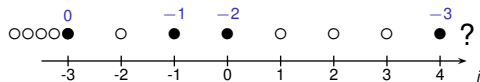
Lay down / stand up

ASEP



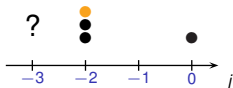
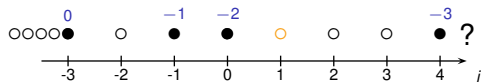
Lay down / stand up

ASEP



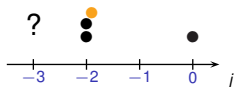
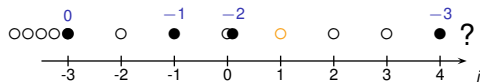
Lay down / stand up

ASEP



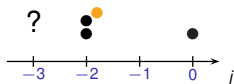
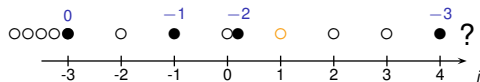
Lay down / stand up

ASEP



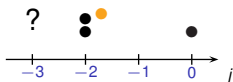
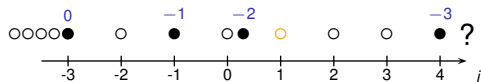
Lay down / stand up

ASEP



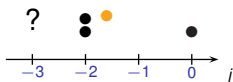
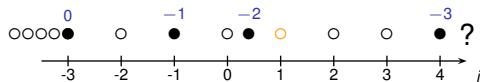
Lay down / stand up

ASEP



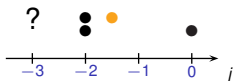
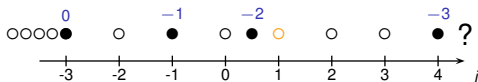
Lay down / stand up

ASEP



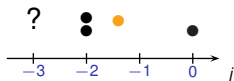
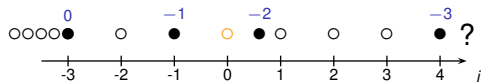
Lay down / stand up

ASEP



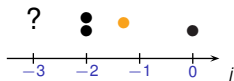
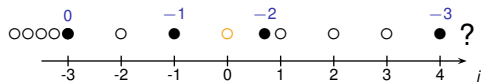
Lay down / stand up

ASEP



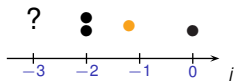
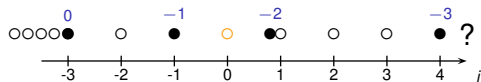
Lay down / stand up

ASEP



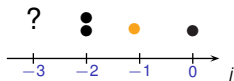
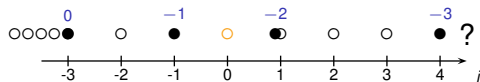
Lay down / stand up

ASEP



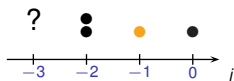
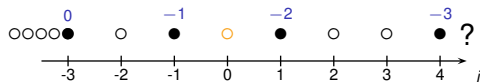
Lay down / stand up

