A gentle introduction to the Exclusion Process: traffic jams, hydrodynamics and fluctuations

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Arriving to a traffic jam

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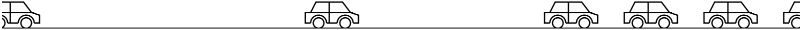

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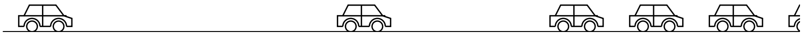

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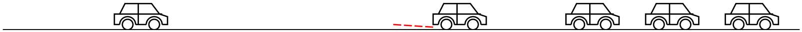

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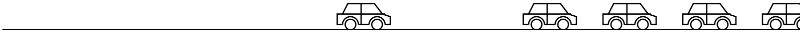

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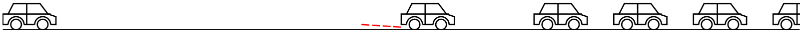

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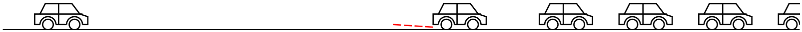

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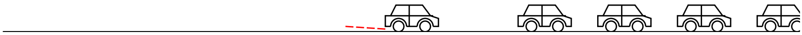

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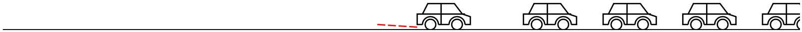

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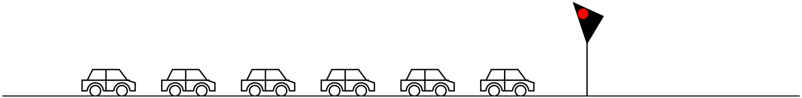
Arriving to a traffic jam is always sharp.

This is one aspect that makes motorways dangerous places.

Leaving a traffic jam



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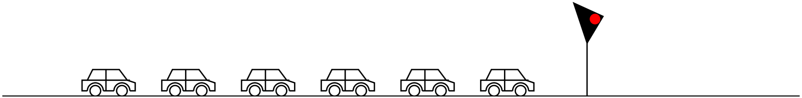

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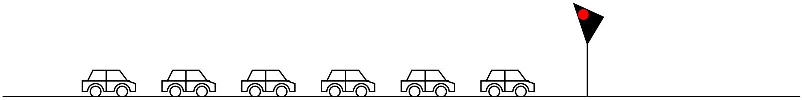

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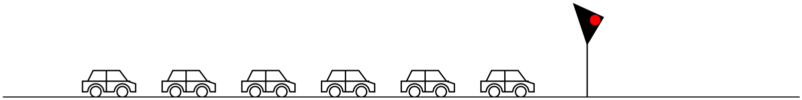

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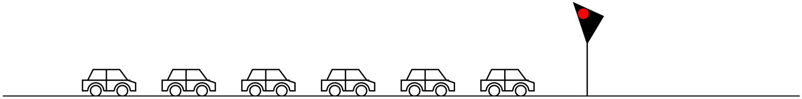

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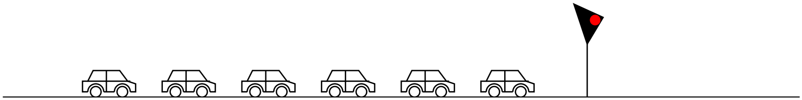

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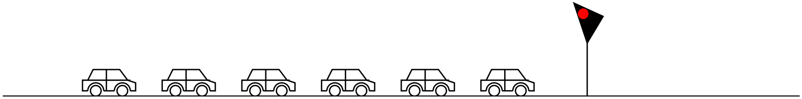

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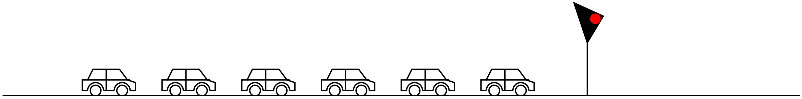

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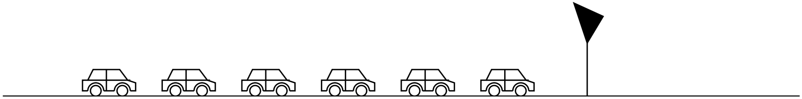

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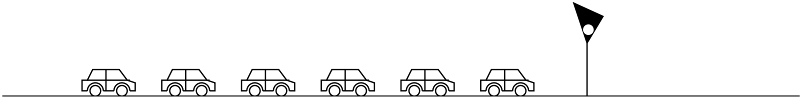

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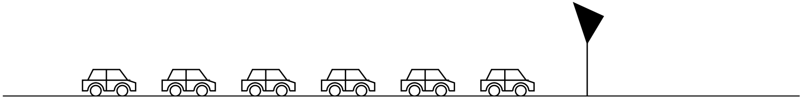

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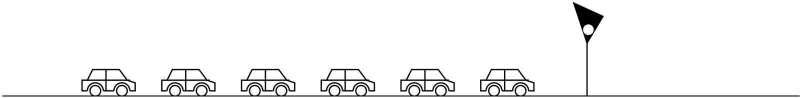

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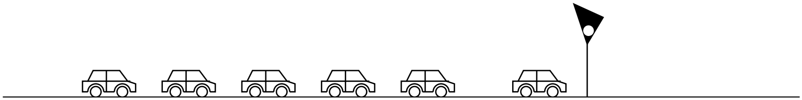

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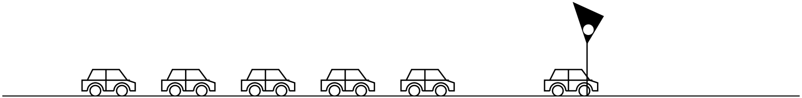

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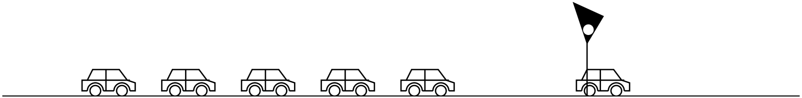

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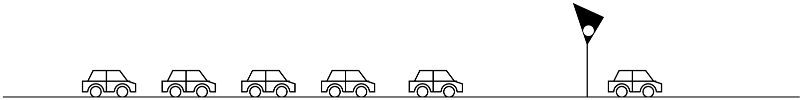

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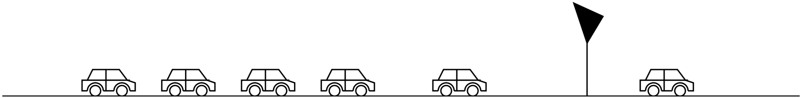

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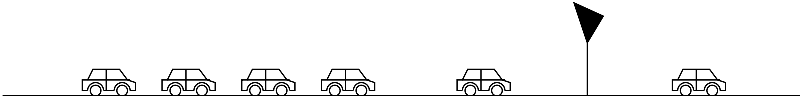

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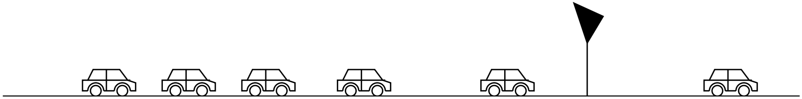

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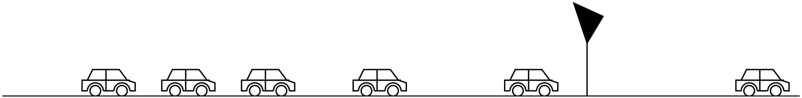

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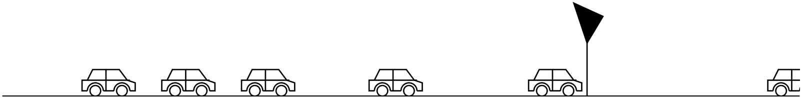

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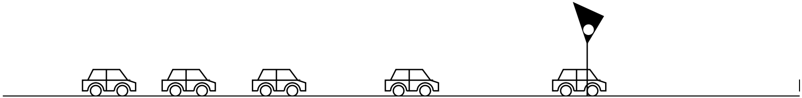

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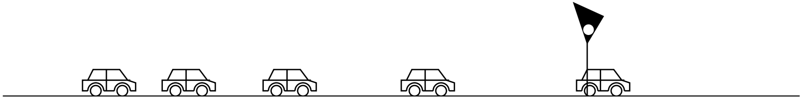

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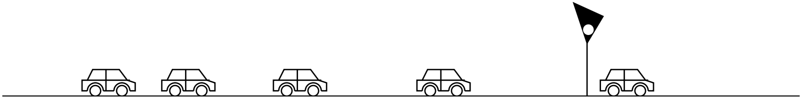

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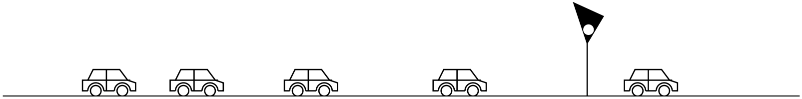

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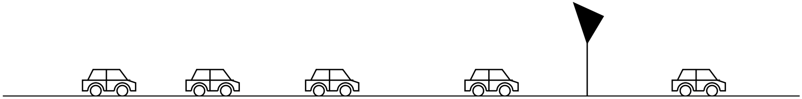

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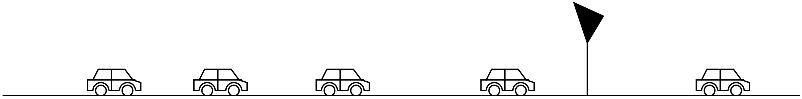

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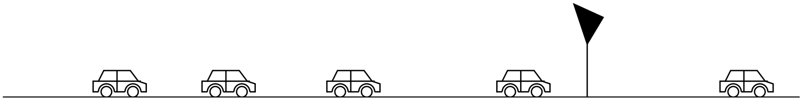

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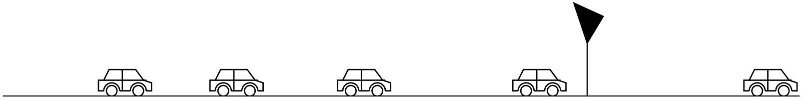

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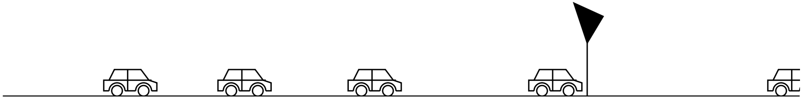

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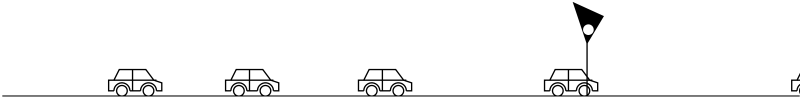

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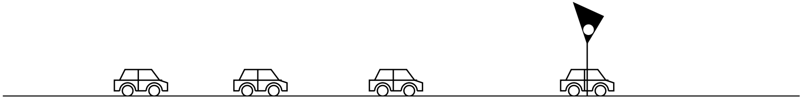

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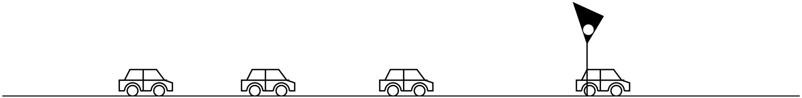

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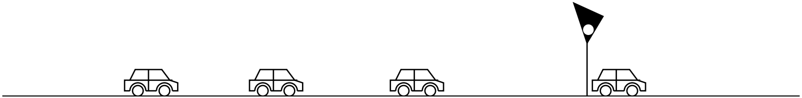

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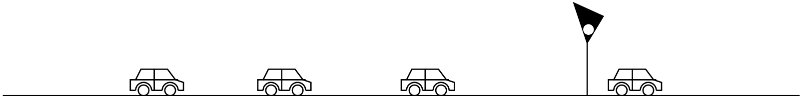

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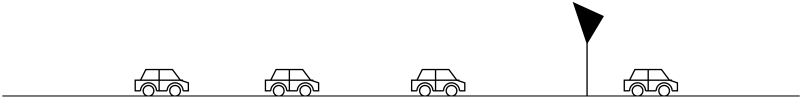

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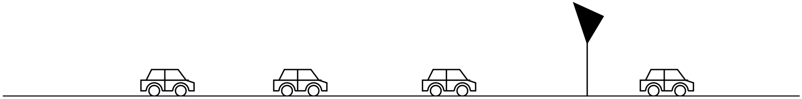

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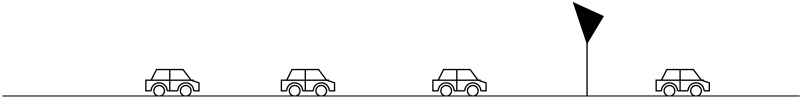

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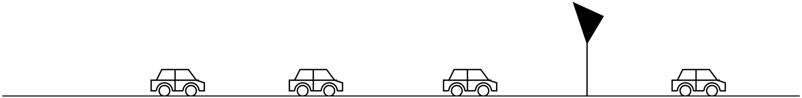

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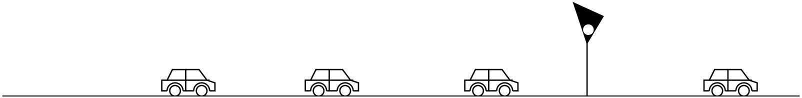

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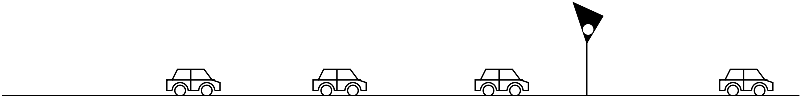

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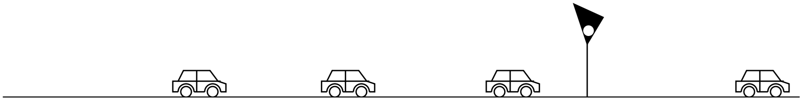

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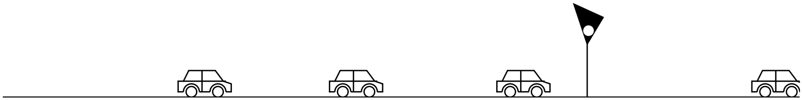

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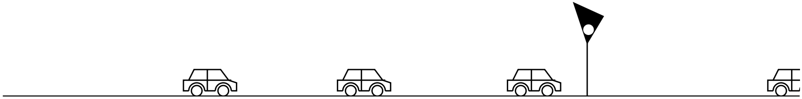

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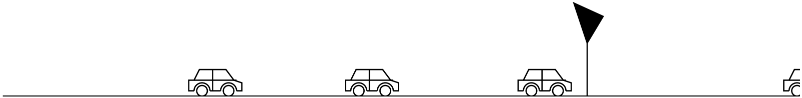

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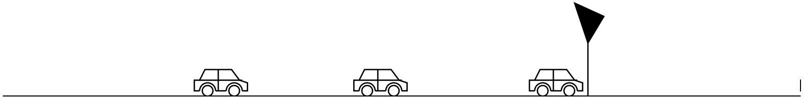

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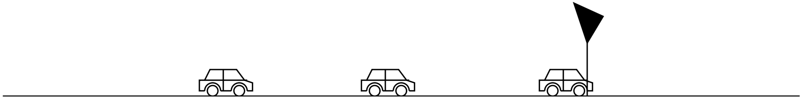

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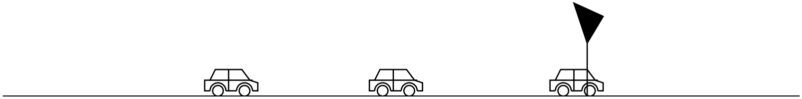

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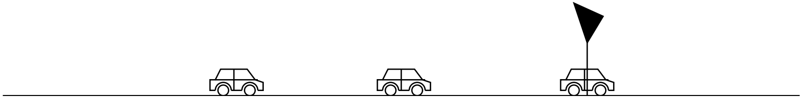

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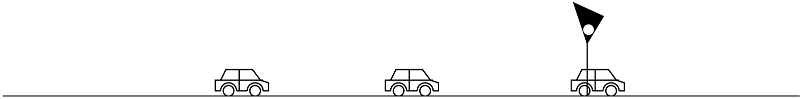

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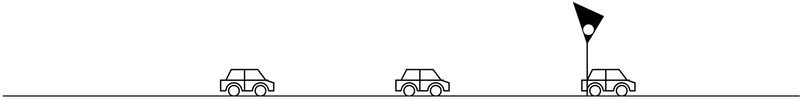

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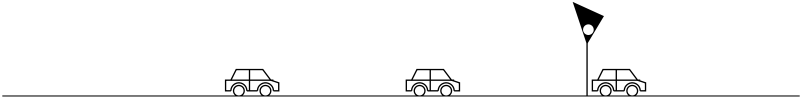

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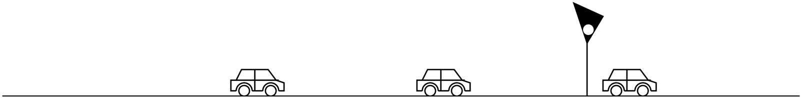

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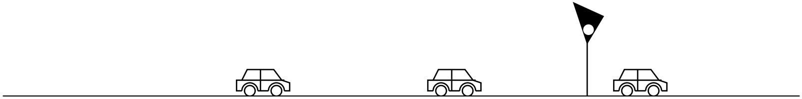

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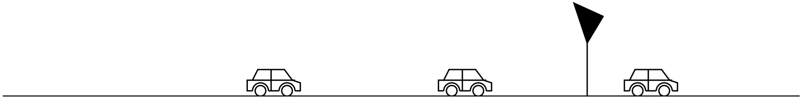

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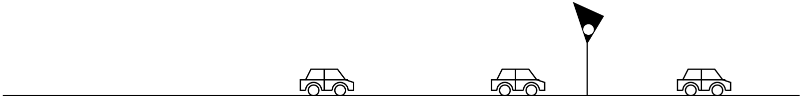

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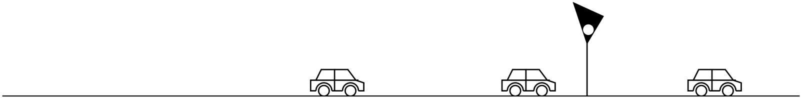

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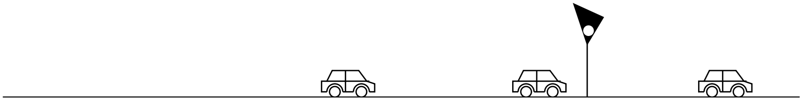

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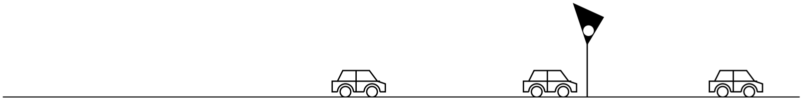

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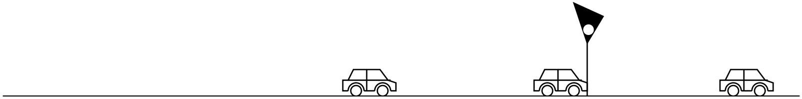

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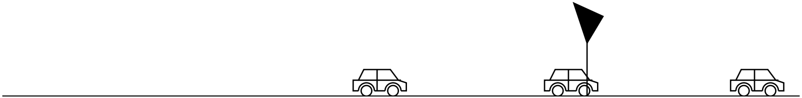

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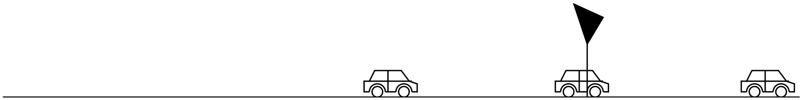

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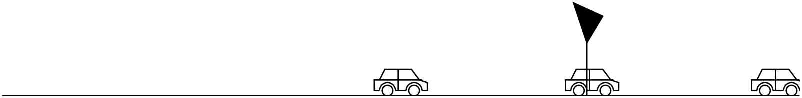

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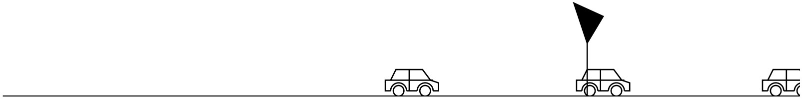

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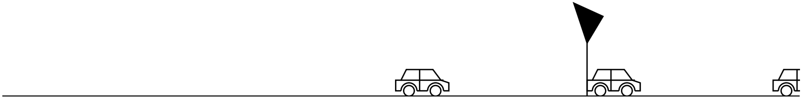

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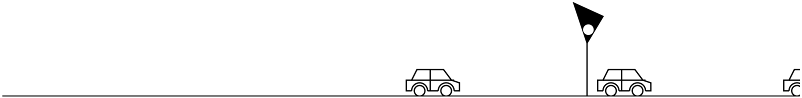

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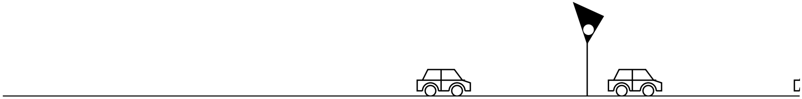

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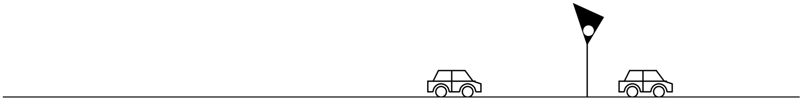

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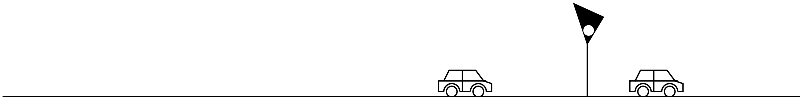

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Continuous, long acceleration for those starting from the rear

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Why is there such a difference between the two ends of a traffic jam?

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Why is there such a difference between the two ends of a traffic jam?

Totally asymmetric simple exclusion process: an explanation

Being ageless

We first seek a random time that does not remember its past.

Let $\tau > 0$ be a random time such that

$$\mathbf{P}\{\tau > t\} = \mathrm{e}^{-t} \qquad \text{for all } t > 0. \qquad \text{(Exponential distribution)}$$

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The same as $\mathbf{P}\{\tau > s\}$, regardless of t!

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We have found the secret of being ageless.

Being ageless

 $\leftarrow$ This will be the ageless alarm clock that rings at time τ

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$\rightsquigarrow$ What is the probability that an ⏰ rings within a small time t?

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$$\mathbf{P}\{\tau \le t\} = 1 - \mathbf{P}\{\tau > t\} = 1 - \mathrm{e}^{-t} = 1 - (1 - t) + \mathfrak{o}(t) = t + \mathfrak{o}(t).$$

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$\rightarrow$ More ⏰ 's, even smaller probability.

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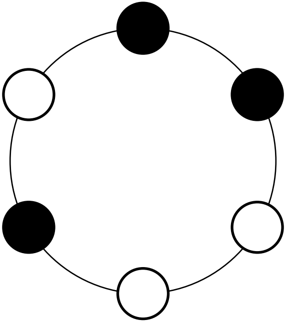
$$\begin{aligned}\mathbf{P}\{\text{none of them ring}\} &= \mathbf{P}\{\tau > t\}^k \\ &= \mathrm{e}^{-kt} \\ &= (1 - kt) + \mathfrak{o}(t).\end{aligned}$$

The Totally Asymmetric Simple Exclusion Process

TASEP



m balls in N possible slots.

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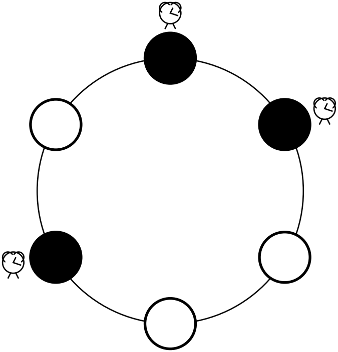

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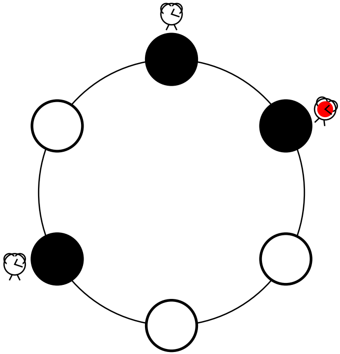

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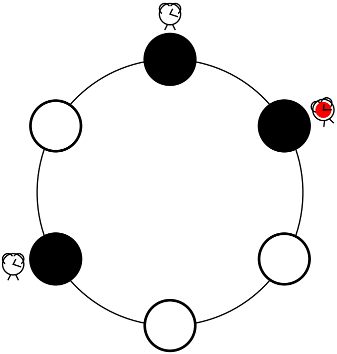

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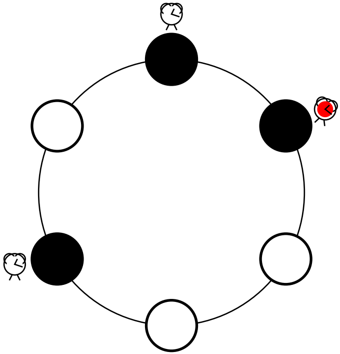

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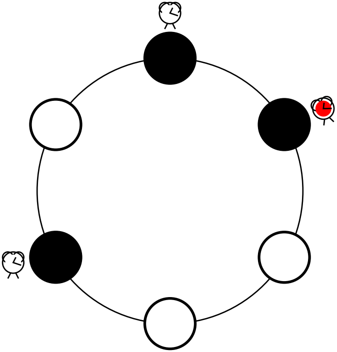

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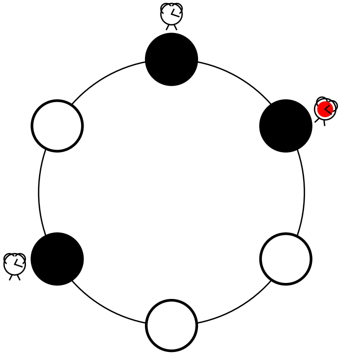

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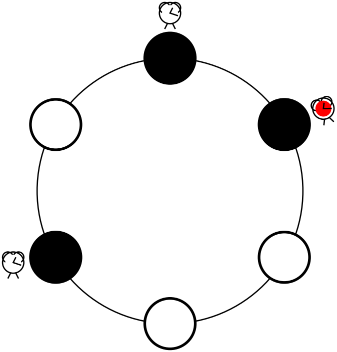

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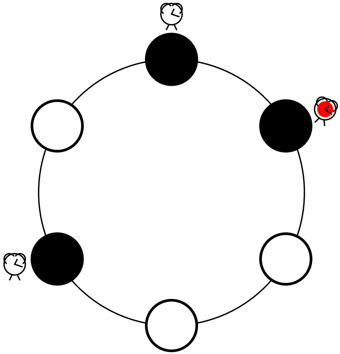

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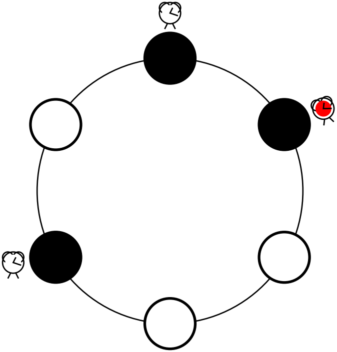

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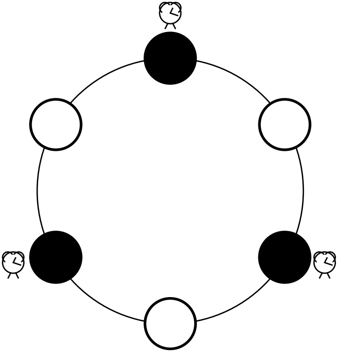

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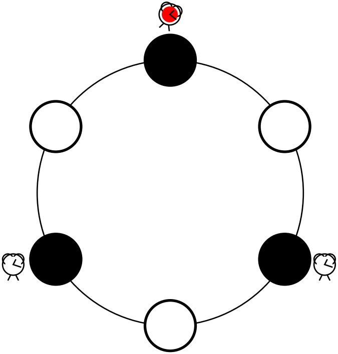

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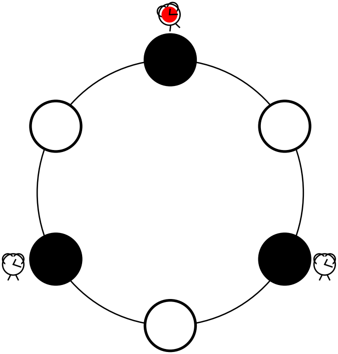

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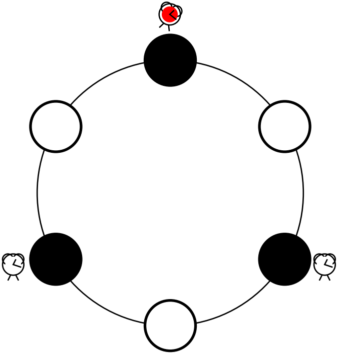

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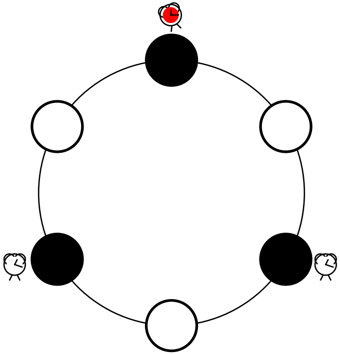

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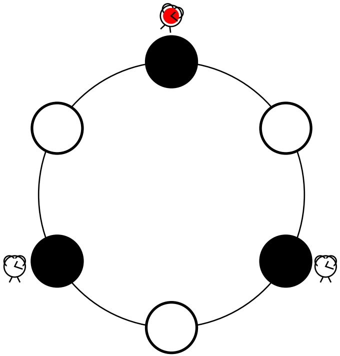

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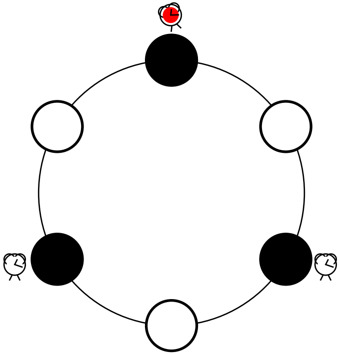

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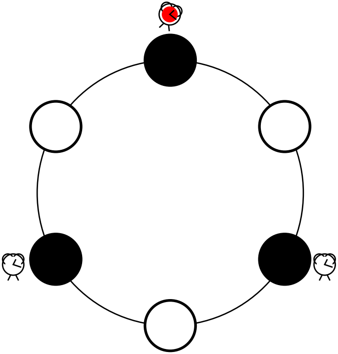

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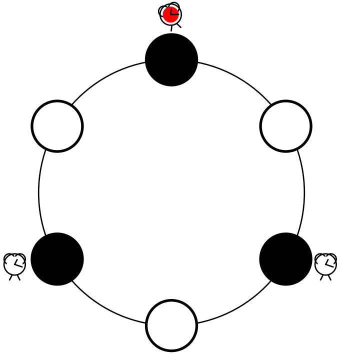

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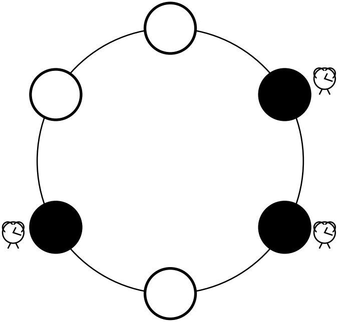

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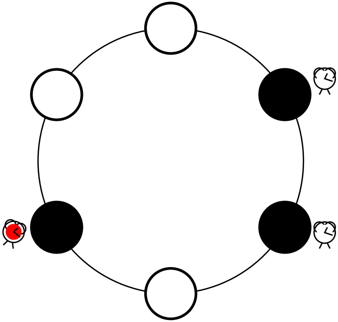

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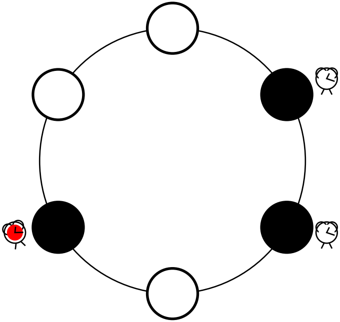

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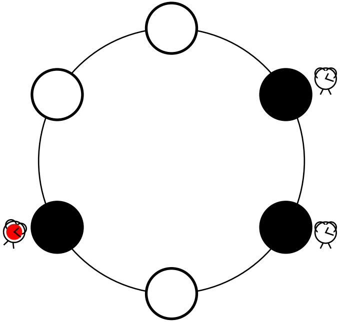

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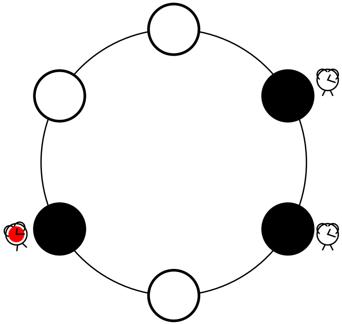

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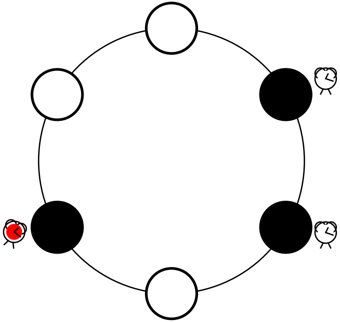

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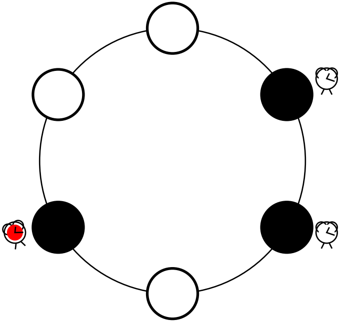

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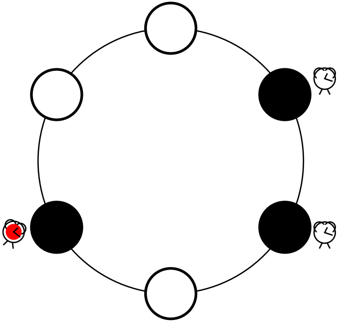

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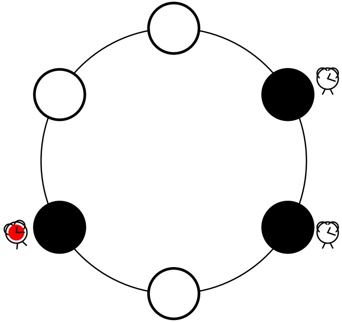

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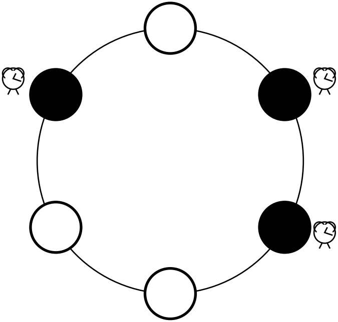

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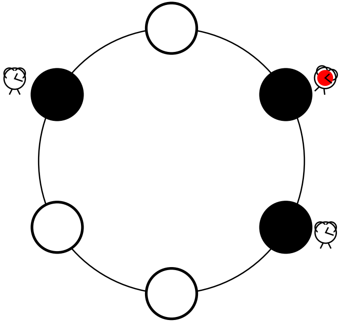

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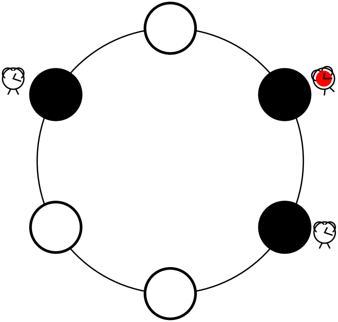

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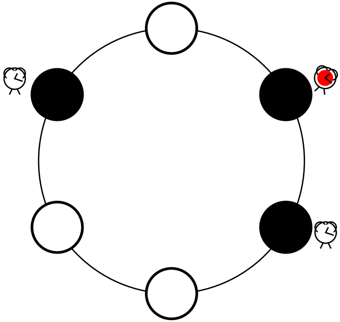

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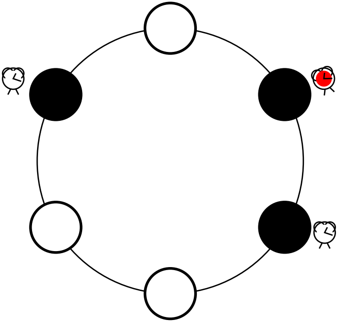

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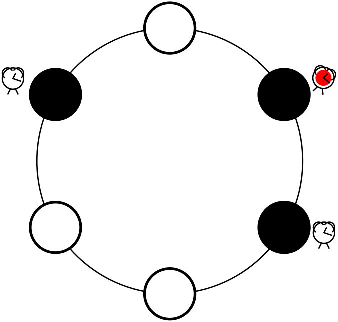

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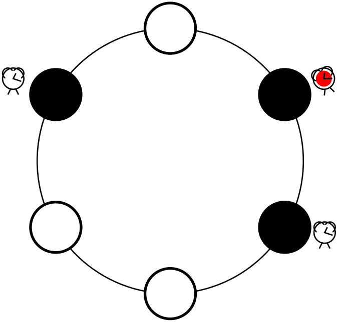
Each listening to its own  . When that rings, the ball tries to jump to the right.

The Totally Asymmetric Simple Exclusion Process

TASEP



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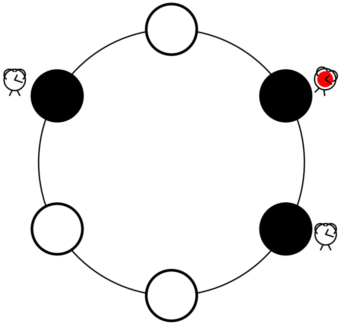

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The Totally Asymmetric Simple Exclusion Process

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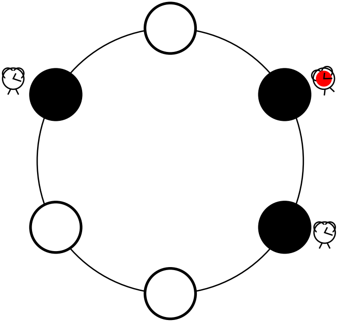

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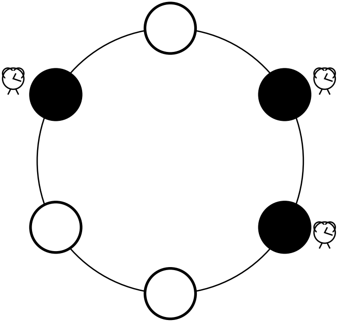

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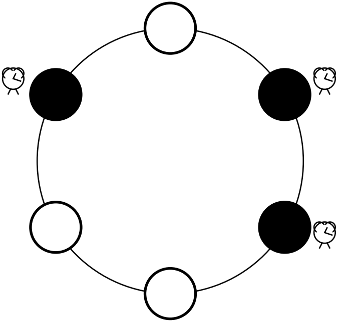
The Totally Asymmetric Simple Exclusion Process

TASEP



m balls in N possible slots.

Each listening to its own . When that rings, the ball tries to jump to the right. But sometimes it's blocked.
Ageless, independent 's ⇒ if we know the present, no need to know the past. *Markov property*, makes things handy.

Stationary distribution

Random process $\rightsquigarrow$ need to talk about *distributions*.

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What is the stationary distribution the one that's unchanged in time?

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With N and m fixed, the distribution that gives equal chance to each (m-ball) configuration, is stationary.

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In this case every configuration occurs with probability $1/\binom{N}{m}$.

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1st remark.

In this case every configuration occurs with probability $1/\binom{N}{m}$.