ASEP



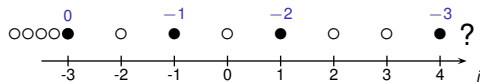
Lay down / stand up

ASEP

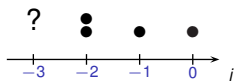


Lay down / stand up

ASEP

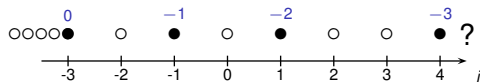


AZRP

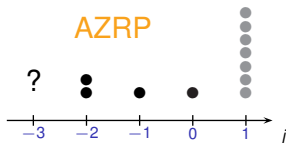


Lay down / stand up

ASEP

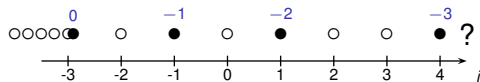


AZRP

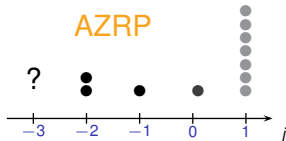


Lay down / stand up

ASEP

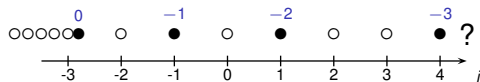


AZRP

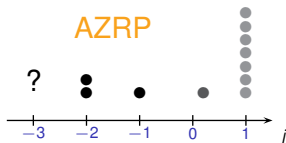


Lay down / stand up

ASEP

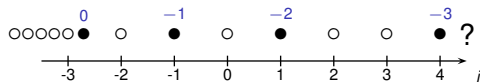


AZRP

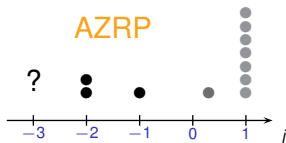


Lay down / stand up

ASEP

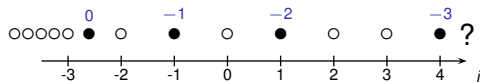


AZRP

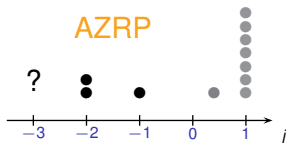


Lay down / stand up

ASEP

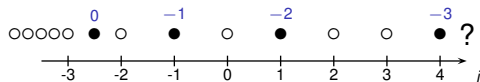


AZRP

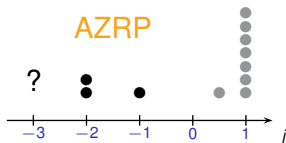


Lay down / stand up

ASEP

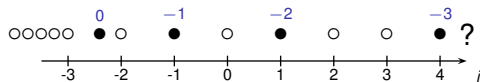


AZRP

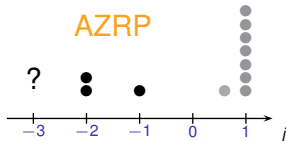


Lay down / stand up

ASEP

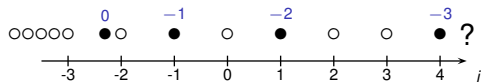


AZRP

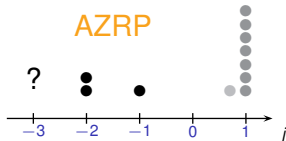


Lay down / stand up

ASEP

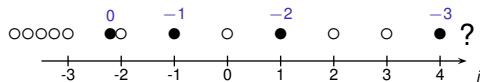


AZRP

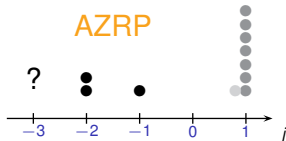


Lay down / stand up

ASEP

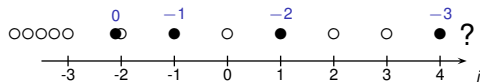


AZRP

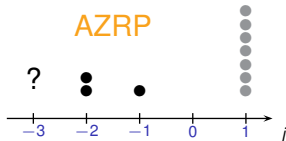


Lay down / stand up

ASEP

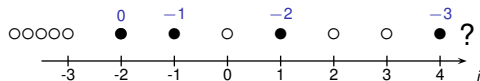


AZRP

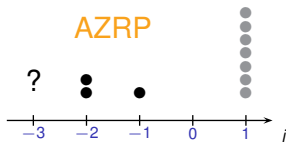


Lay down / stand up

ASEP



AZRP

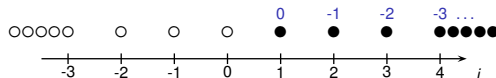
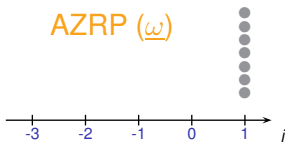


$$\text{ASEP} \stackrel{T^n}{=} \text{AZRP}$$

$$\underline{\nu}^n \stackrel{T^n}{=} \prod_{i \leq 0} \text{Geometric} \left(1 - \left(\frac{p}{q} \right)^{i-1} \right)$$

since stationary distributions of countable irreducible Markov chains are unique.

Jacobi triple product

ASEP ($\underline{\eta}$)AZRP ($\underline{\omega}$)

$$\eta_j = \mathbf{1}\{j \geq 1\}, \quad N(\underline{\eta}) = 0, \quad \omega_j \equiv 0.$$

$$\underline{\nu}^0(\underline{\eta}) = \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c) \cdot 0}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c) \cdot (1-1)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{0^2+0}{2} - c \cdot 0}}$$

$$\underline{\mu}(\underline{\omega}) = \prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right)$$

Jacobi triple product

$$\prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right) \cdot \prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right) \cdot \prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) = \sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}$$

LHS:

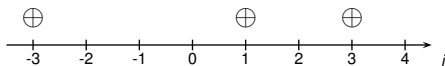
$$\begin{aligned} \prod_{i=1}^{\infty} \left(1 - \left(\frac{q}{p}\right)^i\right) \cdot \left(1 + \left(\frac{q}{p}\right)^{i-1+c}\right) \cdot \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) \\ = \prod_{i=1}^{\infty} \left(1 - x^{2i}\right) \left(1 + \frac{x^{2i-1}}{y^2}\right) \left(1 + x^{2i-1}y^2\right) \end{aligned}$$

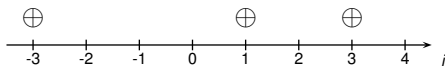
with $x = \left(\frac{q}{p}\right)^{\frac{1}{2}}$, $y = \left(\frac{q}{p}\right)^{\frac{1}{4}-\frac{c}{2}}$.

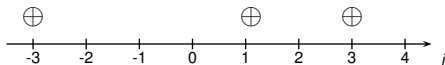
RHS:

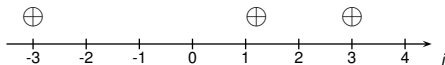
$$\sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2}{2}} \left(\frac{q}{p}\right)^{m\left(\frac{1}{2}-c\right)} = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

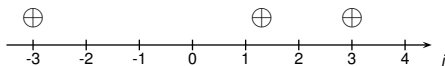


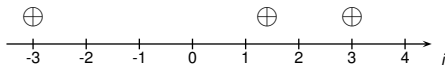
\oplus \ominus \emptyset models

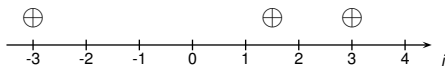
\oplus \ominus \emptyset models \oplus to the right

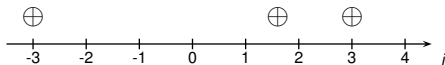
\oplus \ominus \emptyset models \oplus to the right

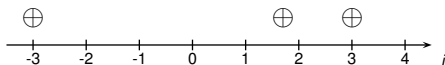
\oplus \ominus \emptyset models \oplus to the right