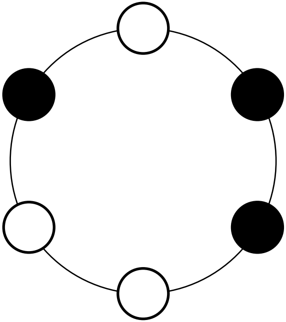
2nd remark.

With fixed N, m, there is no other stationary distribution.

Stationary distribution

Almost proof

A state ω



Stationary distribution

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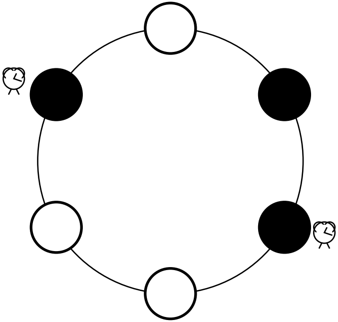
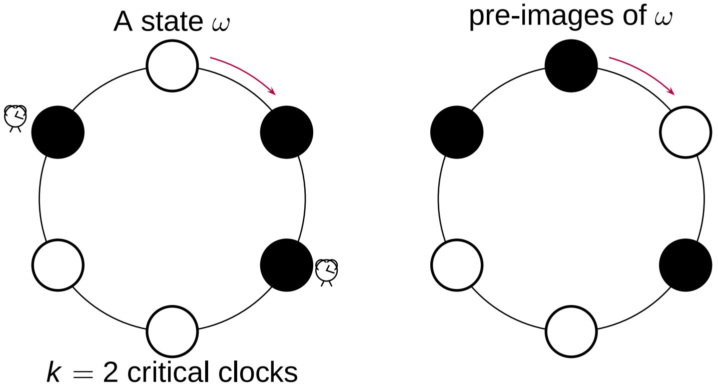


$k = 2$ critical clocks

Stationary distribution

Almost proof



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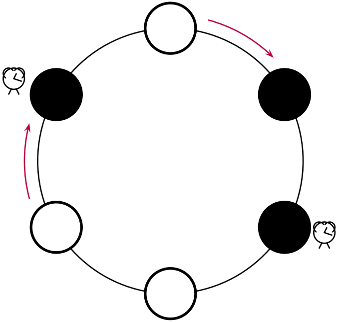
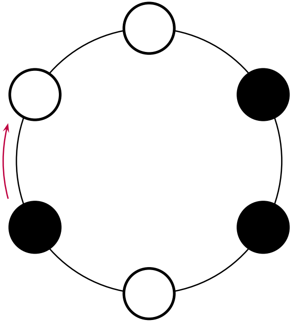


$k = 2$ critical clocks

pre-images of ω



$k = 2$ pre-images

Stationary distribution

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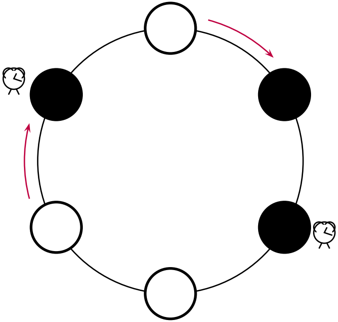
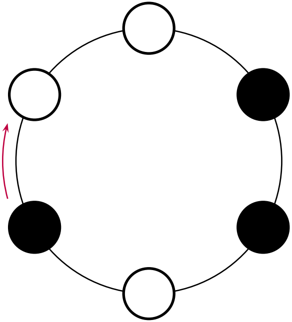
A state ω pre-images of ω





$k = 2$ critical clocks $k = 2$ pre-images

The number of critical clocks for $\omega =$ the number of pre-images of $\omega = k$

Stationary distribution

Almost proof

Suppose that each configuration has the same probability p at time s. What is the probability of the state ω after a small time t?

Stationary distribution

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$$\mathbf{P}\{\omega \text{ at time } s+t\}$$

Stationary distribution

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Suppose that each configuration has the same probability p at time s. What is the probability of the state ω after a small time t?

$$\begin{aligned}
&\mathbf{P}\{\omega \text{ at time } s+t\} \\
&= \mathbf{P}\{\omega \text{ at time } s \text{ and no jumps within time } t\} \\
&\quad + \mathbf{P}\{\text{was a pre-image of } \omega \text{ at time } s\text{, and jumps to } \omega\} \\
&\quad + \mathfrak{o}(t) \text{ (at least two jumps occur within the small time } t\text{)}
\end{aligned}$$

Stationary distribution

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$+ \mathfrak{o}(t)$ (at least two jumps occur within the small time t)

$= \mathbf{P}\{\omega$ at time s and none of the k critical  's ring$\}$

$+ \sum_{\eta \text{ is a pre-image of } \omega} \mathbf{P}\{\eta$ at time s and the right critical  rings$\}$

$$+ \mathfrak{o}(t)$$

Stationary distribution

Almost proof

$\mathbf{P}\{\omega \text{ at time } s+t\}$

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Stationary distribution

Almost proof

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\end{aligned}$$

$$\text{'s ring}\}$$

$$+ \sum_{\eta \text{ is a pre-image of } \omega} \mathbf{P}\{\eta \text{ at time } s \text{ and the right critical }$$

$$\text{rings}\}$$

$$\begin{aligned}
&+ \mathfrak{o}(t)\\
&= p\cdot(1-kt) + \sum_{\eta \text{ is a pre-image of } \omega} p\cdot t + \mathfrak{o}(t)
\end{aligned}$$

Stationary distribution

Almost proof

$$
\begin{aligned}
&\mathbf{P}\{\omega \text{ at time } s+t\} \\
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&\quad + \mathfrak{o}(t) \\
&= p\cdot(1-kt) + \sum_{\eta \text{ is a pre-image of } \omega} p\cdot t + \mathfrak{o}(t) \\
&= p\cdot(1-kt) + k\cdot p\cdot t + \mathfrak{o}(t)
\end{aligned}
$$

Stationary distribution

Almost proof

$$\mathbf{P}\{\omega \text{ at time } s+t\}$$

$$= \mathbf{P}\{\omega \text{ at time } s \text{ and none of the } k \text{ critical ⏰'s ring}\}$$

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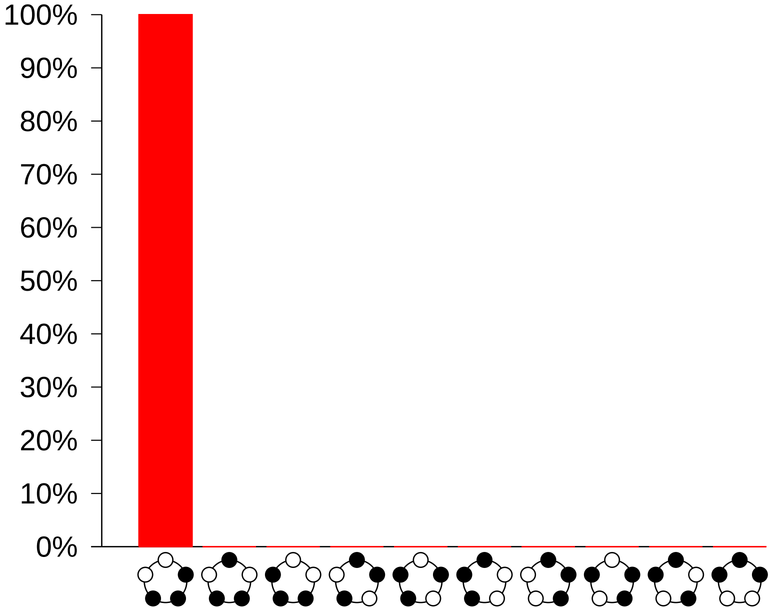
Take, say, 1 sec. and $t = \frac{1}{n}$. Then the errors $\mathfrak{o}(t) = \mathfrak{o}\left(\frac{1}{n}\right)$ stay small even if summed up: $\sum_{k=1}^{n} \mathfrak{o}\left(\frac{1}{n}\right) \to 0$ for large n. □

Stationary distribution



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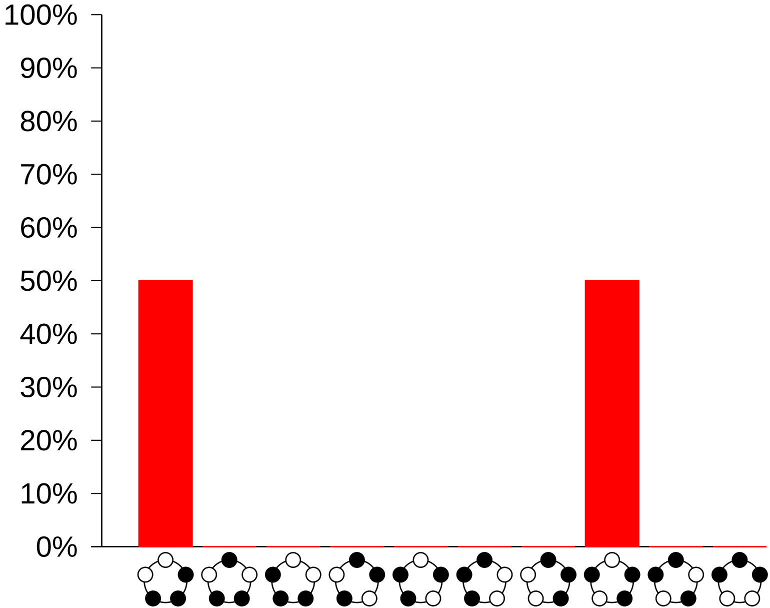

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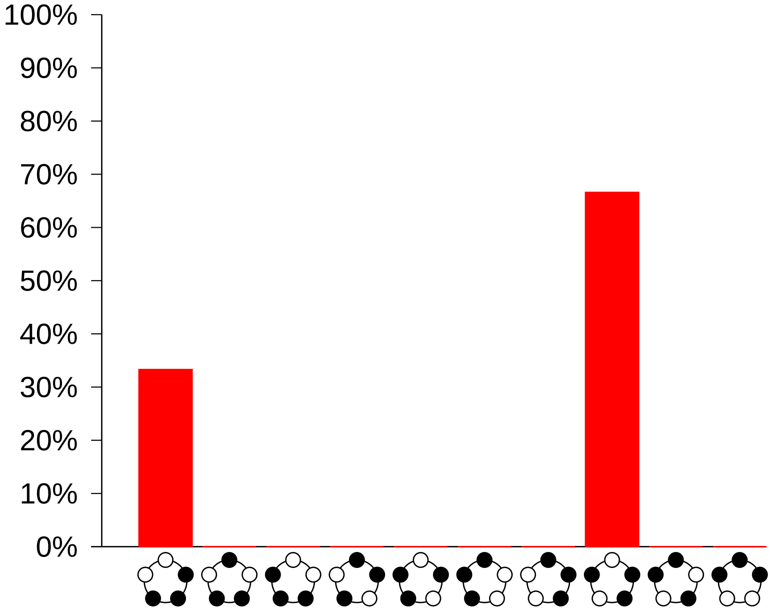

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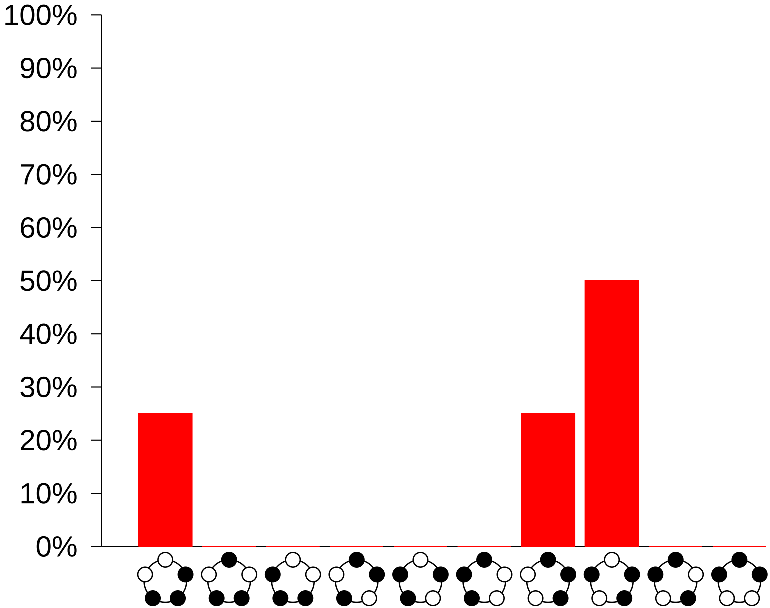

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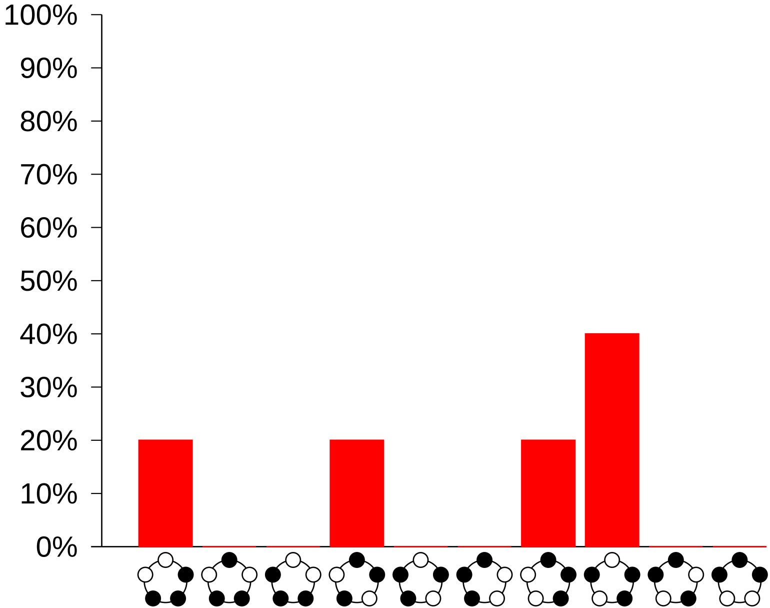

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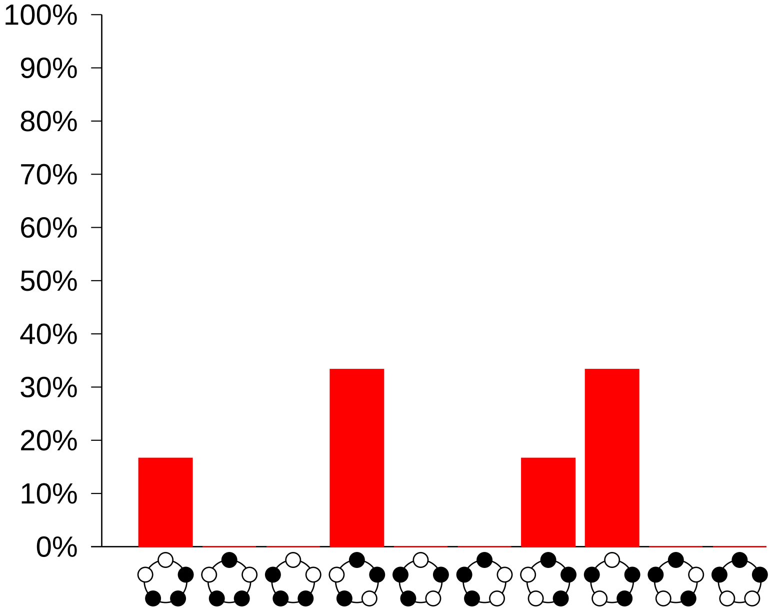

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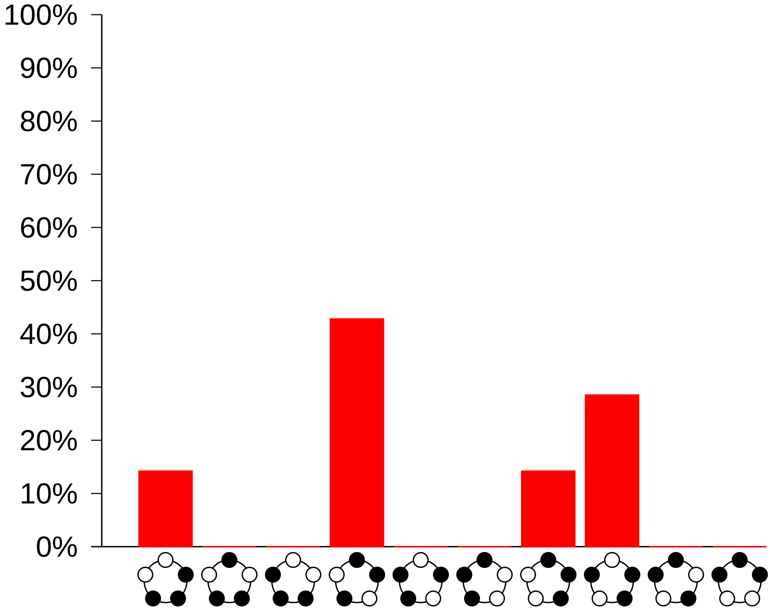

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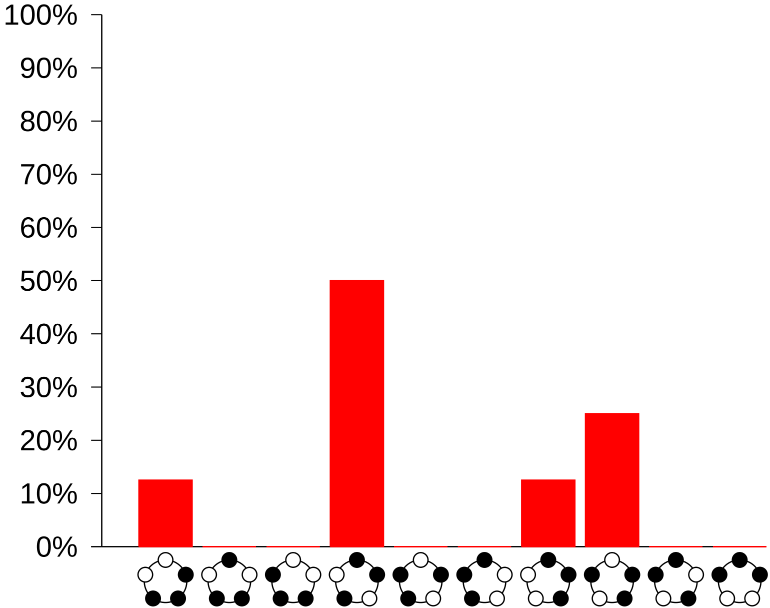

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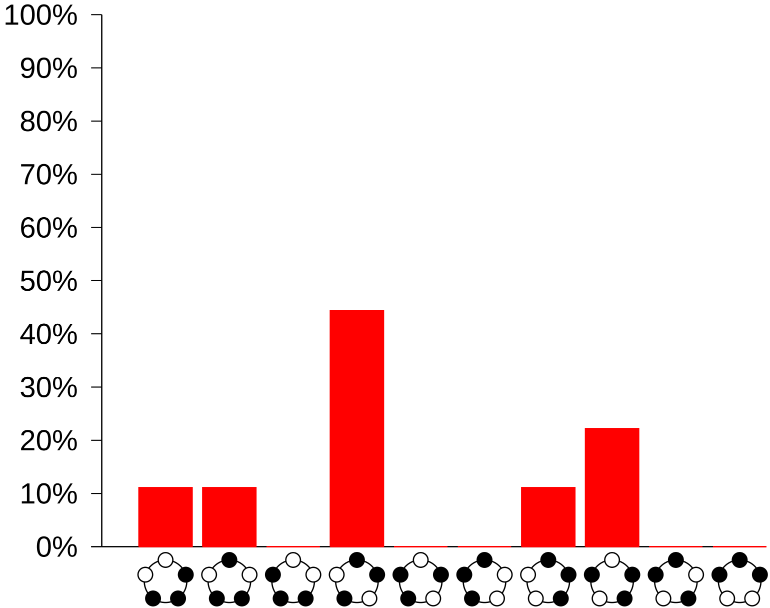