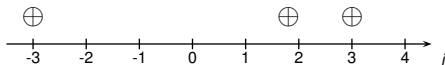
\oplus \ominus \emptyset models \oplus to the right

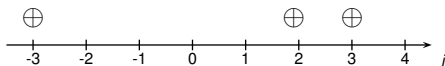
\oplus \ominus \emptyset models \oplus to the right

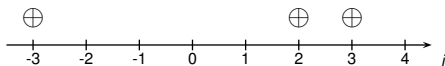
\oplus \ominus \emptyset models \oplus to the right

\oplus \ominus \emptyset models \oplus to the right

\oplus \ominus \emptyset models \oplus to the right

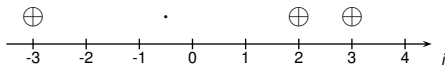
\oplus \ominus \emptyset models \oplus to the right

\oplus \ominus \emptyset models \oplus to the right

\oplus \ominus \emptyset models \oplus to the right

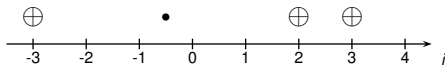
\oplus \ominus \emptyset models

pair creation from vacuum



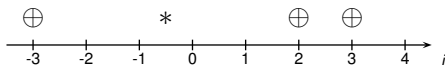
\oplus \ominus \emptyset models

pair creation from vacuum



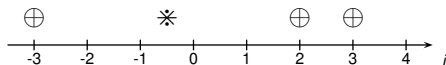
\oplus \ominus \emptyset models

pair creation from vacuum



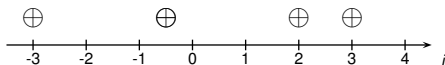
\oplus \ominus \emptyset models

pair creation from vacuum



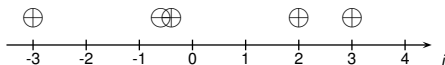
\oplus \ominus \emptyset models

pair creation from vacuum



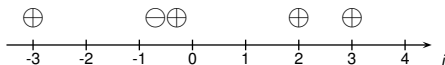
\oplus \ominus \emptyset models

pair creation from vacuum



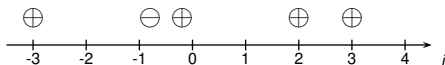
\oplus \ominus \emptyset models

pair creation from vacuum



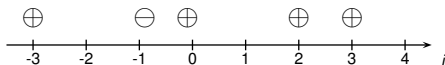
\oplus \ominus \emptyset models

pair creation from vacuum



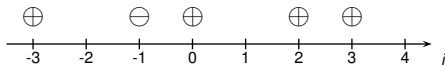
\oplus \ominus \emptyset models

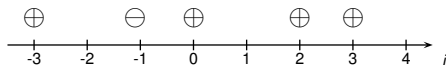
pair creation from vacuum



\oplus \ominus \emptyset models

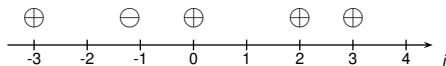
pair creation from vacuum

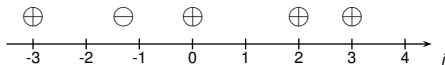


\oplus \ominus \emptyset models \ominus to the left

\oplus \ominus \emptyset models

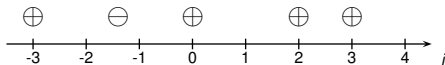
\ominus to the left

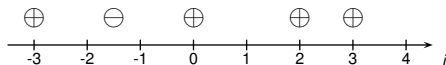


\oplus \ominus \emptyset models \ominus to the left

\oplus \ominus \emptyset models

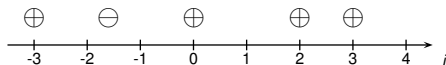
\ominus to the left



\oplus \ominus \emptyset models \ominus to the left

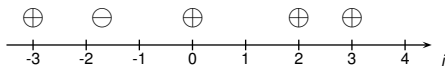
\oplus \ominus \emptyset models

\ominus to the left



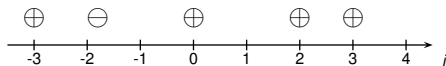
\oplus \ominus \emptyset models

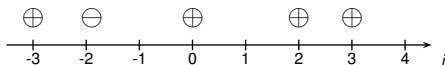
\ominus to the left

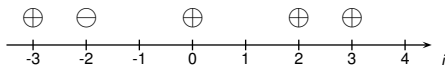


\oplus \ominus \emptyset models

\ominus to the left

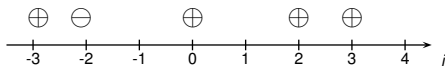


\oplus \ominus \emptyset models \ominus to the left

\oplus \ominus \emptyset models \ominus to the left

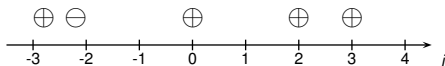
\oplus \ominus \emptyset models

annihilation



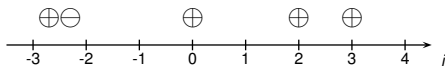
\oplus \ominus \emptyset models

annihilation



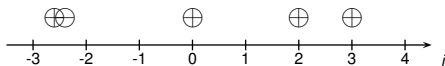
\oplus \ominus \emptyset models

annihilation



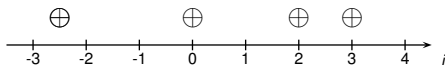
\oplus \ominus \emptyset models

annihilation



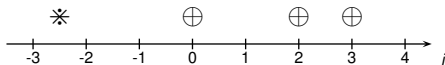
\oplus \ominus \emptyset models

annihilation



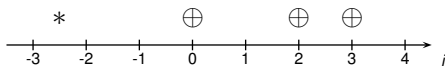
\oplus \ominus \emptyset models

annihilation



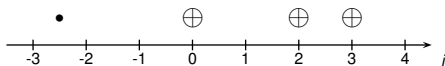
\oplus \ominus \emptyset models

annihilation



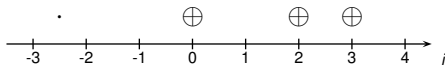
\oplus \ominus \emptyset models

annihilation



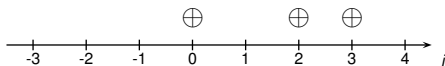
\oplus \ominus \emptyset models

annihilation



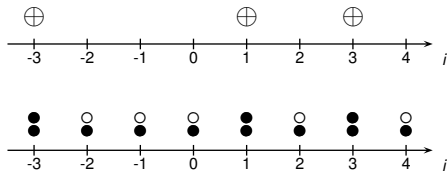
\oplus \ominus \emptyset models

annihilation



\oplus \ominus \emptyset models

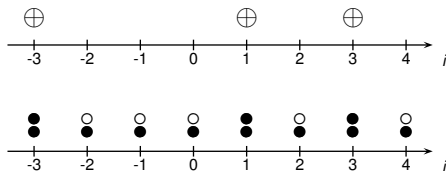
$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

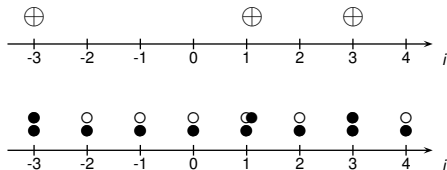
\oplus to the right



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

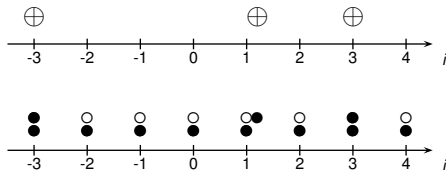
\oplus to the right



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

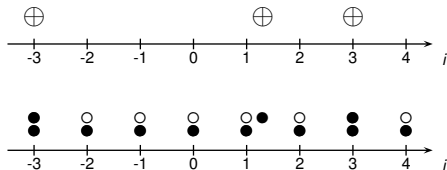
\oplus to the right



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

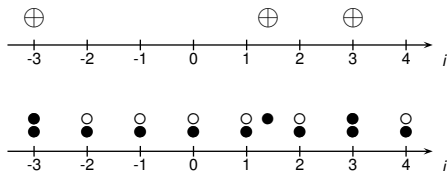
\oplus to the right



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

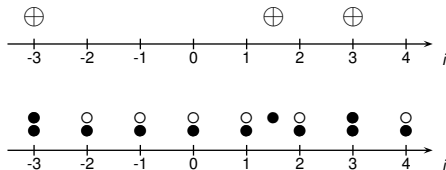
\oplus to the right



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

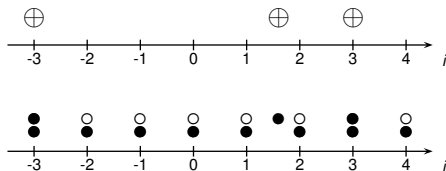
\oplus to the right



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

\oplus to the right

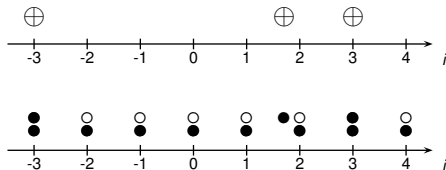


\oplus \ominus \emptyset models

$\oplus \rightsquigarrow \bullet$

$\emptyset \rightsquigarrow \circ$

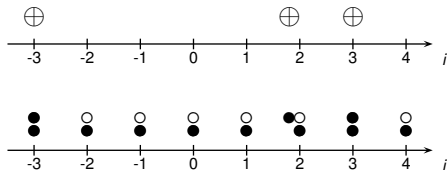
$\ominus \rightsquigarrow \circ$

 \oplus to the right


\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

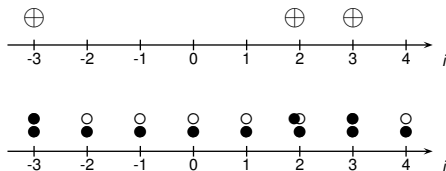
\oplus to the right



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

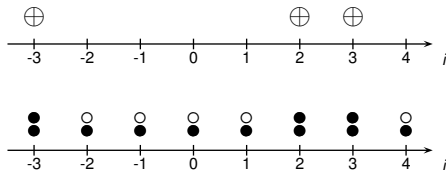
\oplus to the right



\oplus \ominus \emptyset models

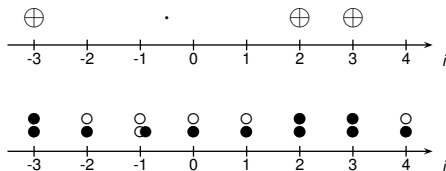
$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