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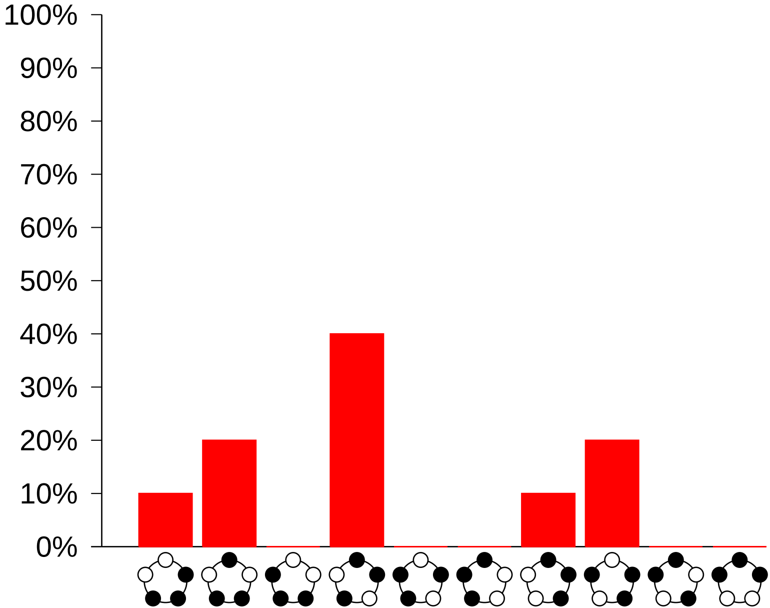

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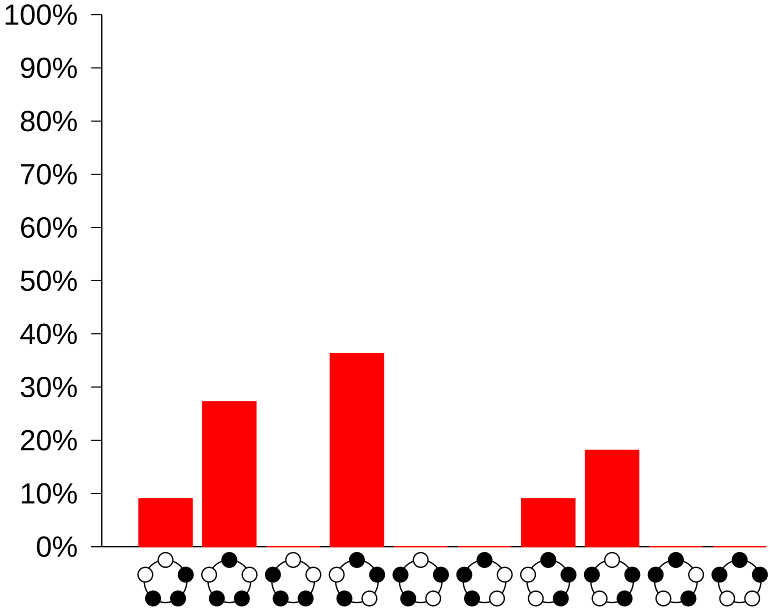

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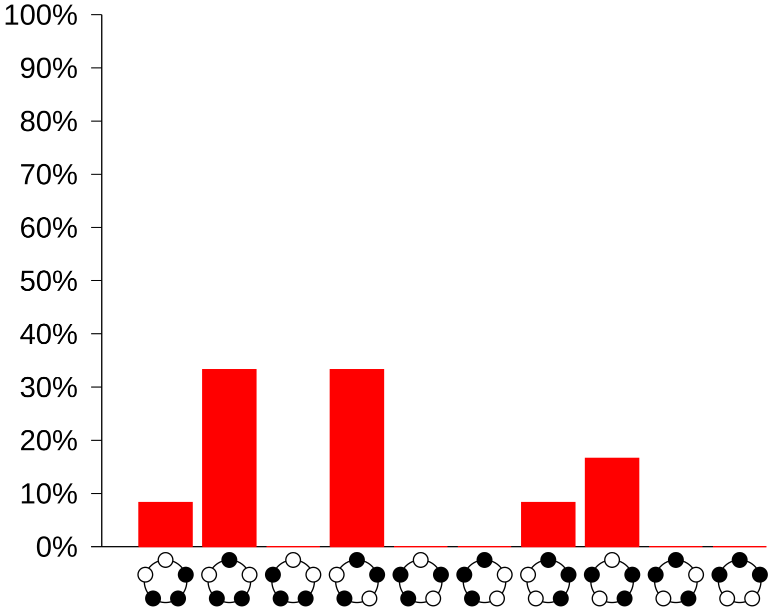

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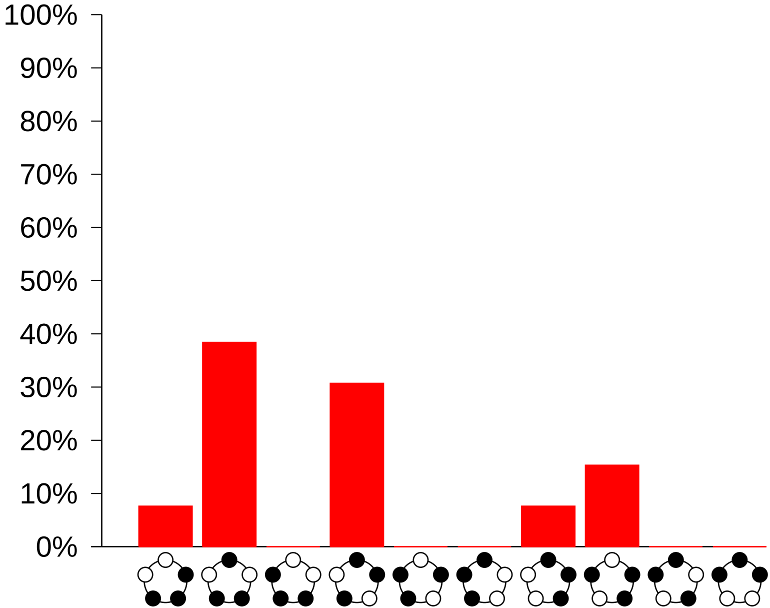

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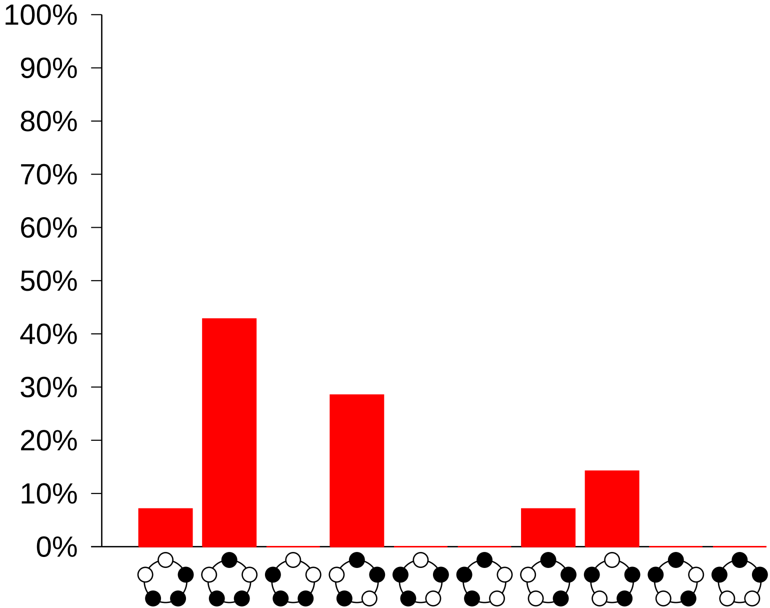

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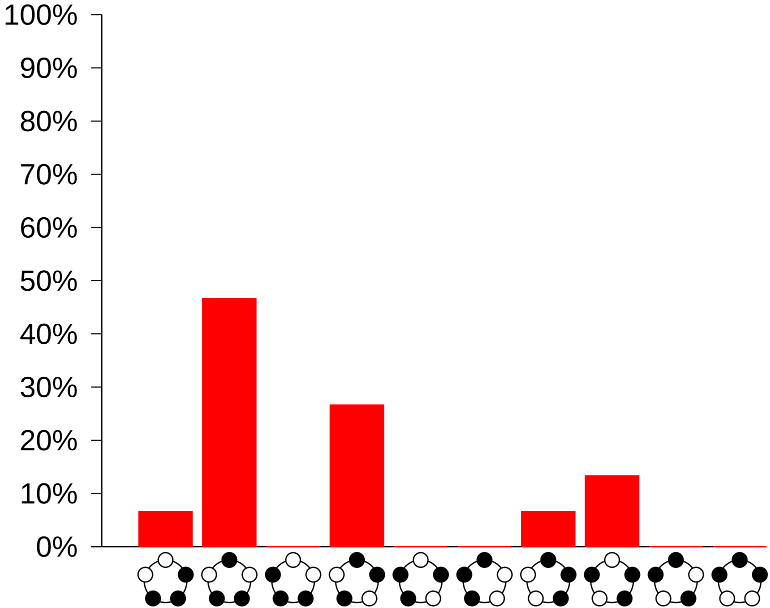

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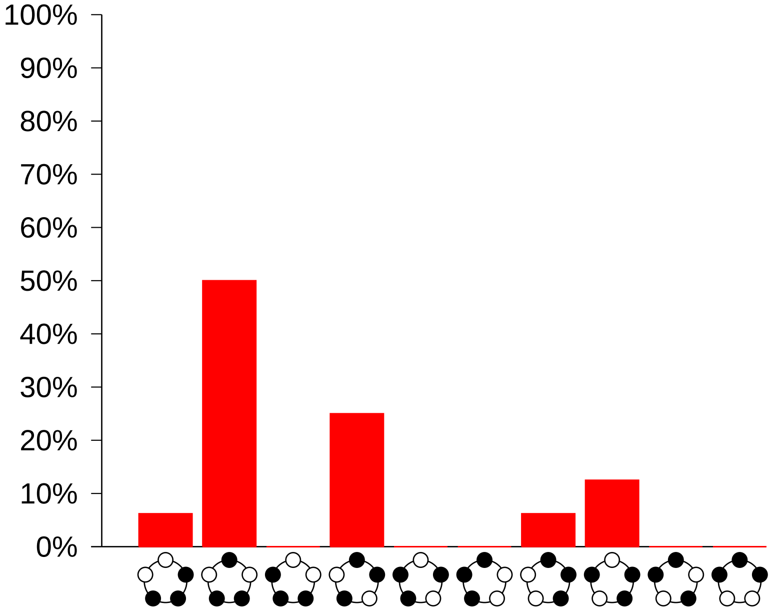

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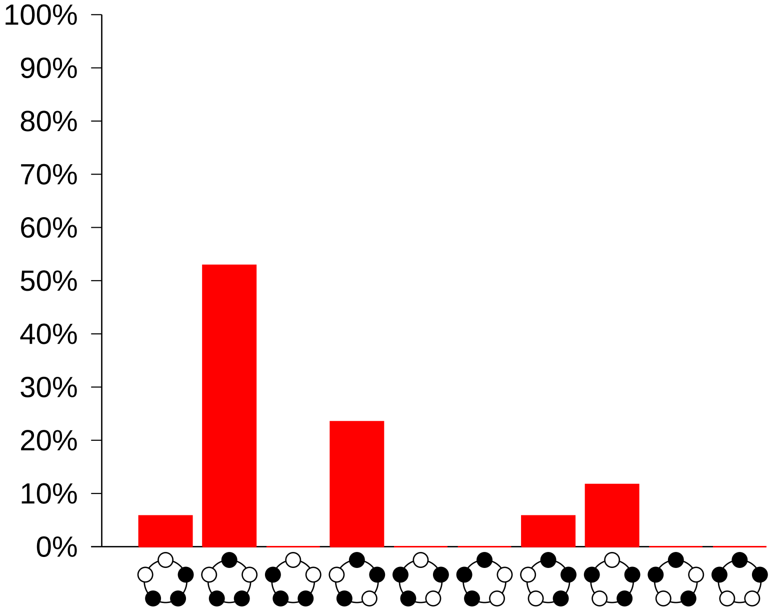

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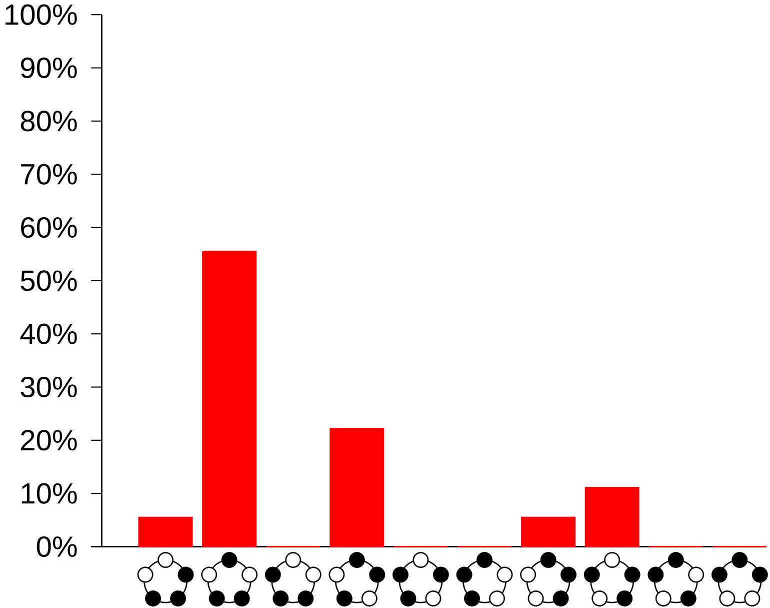

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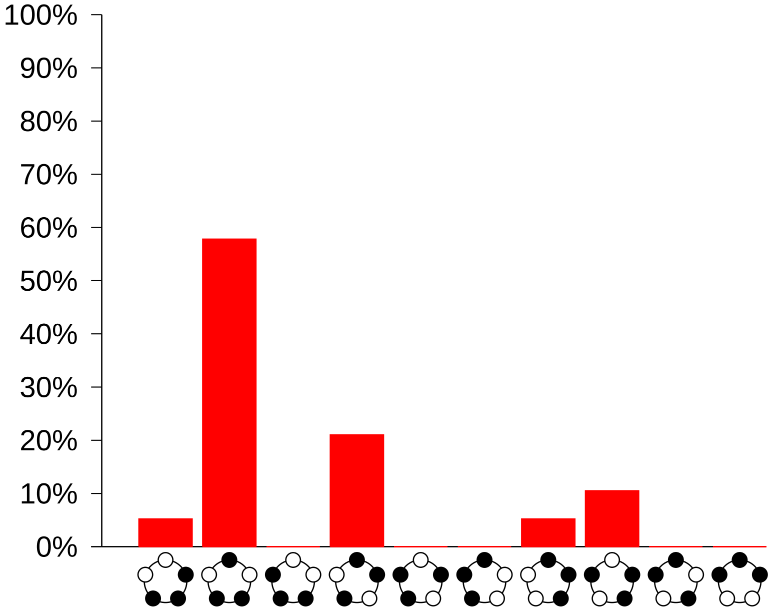

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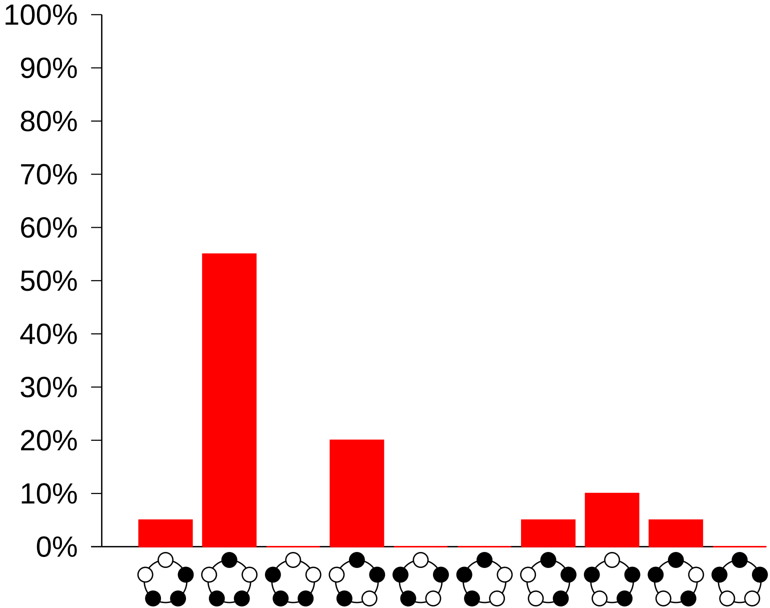

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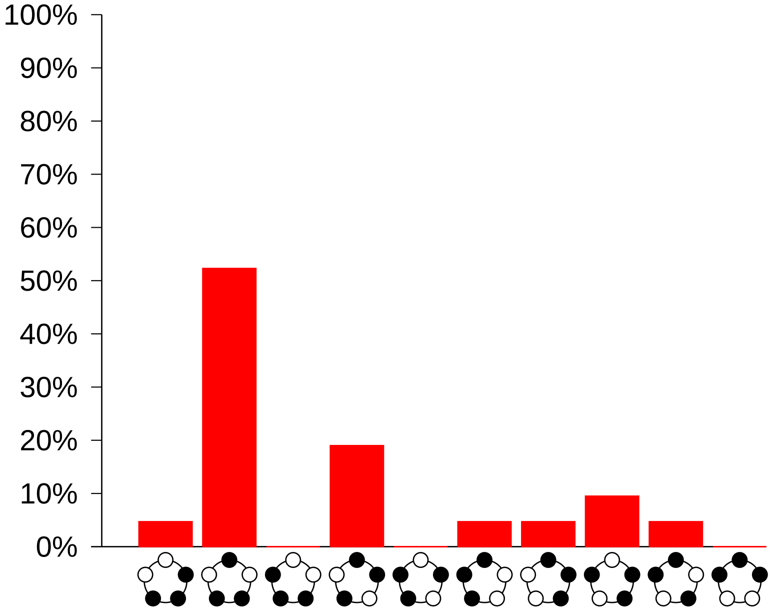

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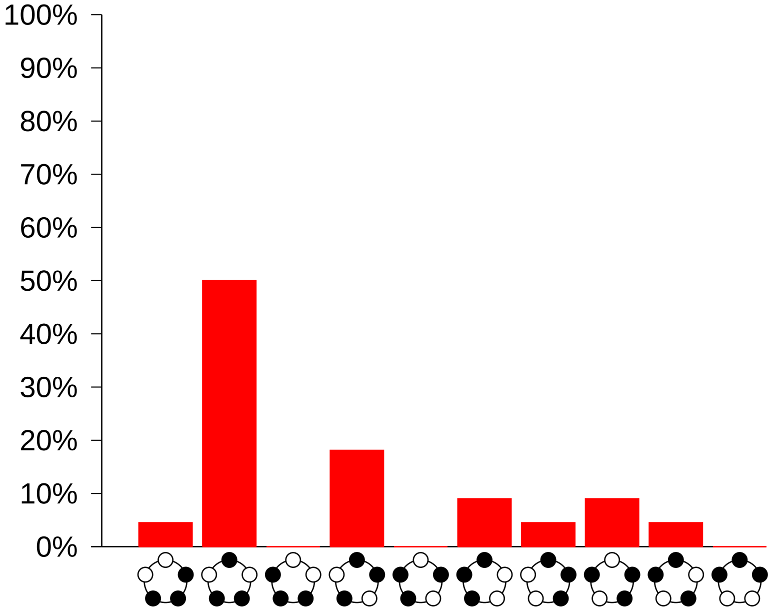