\oplus to the right



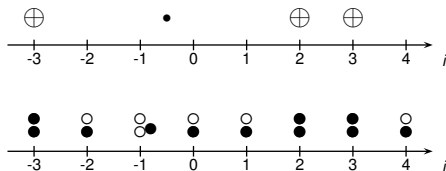
\oplus \ominus \emptyset models


pair creation from vacuum



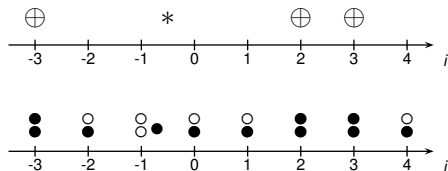
\oplus \ominus \emptyset models


pair creation from vacuum



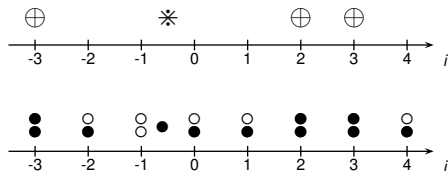
\oplus \ominus \emptyset models


pair creation from vacuum



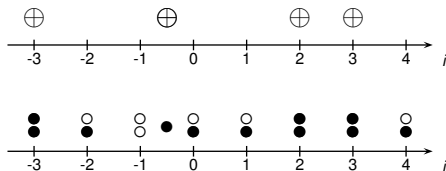
\oplus \ominus \emptyset models


pair creation from vacuum



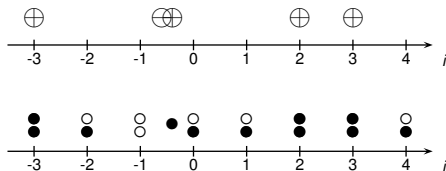
\oplus \ominus \emptyset models


pair creation from vacuum



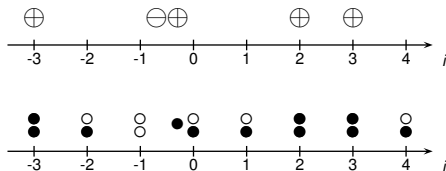
\oplus \ominus \emptyset models


pair creation from vacuum



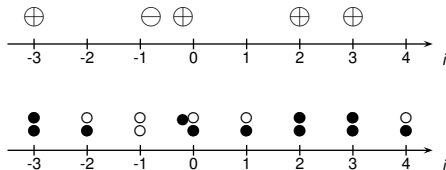
\oplus \ominus \emptyset models


pair creation from vacuum



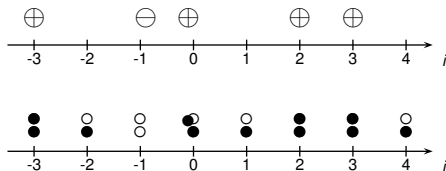
\oplus \ominus \emptyset models


pair creation from vacuum



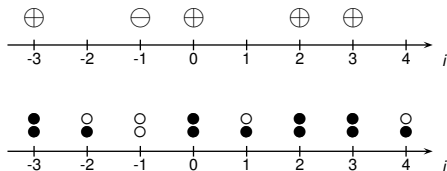
\oplus \ominus \emptyset models


pair creation from vacuum



\oplus \ominus \emptyset models

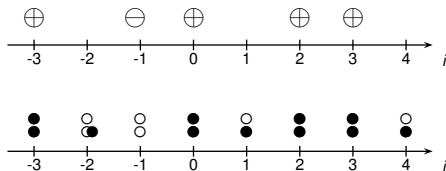

pair creation from vacuum



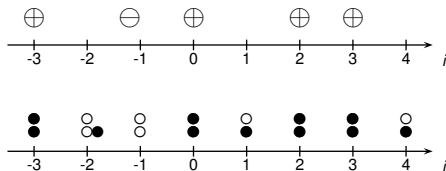
\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