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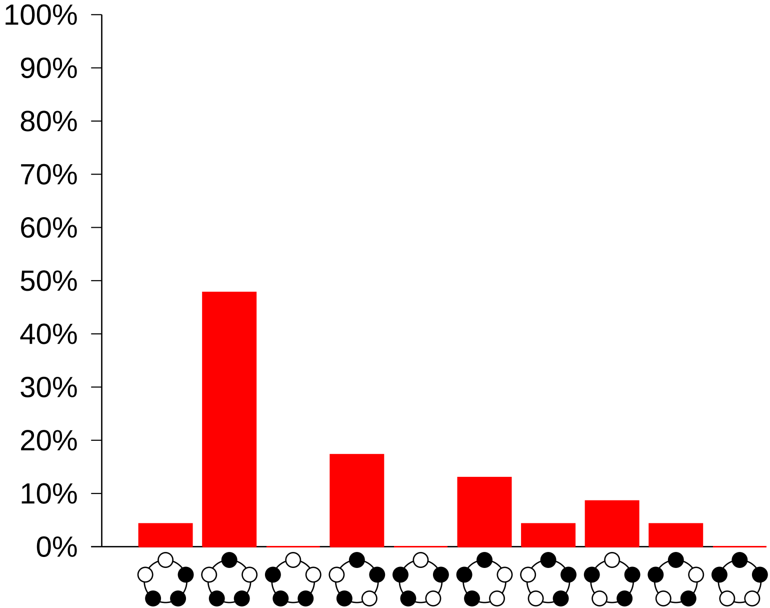

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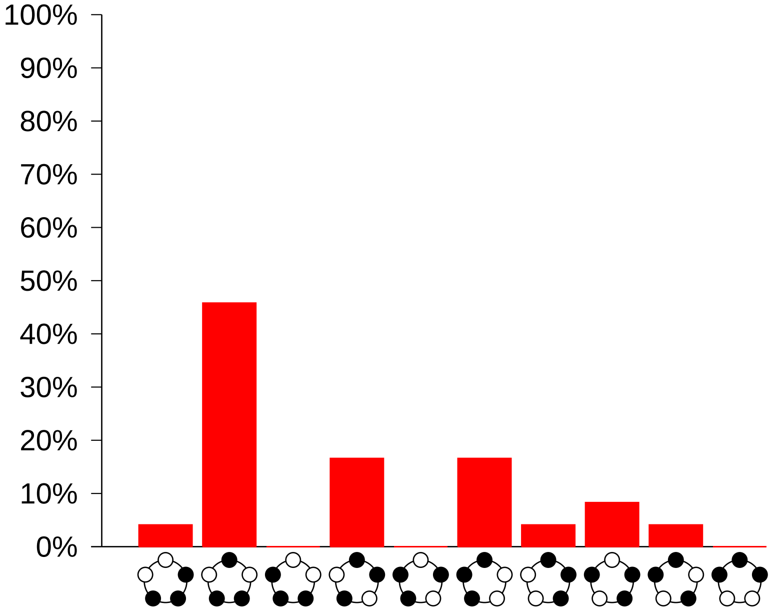

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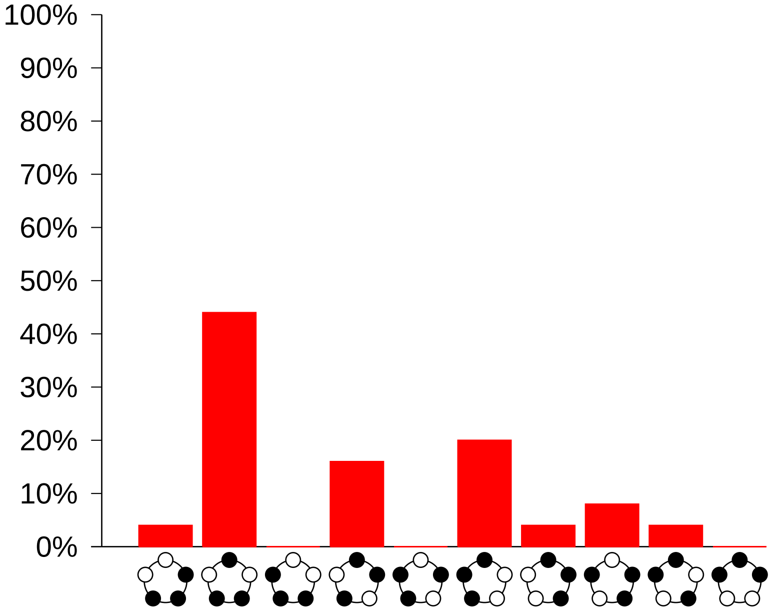

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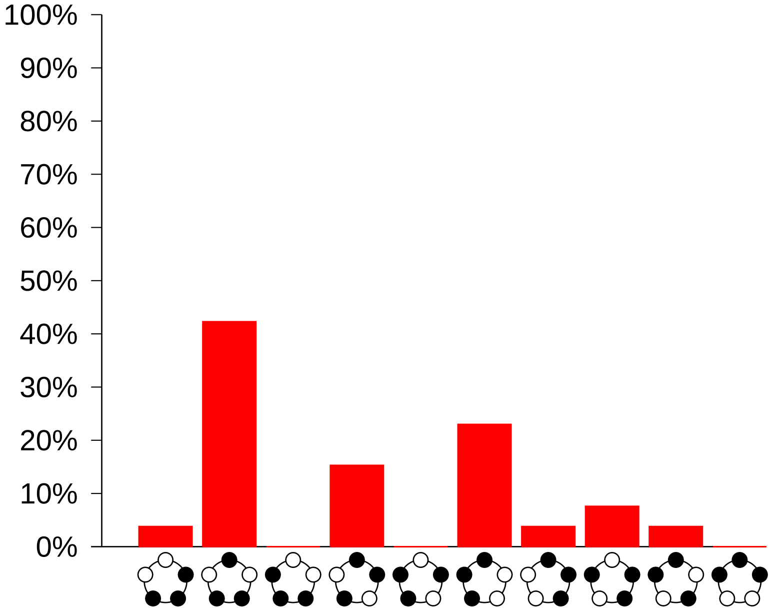

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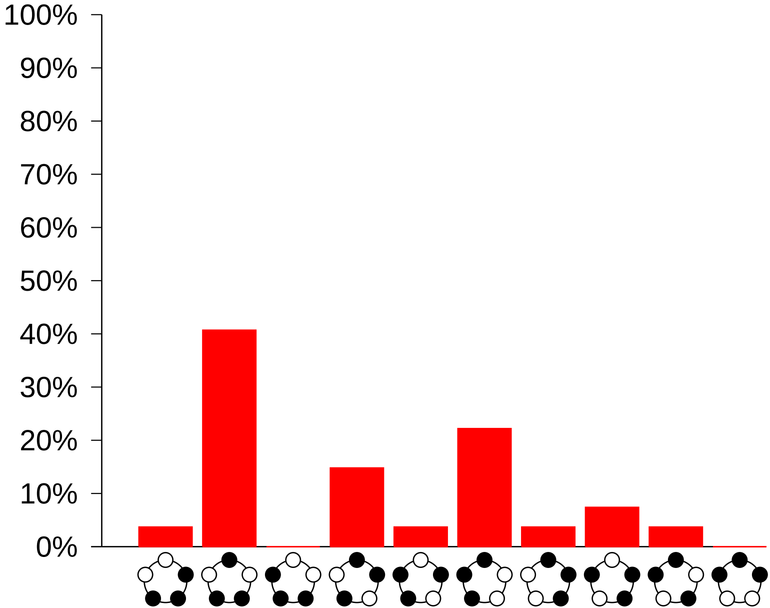

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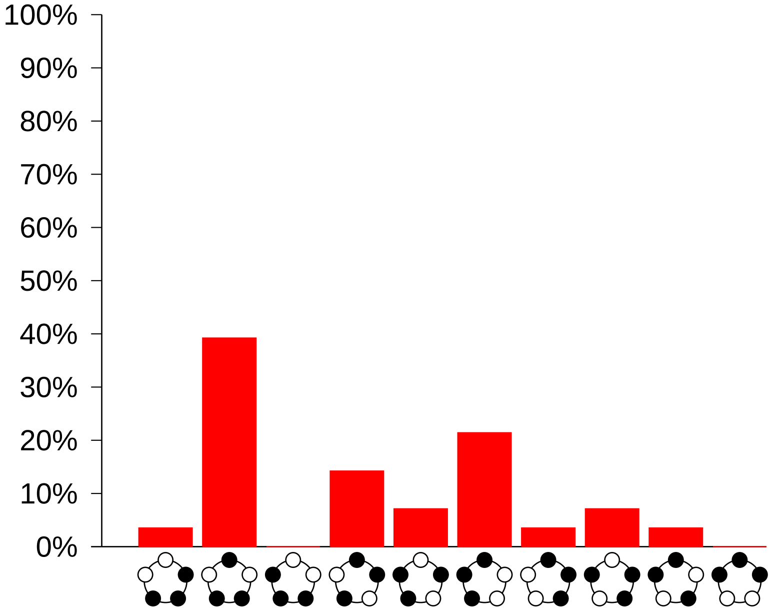

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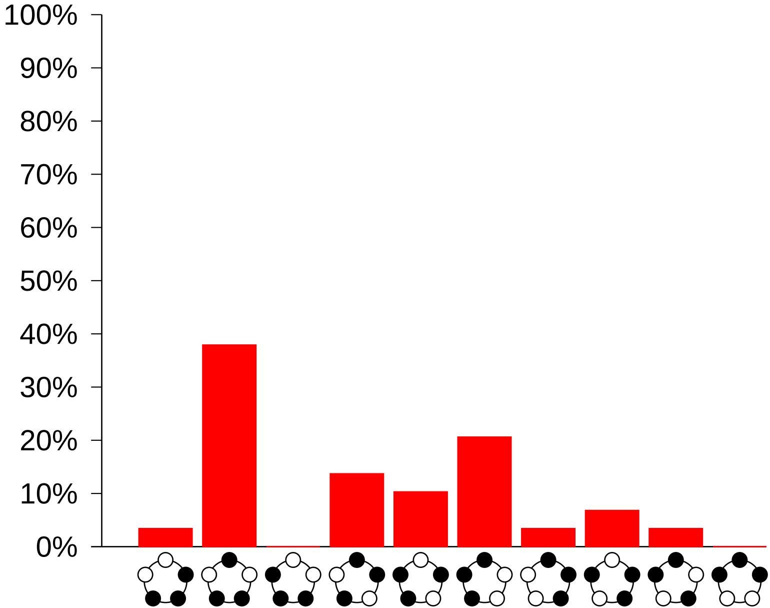

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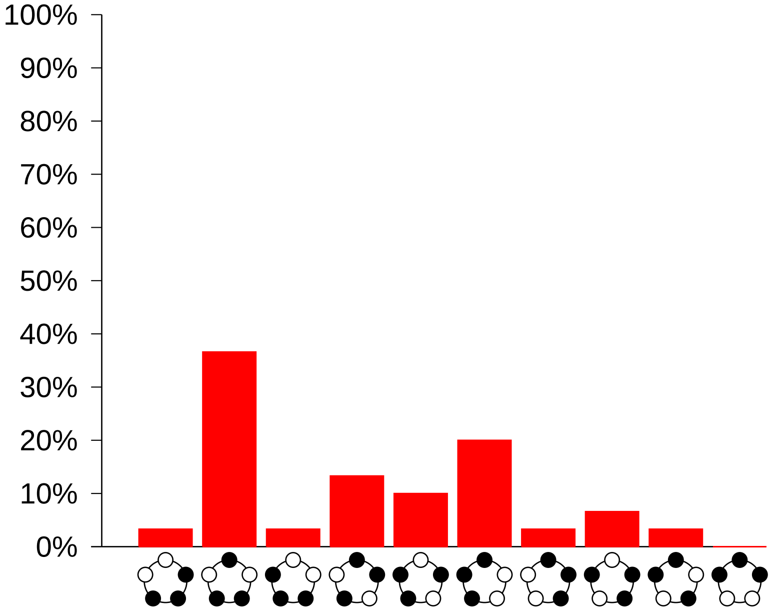

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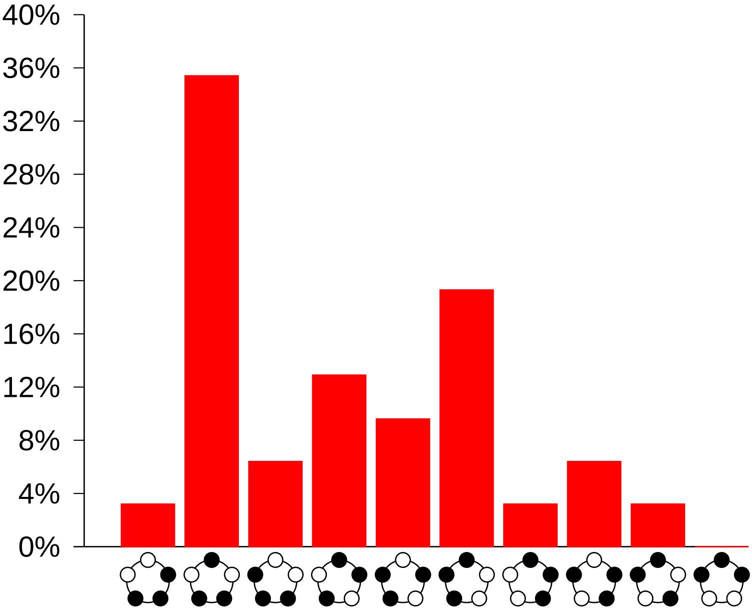

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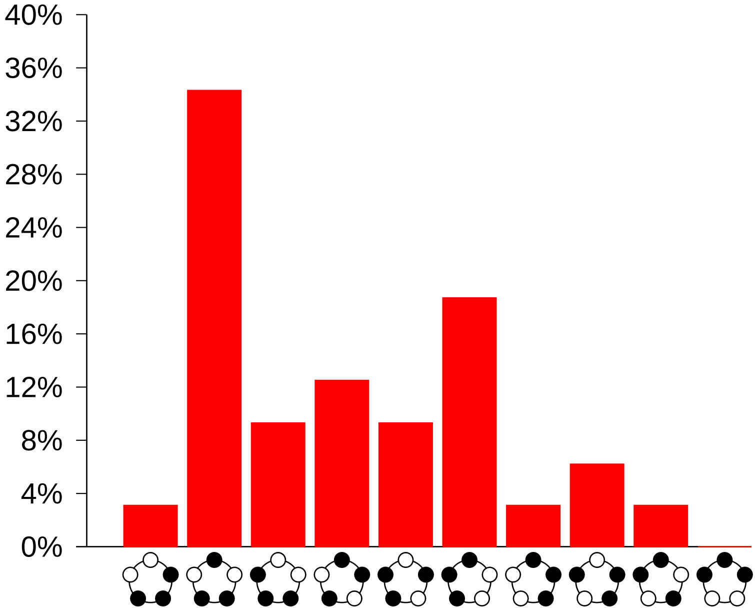

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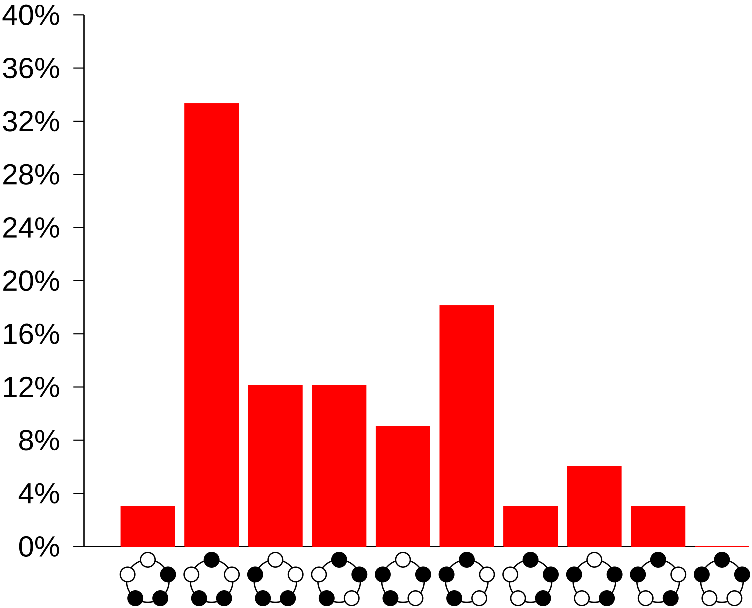

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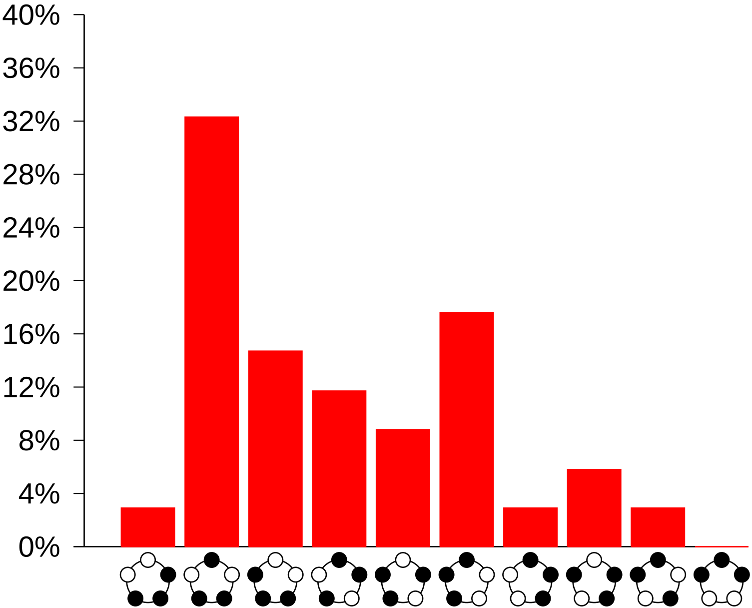

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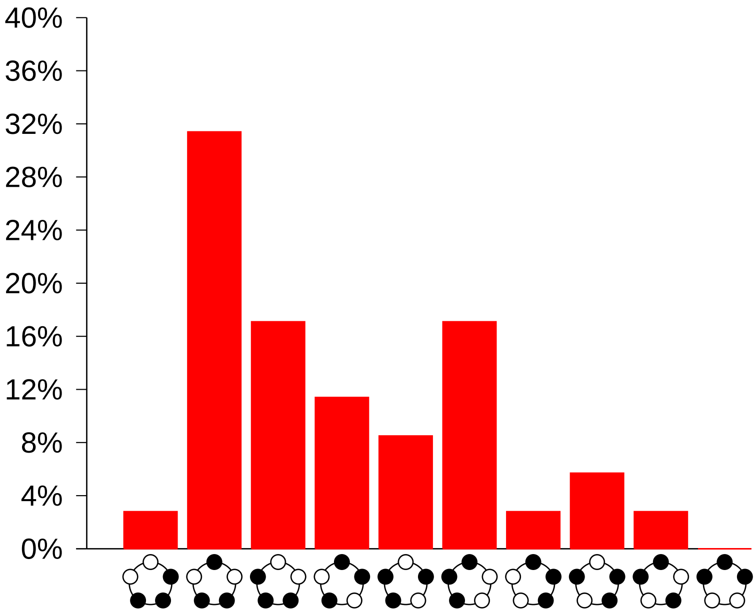