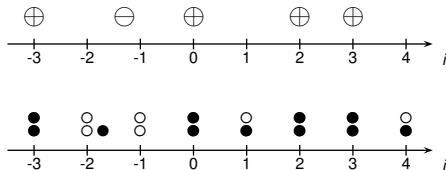
\ominus to the left



\oplus \ominus \emptyset models

 \ominus to the left


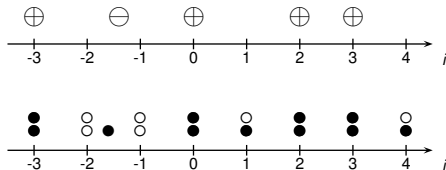
\oplus \ominus \emptyset models

 \ominus to the left


\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

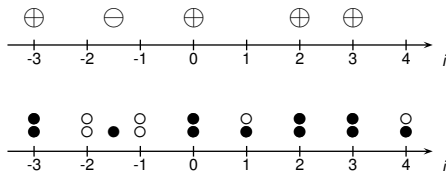
\ominus to the left



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

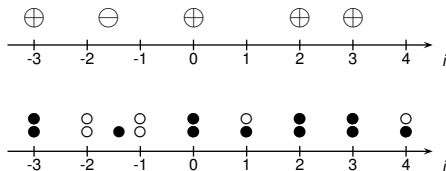
\ominus to the left



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

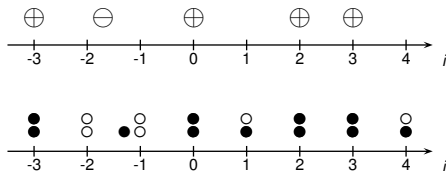
\ominus to the left



\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

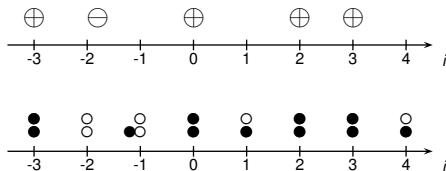
\ominus to the left



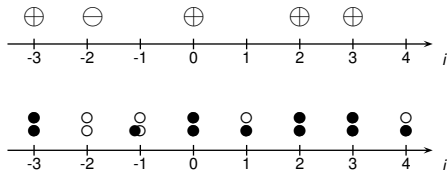
\oplus \ominus \emptyset models

$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

\ominus to the left



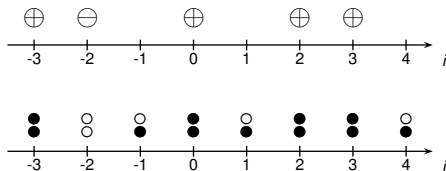
\oplus \ominus \emptyset models

 \ominus to the left


\oplus \ominus \emptyset models

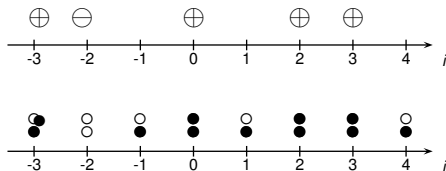
$$\oplus \rightsquigarrow \bullet \bullet \quad \emptyset \rightsquigarrow \circ \bullet \quad \ominus \rightsquigarrow \circ \circ$$

\ominus to the left



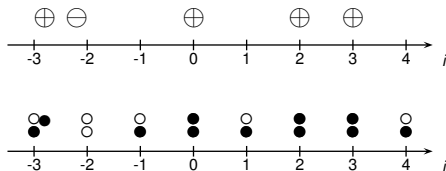
\oplus \ominus \emptyset models


annihilation



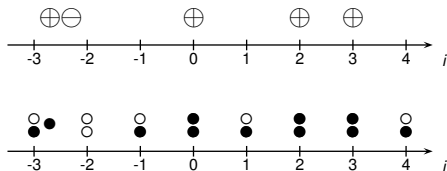
\oplus \ominus \emptyset models


annihilation



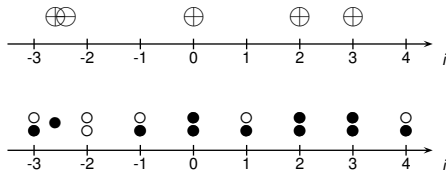
\oplus \ominus \emptyset models


annihilation



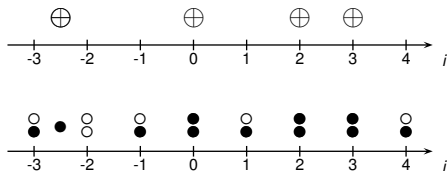
\oplus \ominus \emptyset models


annihilation



\oplus \ominus \emptyset models


annihilation



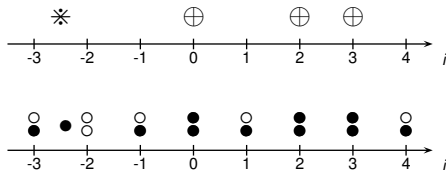
\oplus \ominus \emptyset models

$\oplus \rightsquigarrow \bullet$

$\emptyset \rightsquigarrow \circ$

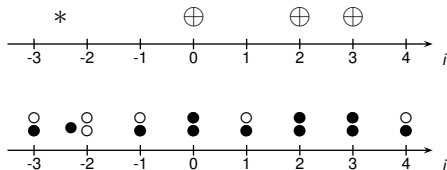
$\ominus \rightsquigarrow \circ$

annihilation



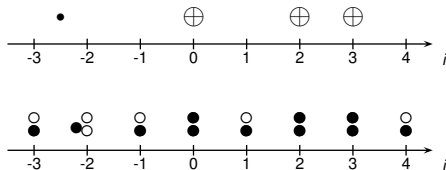
\oplus \ominus \emptyset models


annihilation



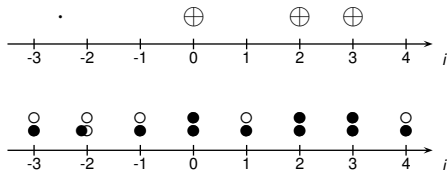
\oplus \ominus \emptyset models


annihilation



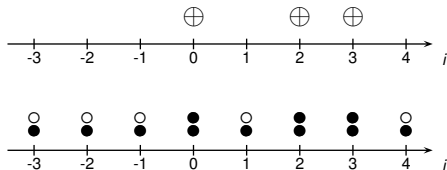
\oplus \ominus \emptyset models

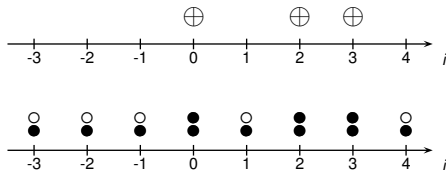

annihilation



\oplus \ominus \emptyset models

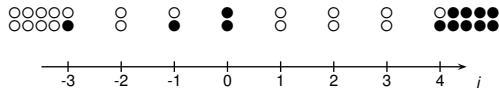

annihilation



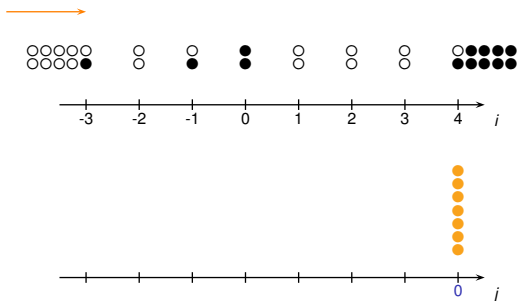
\oplus \ominus \emptyset models


... and the same steps to the left.