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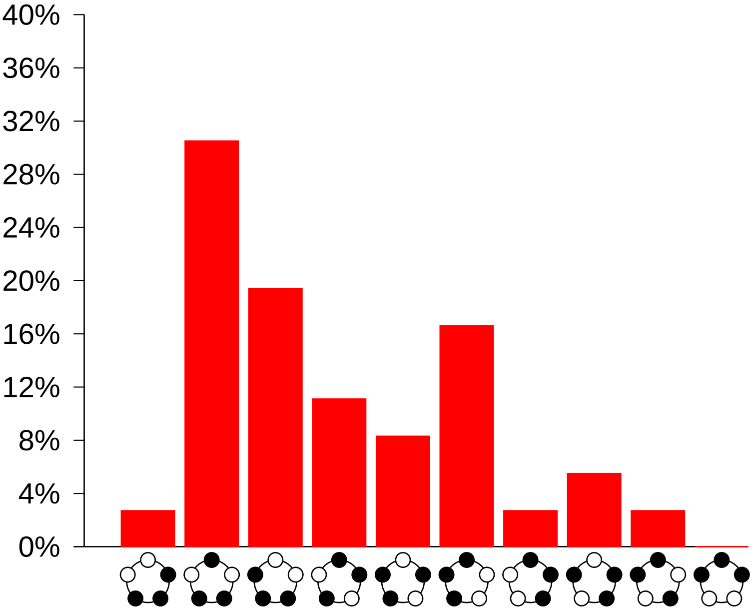

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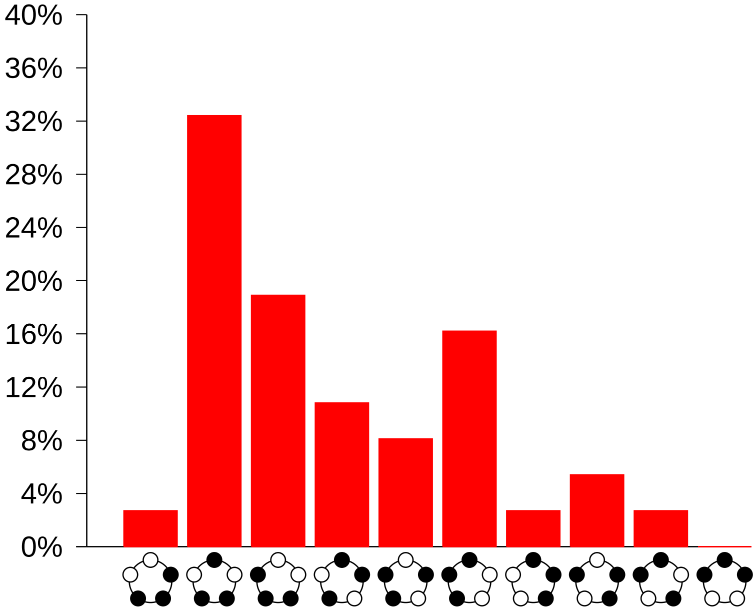

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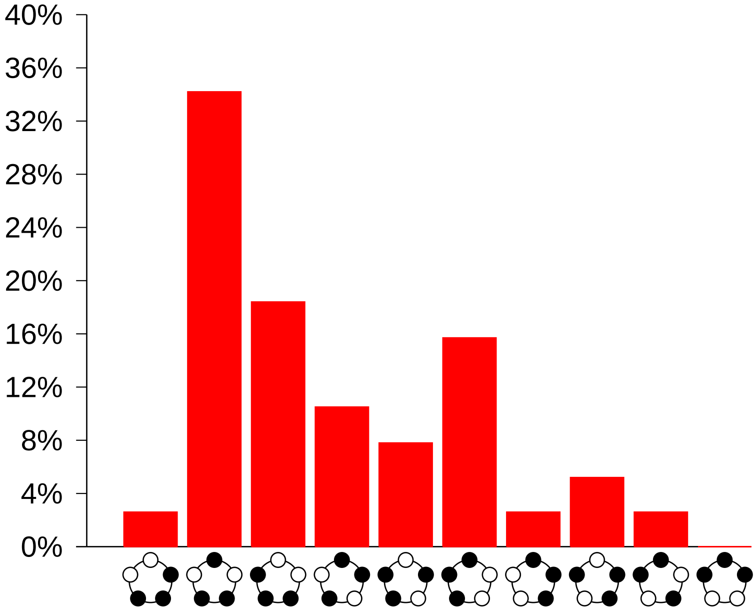

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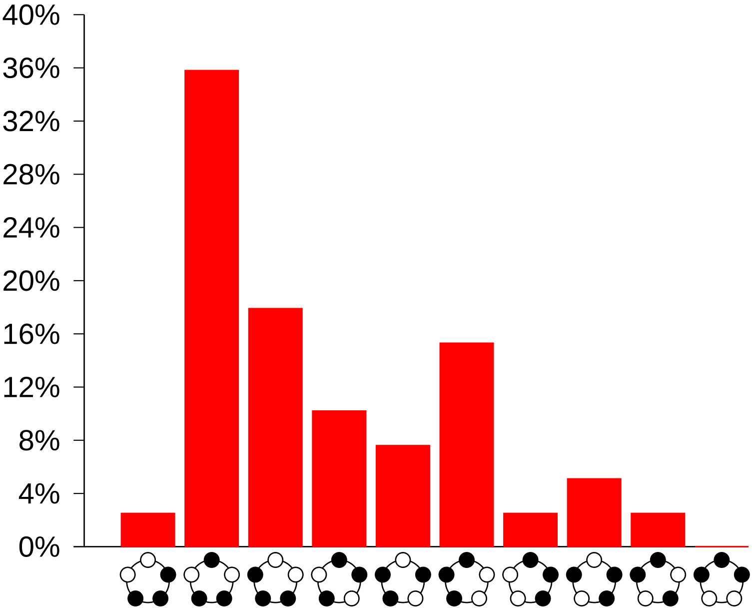

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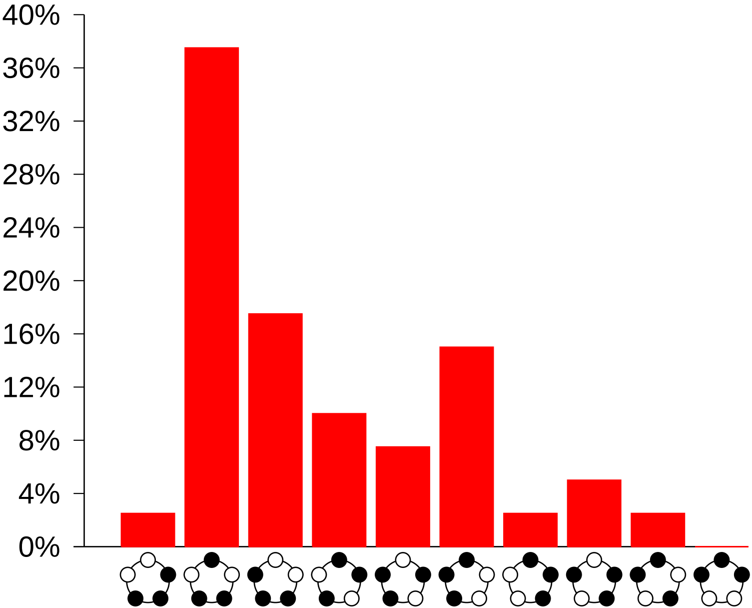

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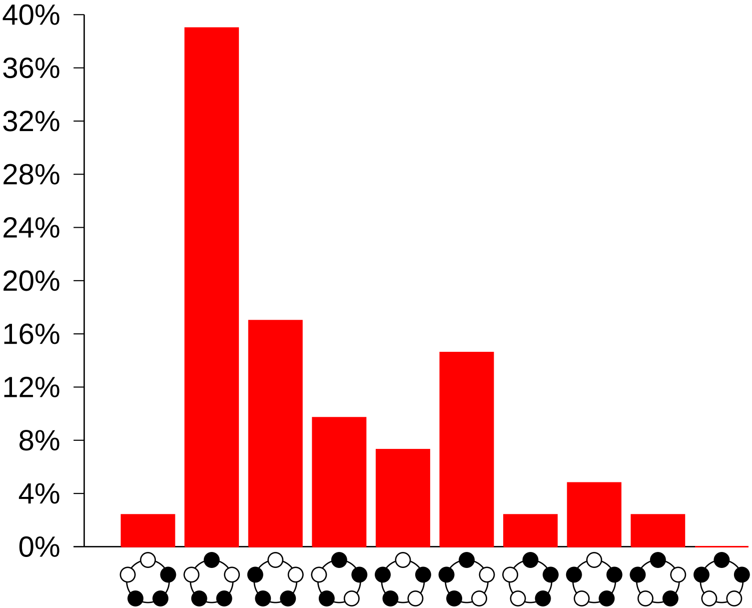
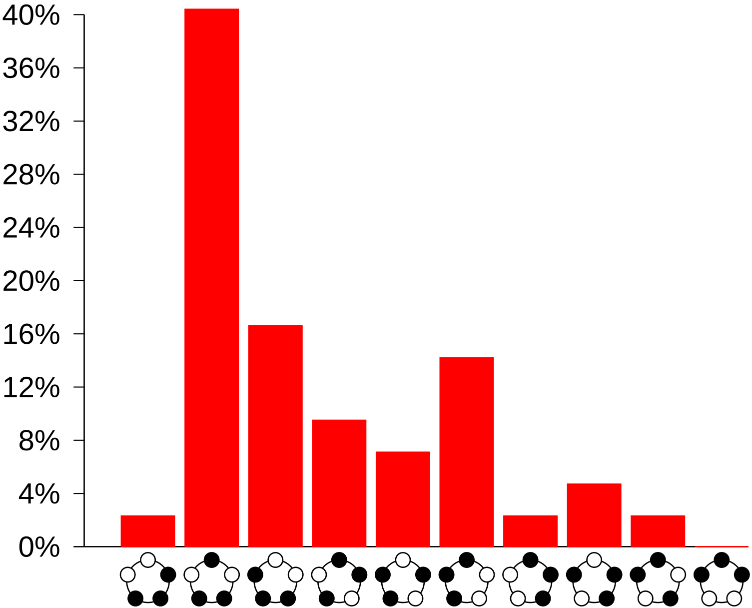


Stationary distribution



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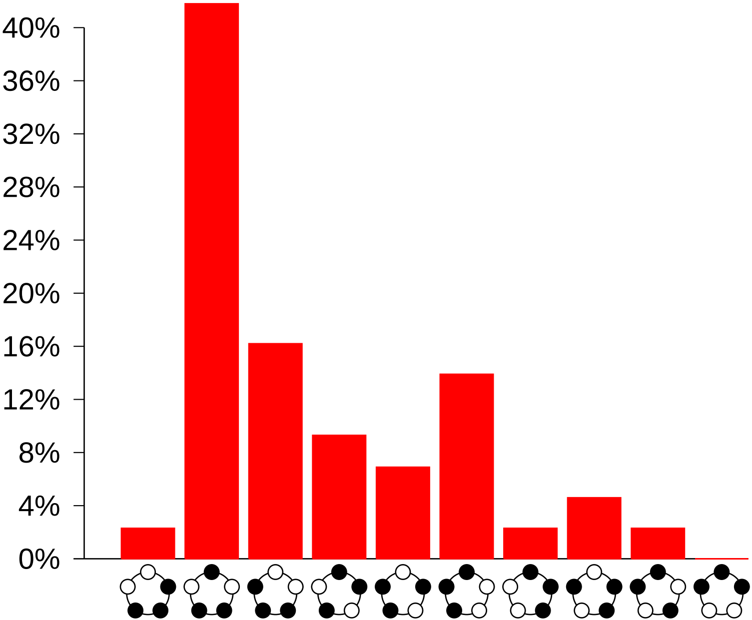

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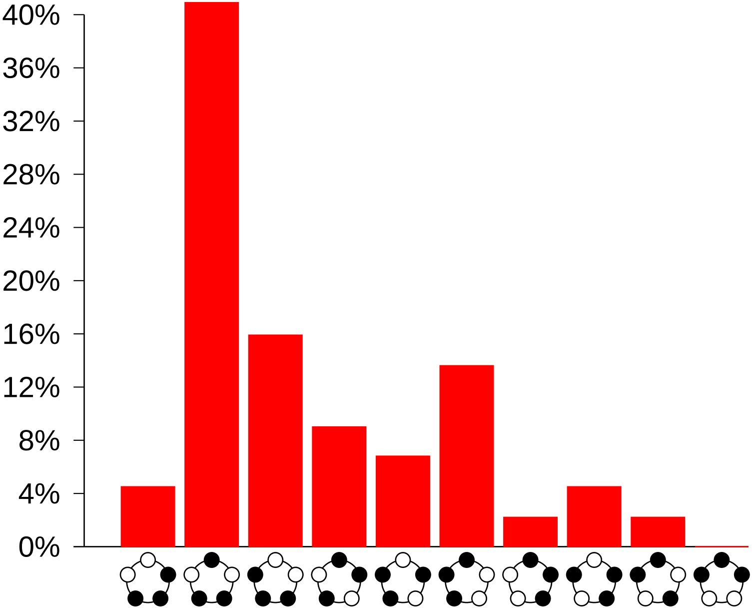

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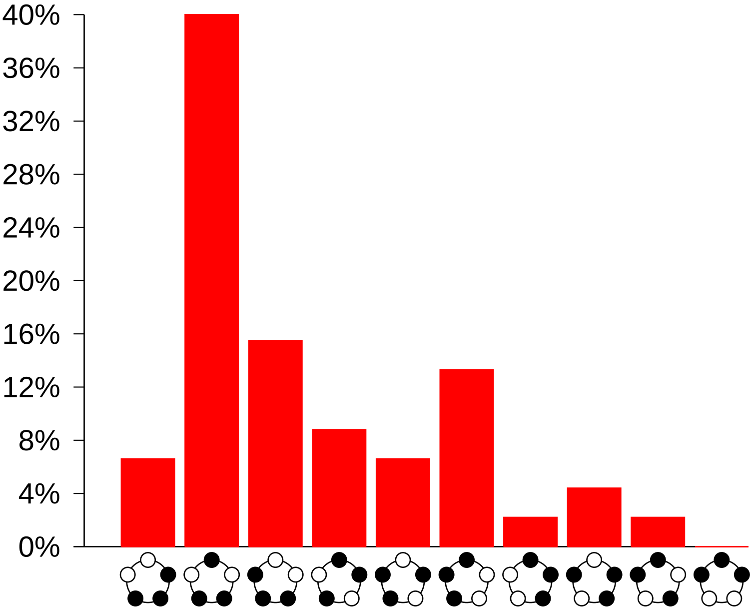

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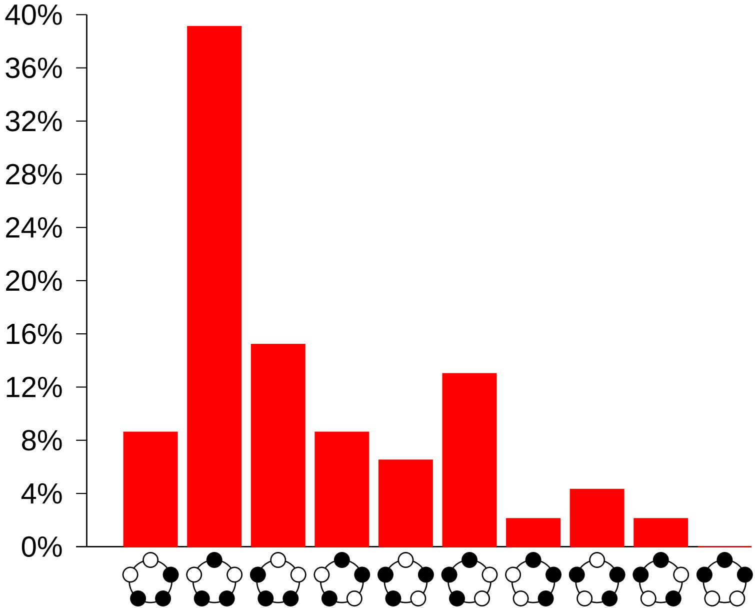

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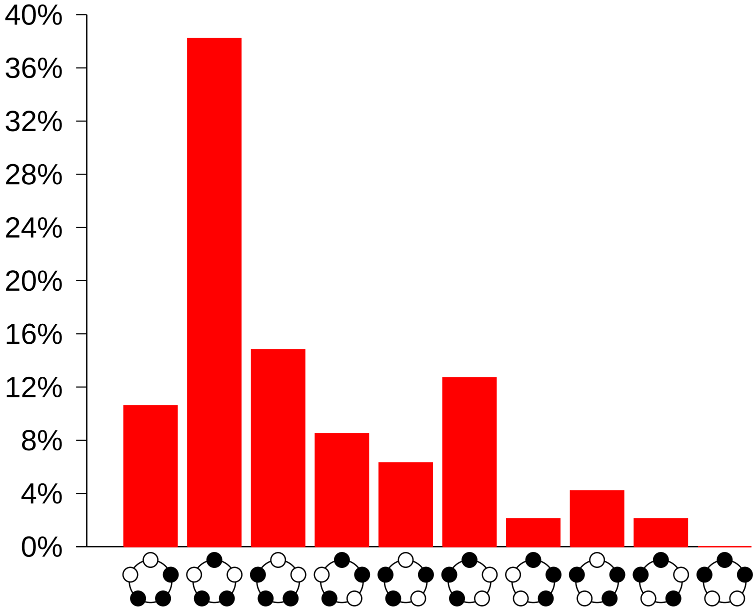

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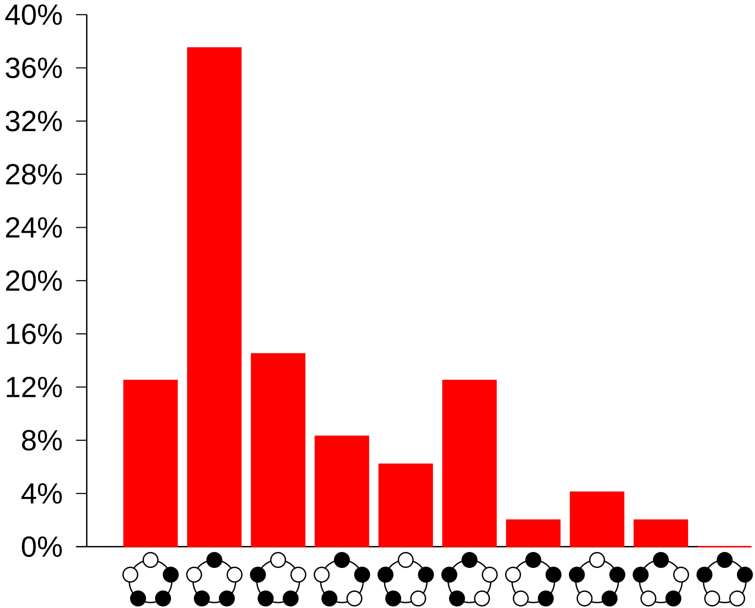

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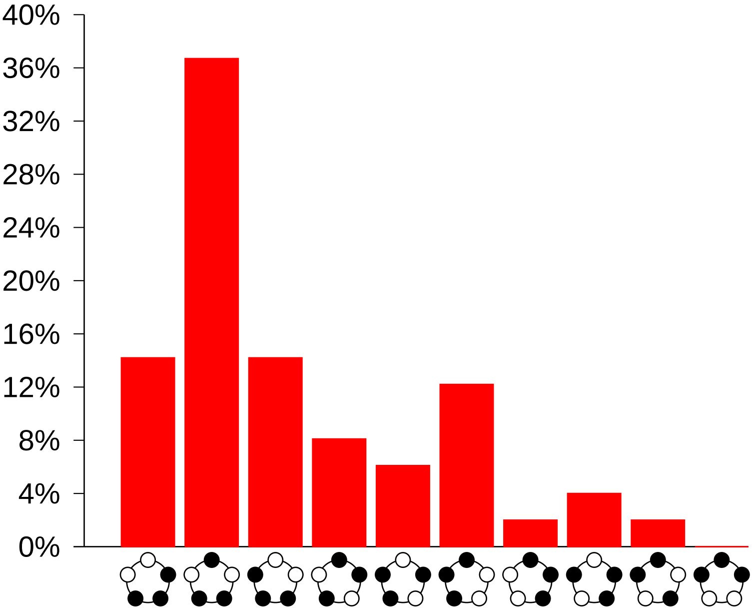

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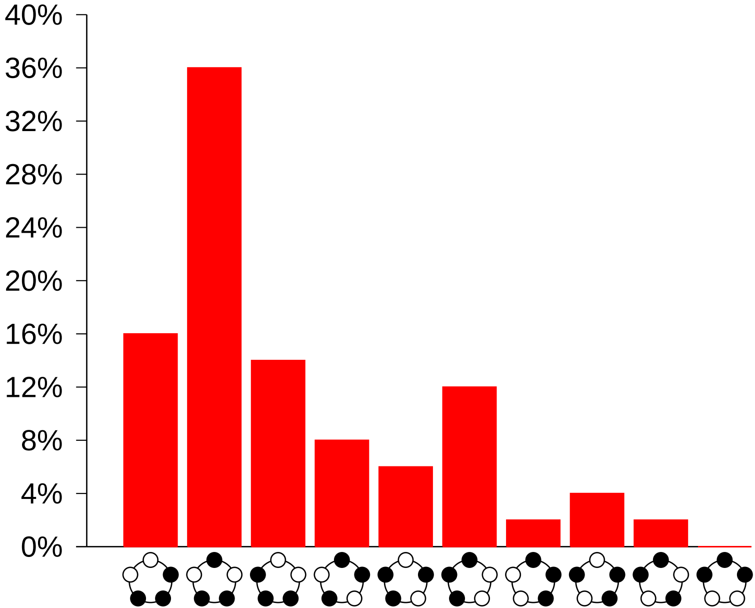

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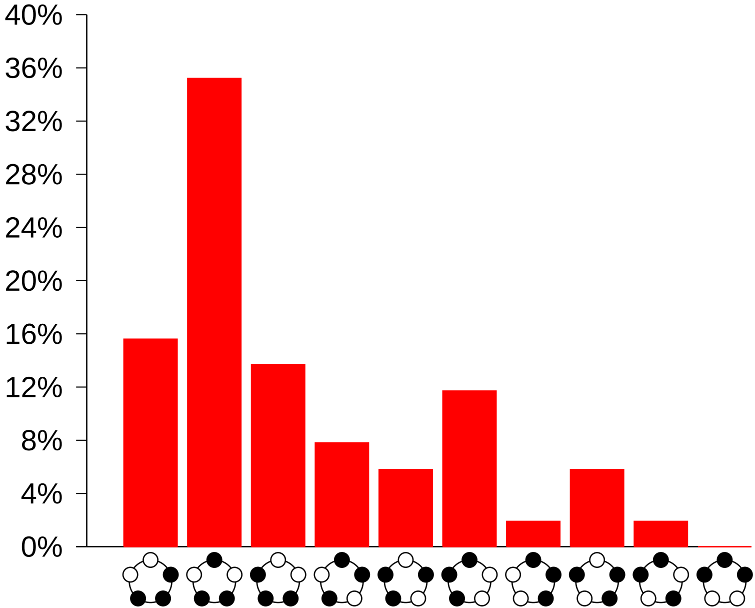

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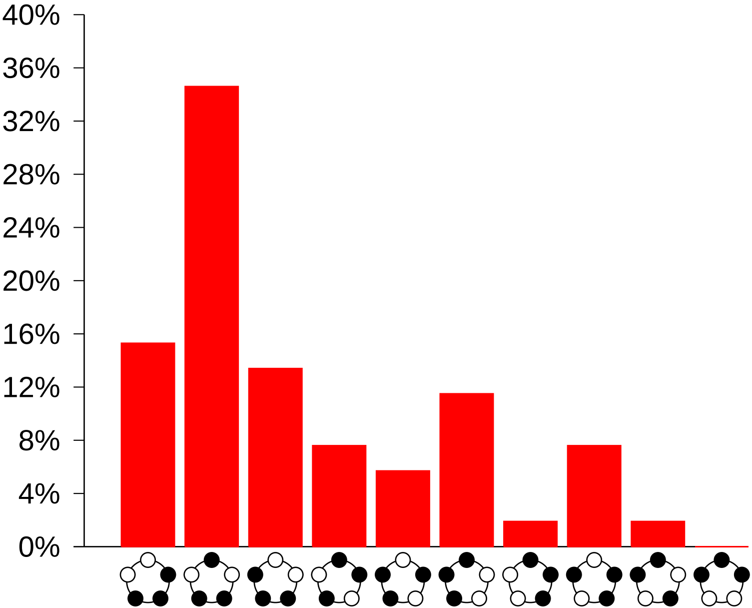

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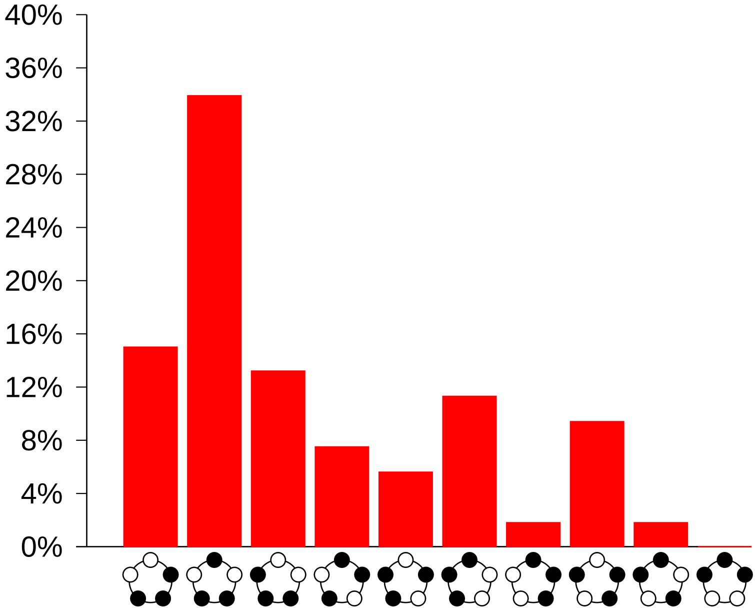

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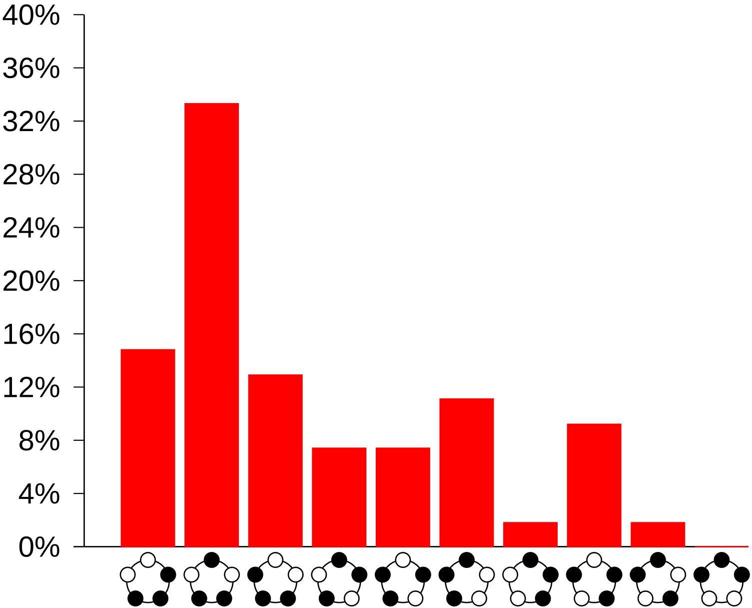

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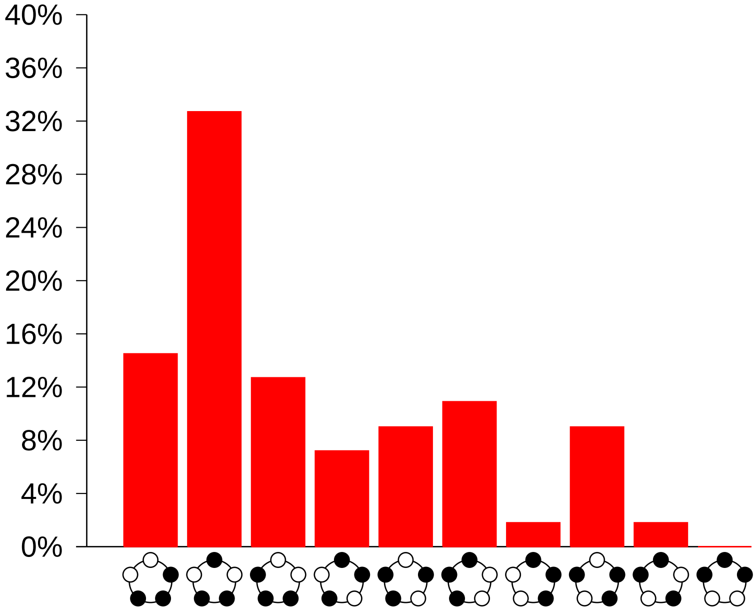

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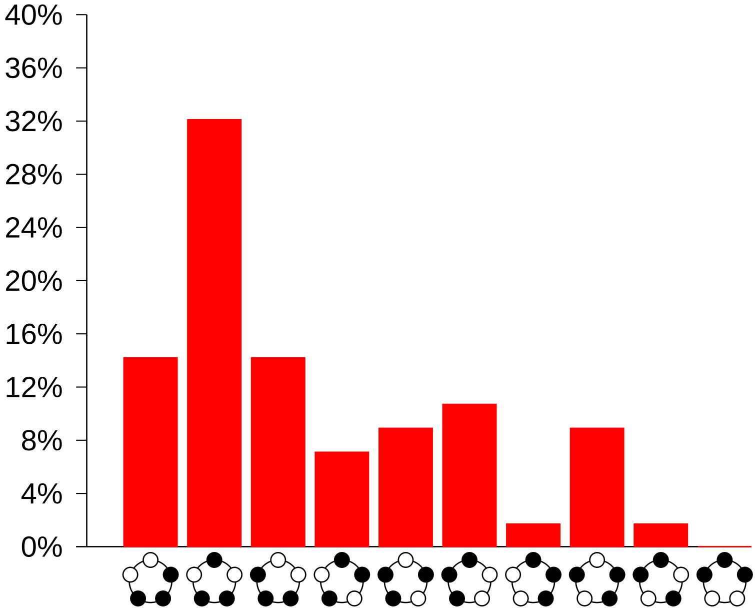

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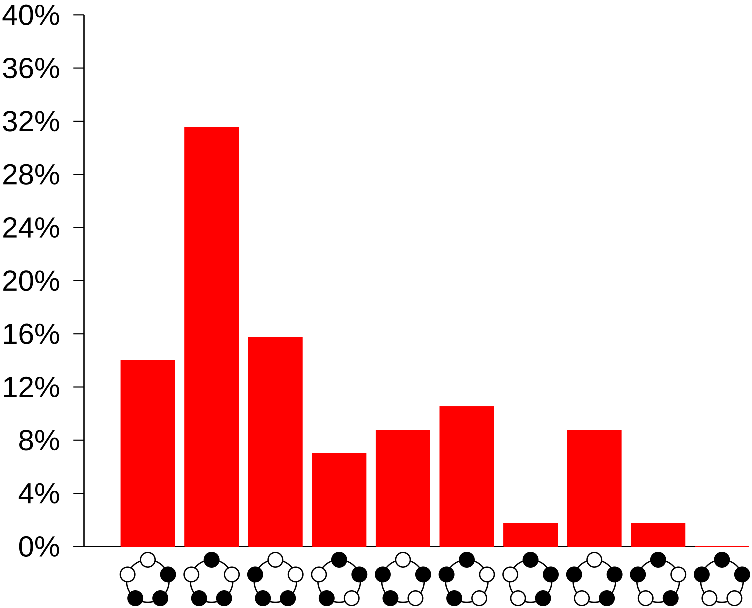

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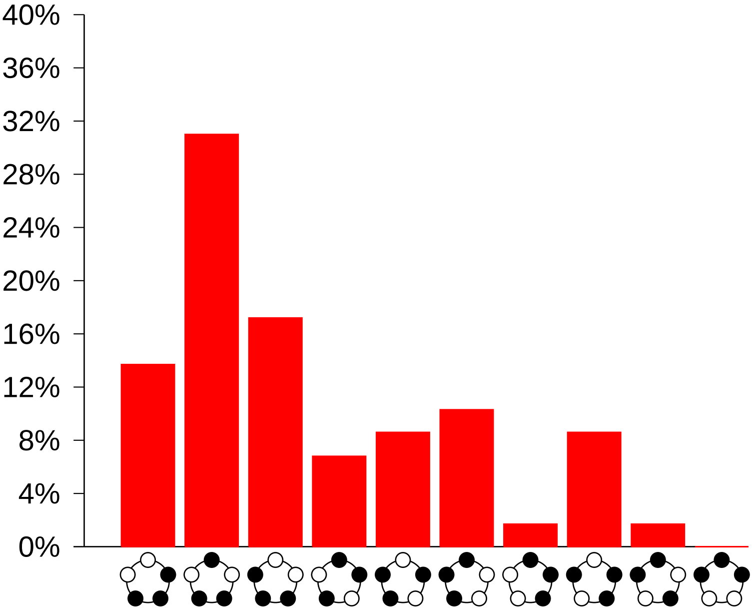

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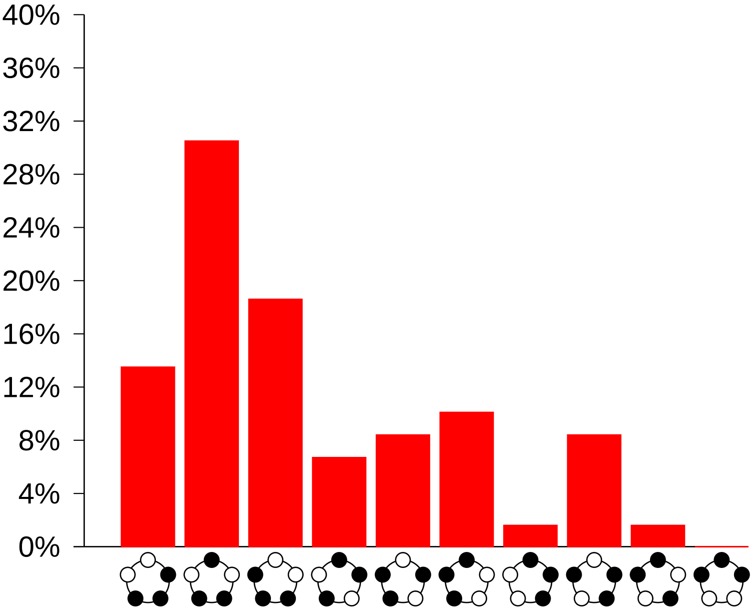

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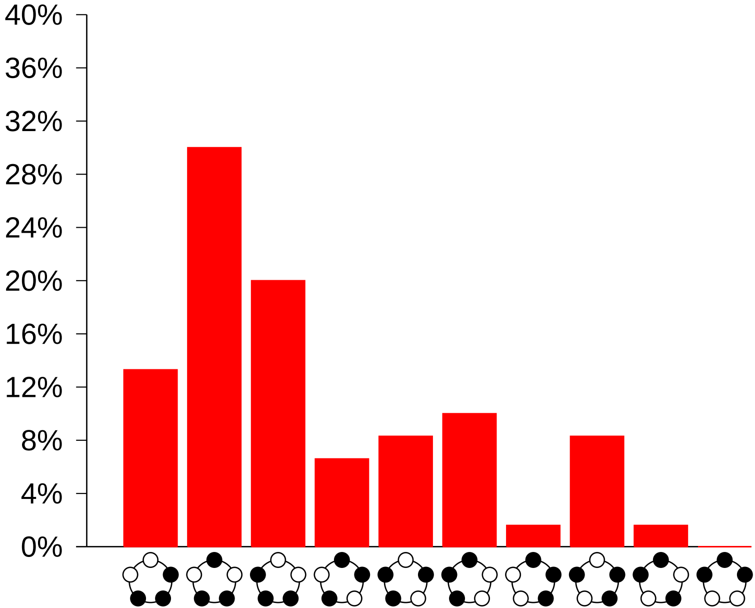

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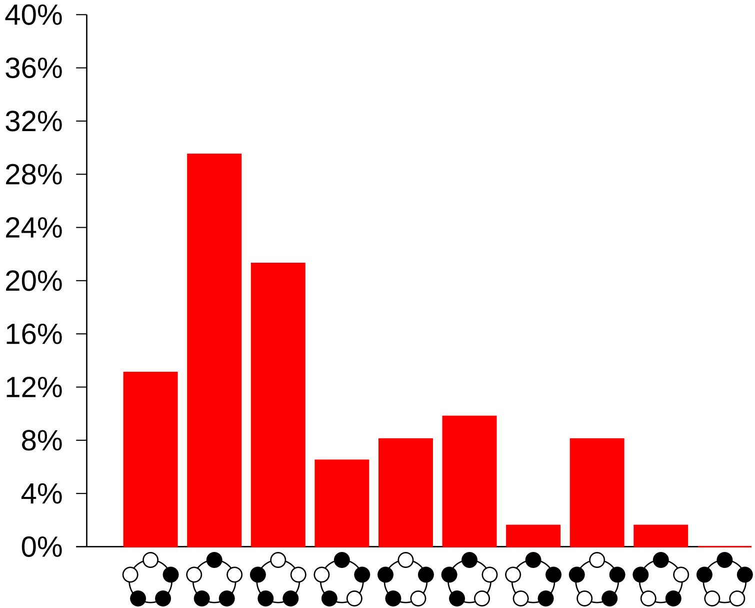

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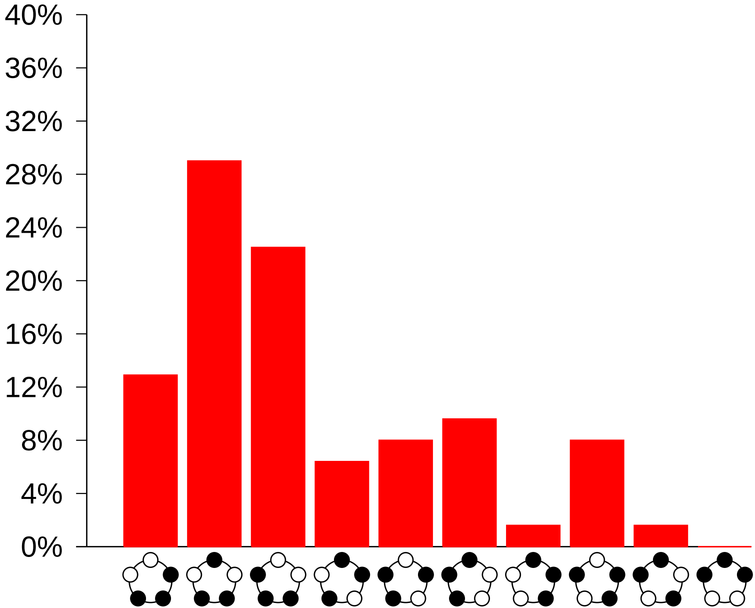