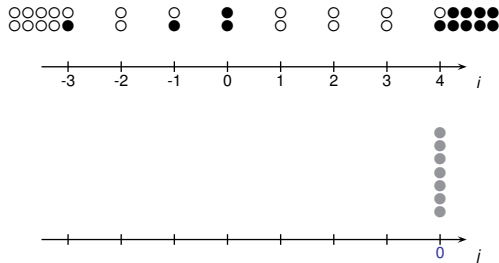
Lay down / stand up a bit differently



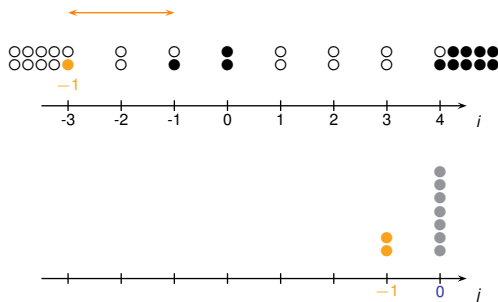
Lay down / stand up a bit differently



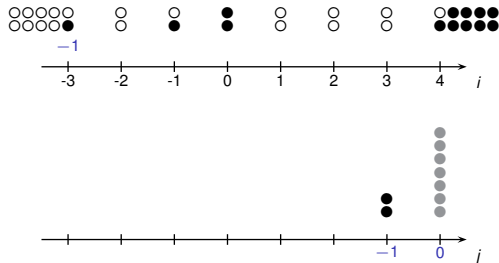
Lay down / stand up a bit differently



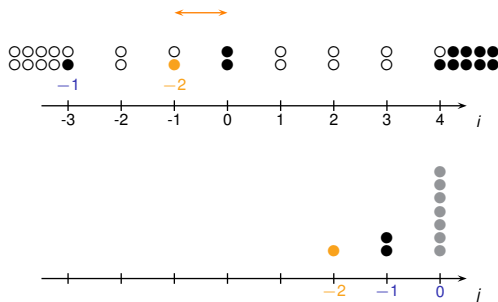
Lay down / stand up a bit differently



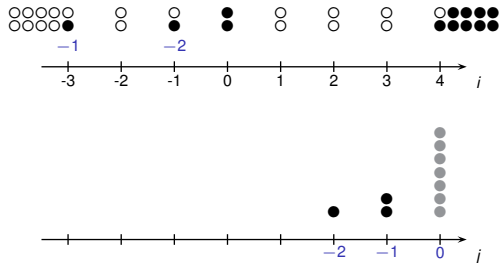
Lay down / stand up a bit differently



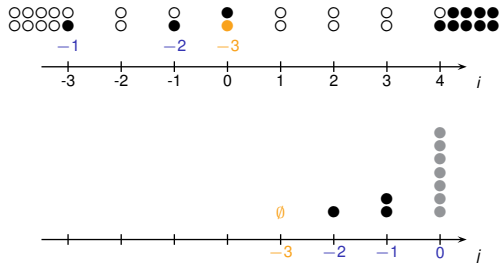
Lay down / stand up a bit differently



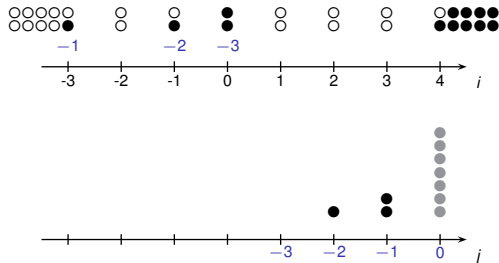
Lay down / stand up a bit differently



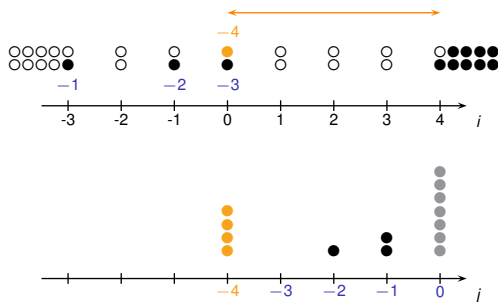
Lay down / stand up a bit differently



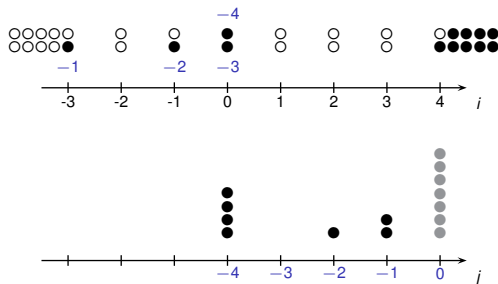
Lay down / stand up a bit differently



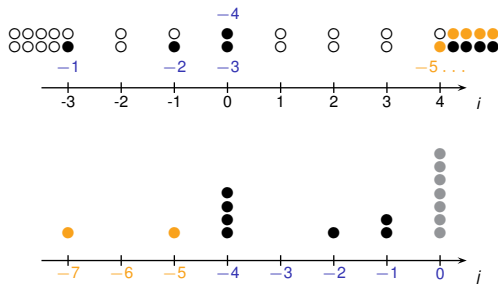
Lay down / stand up a bit differently



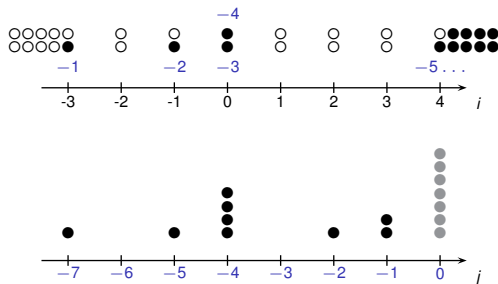
Lay down / stand up a bit differently



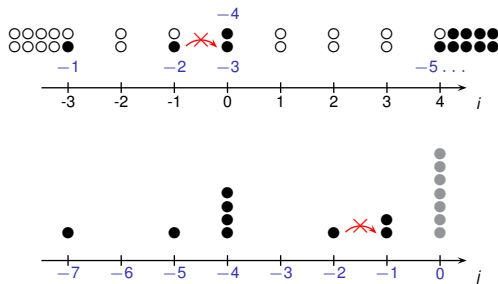
Lay down / stand up a bit differently



Lay down / stand up a bit differently



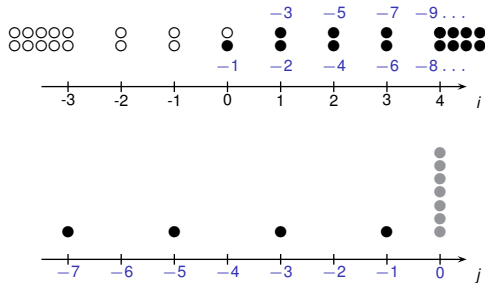
Lay down / stand up a bit differently



No two consecutive 0's!

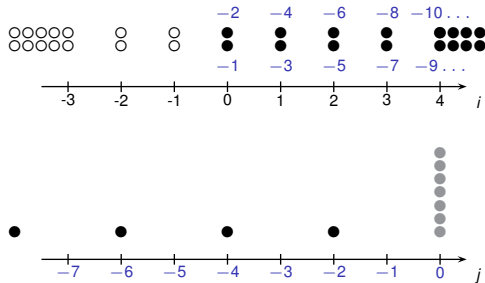
Lay down / stand up a bit differently

Odd ground state:



Lay down / stand up a bit differently

Even ground state:



Lay down / stand up a bit differently

- ▶ *No two consecutive 0's!* is a nonlocal constraint.

Lay down / stand up a bit differently

- ▶ *No two consecutive 0's!* is a nonlocal constraint.
- ▶ The stood up model is nice otherwise. It has reversible product blocking measures.

Lay down / stand up a bit differently

- ▶ *No two consecutive 0's!* is a nonlocal constraint.
- ▶ The stood up model is nice otherwise. It has reversible product blocking measures.
- ▶ *Reversible* measures survive forbidden jumps. Yay!

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

ASEP($q, 1$) Carinci, Giardiná, Redig, Sasamoto:

Jump	Rate	Jump	Rate
$\oplus \rightarrow$	$q^{-1} + q^{-3}$	$\leftarrow \oplus$	$q + q^3$
$\leftarrow \ominus$	$q^{-1} + q^{-3}$	$\ominus \rightarrow$	$q + q^3$
$\emptyset \rightsquigarrow \ominus \oplus$	q^{-3}	$\emptyset \rightsquigarrow \oplus \ominus$	q^3
$\oplus \ominus \rightsquigarrow \emptyset$	$(1 + q^2)(q^{-1} + q^{-3})$	$\ominus \oplus \rightsquigarrow \emptyset$	$(1 + q^{-2})(q + q^3)$

Identity: *Has to do with the odd and even terms of Jacobi's triple product.*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

A nice three-state model:

Jump	Rate	Jump	Rate
$\oplus \rightarrow$	1	$\leftarrow \oplus$	q
$\leftarrow \ominus$	1	$\ominus \rightarrow$	q
$\emptyset \rightsquigarrow \ominus \oplus$	c	$\emptyset \rightsquigarrow \oplus \ominus$	qc
$\oplus \ominus \rightsquigarrow \emptyset$	2	$\ominus \oplus \rightsquigarrow \emptyset$	$2q$

Identity: *Has to do with the square of Jacobi's triple product.*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

2-exclusion:

Jump	Rate	Jump	Rate
$\oplus \rightarrow$	1	$\leftarrow \oplus$	q
$\leftarrow \ominus$	1	$\ominus \rightarrow$	q
$\emptyset \rightsquigarrow \ominus \oplus$	1	$\emptyset \rightsquigarrow \oplus \ominus$	q
$\oplus \ominus \rightsquigarrow \emptyset$	1	$\ominus \oplus \rightsquigarrow \emptyset$	q

Identity: *Looks new and interesting...*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

K-exclusion:

Identity: *Rather nice generalisation using the K^{th} roots of unity.*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

K-exclusion:

Identity: *Rather nice generalisation using the K^{th} roots of unity.*

Zero range:

Identity: *The geometric sum formula. :-D*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

K-exclusion:

Identity: *Rather nice generalisation using the K^{th} roots of unity.*

Zero range:

Identity: *The geometric sum formula. :-D*

Thank you.