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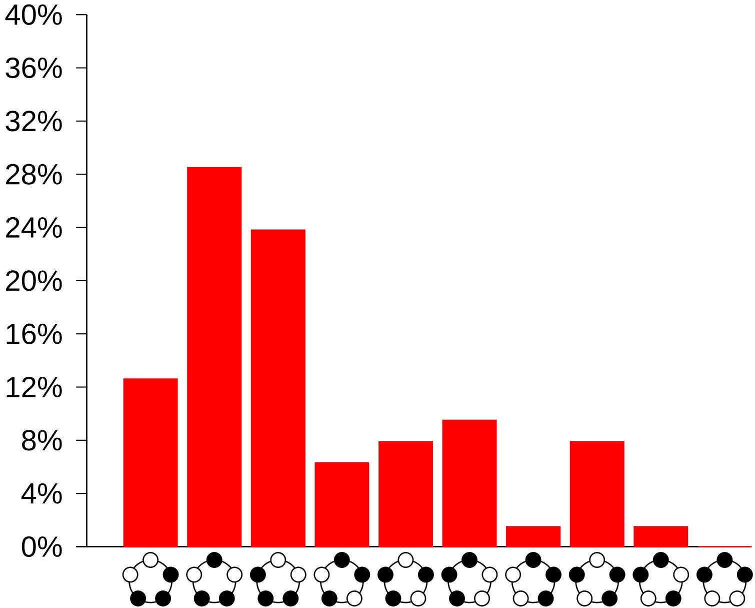

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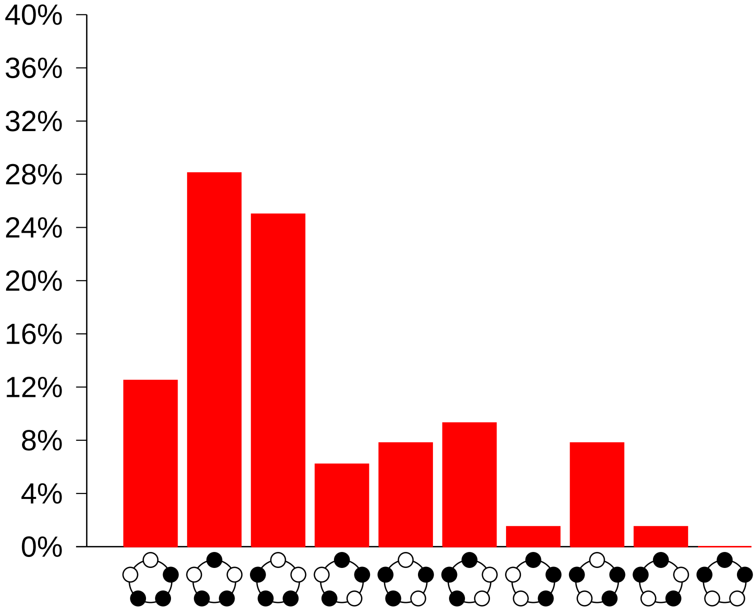

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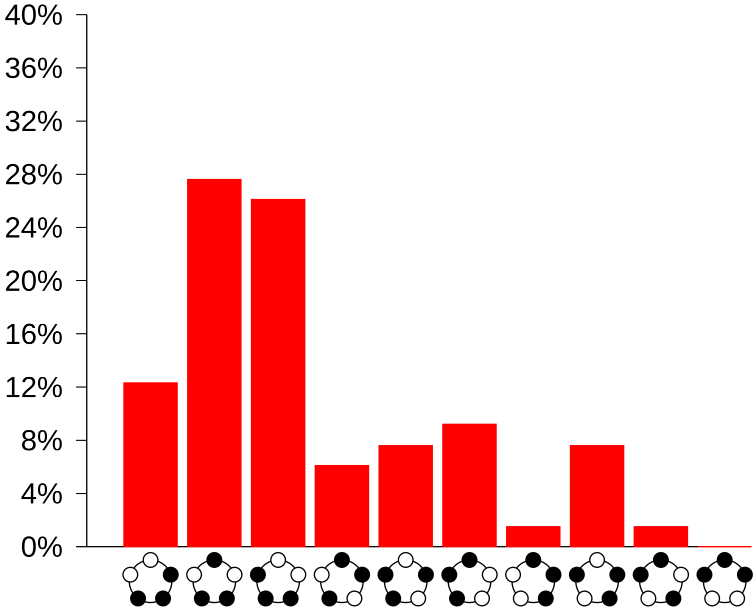

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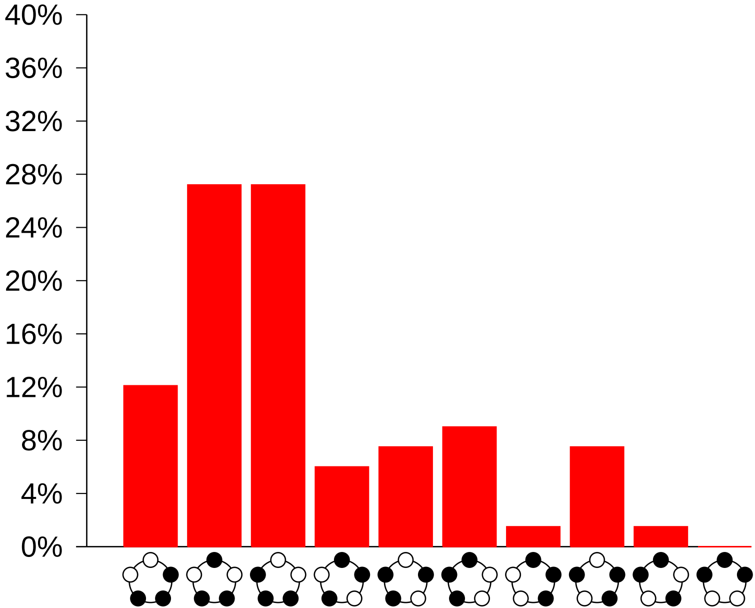

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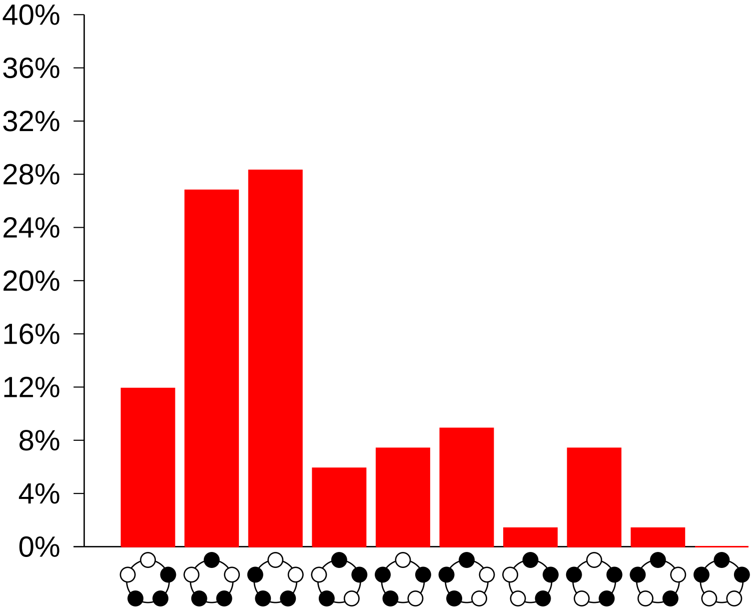

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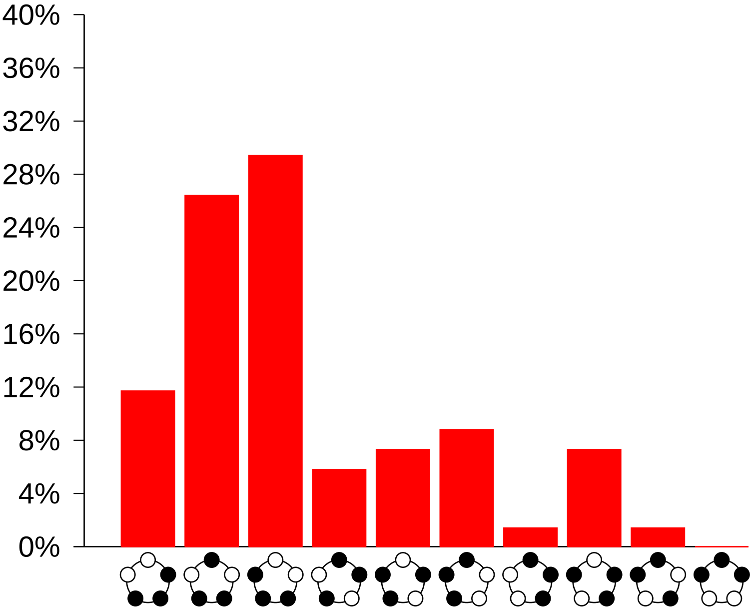

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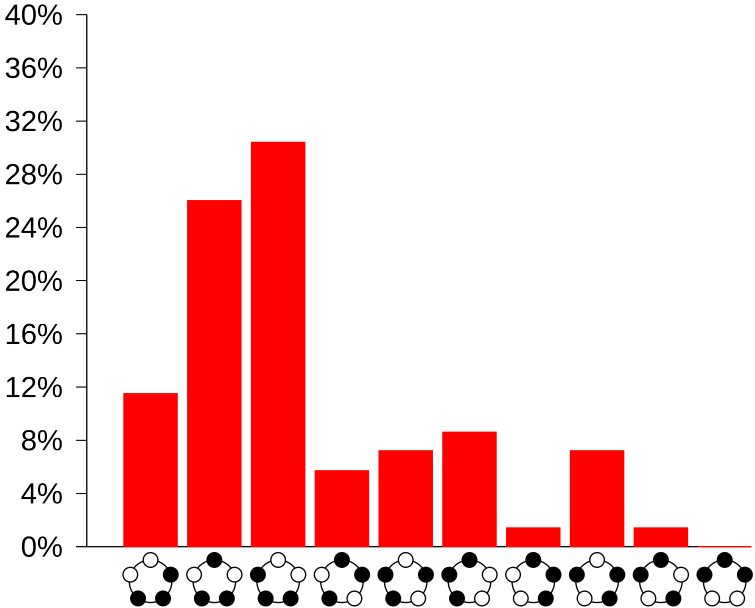

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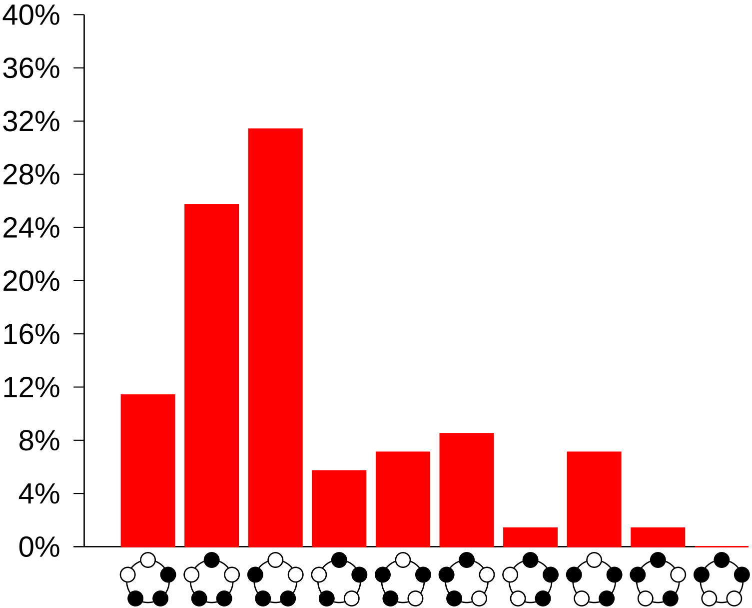

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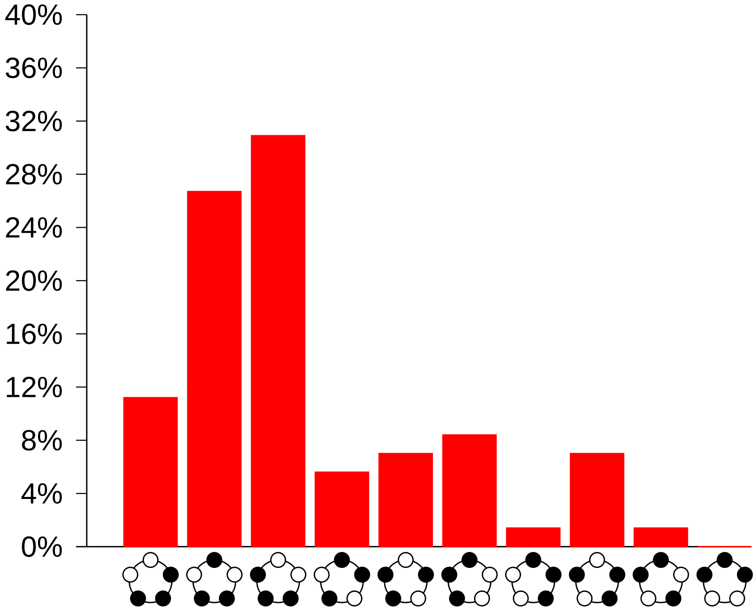

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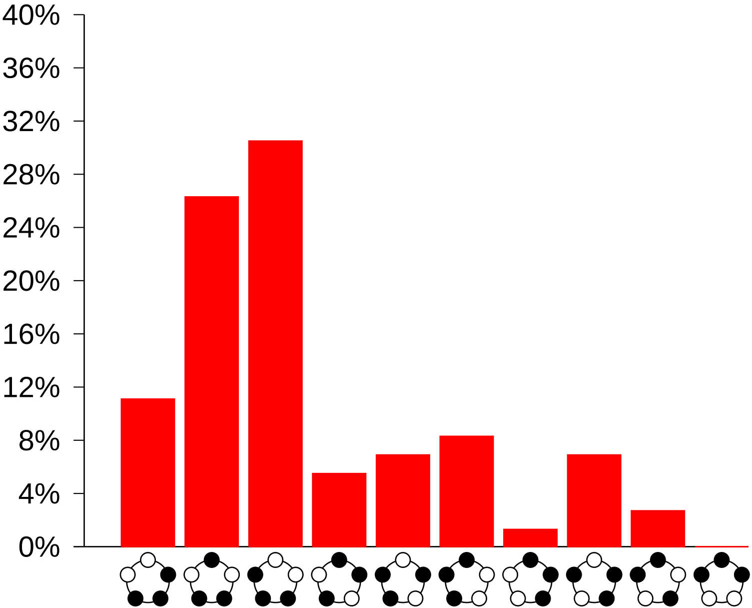

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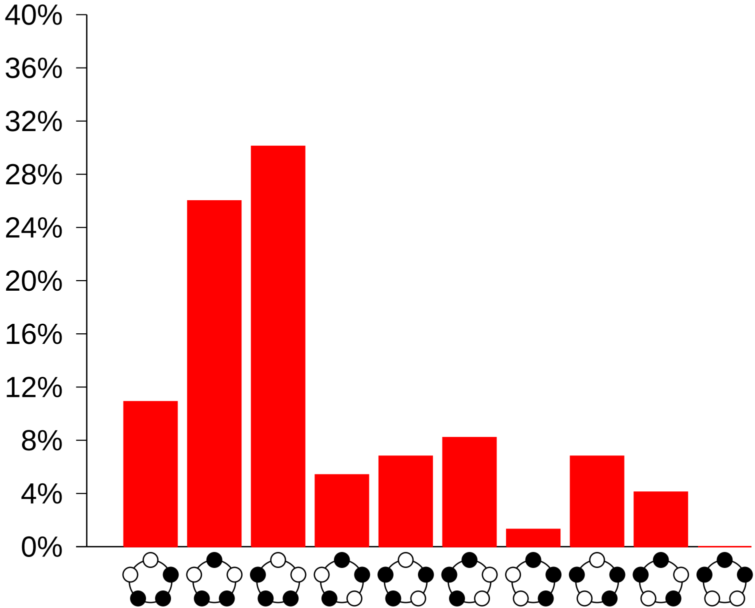

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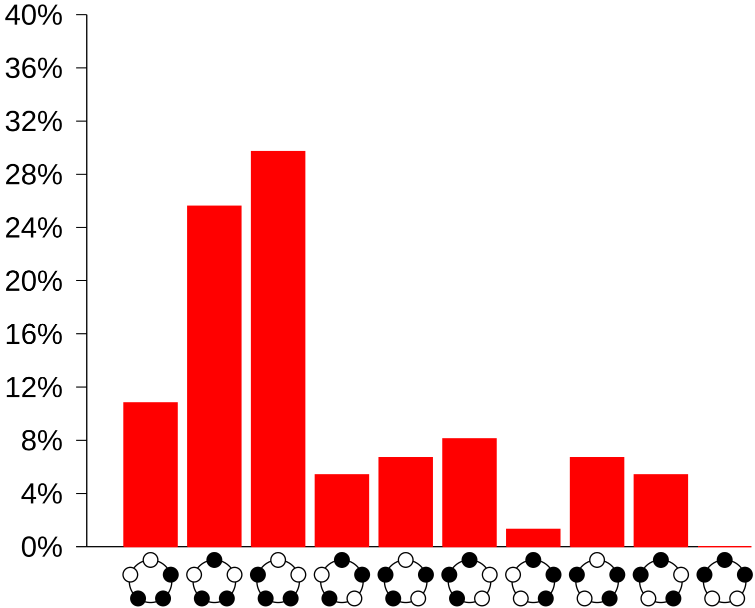

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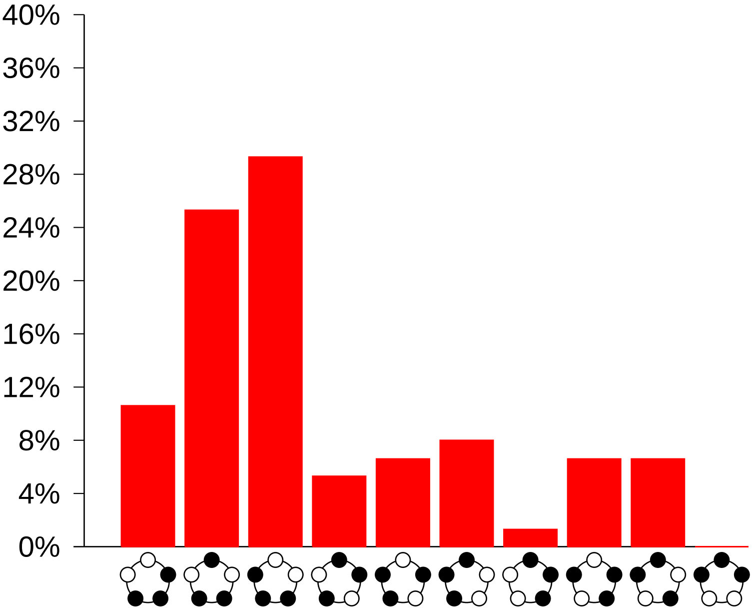

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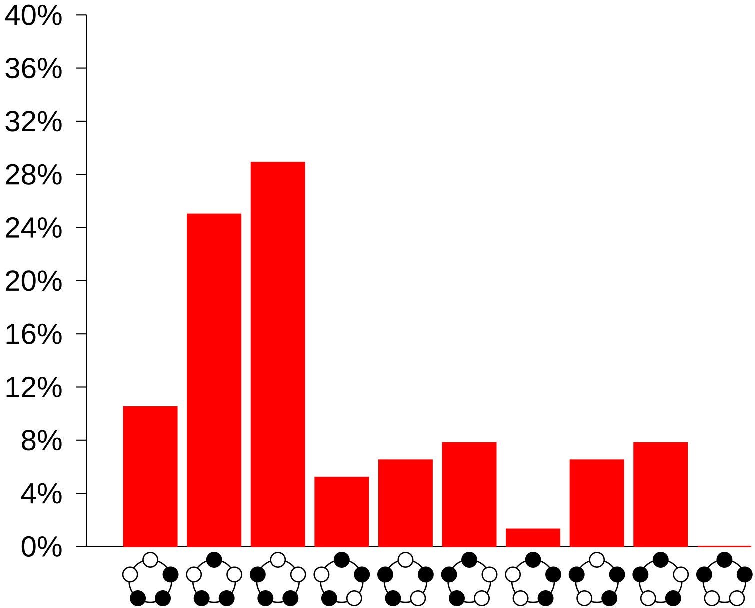

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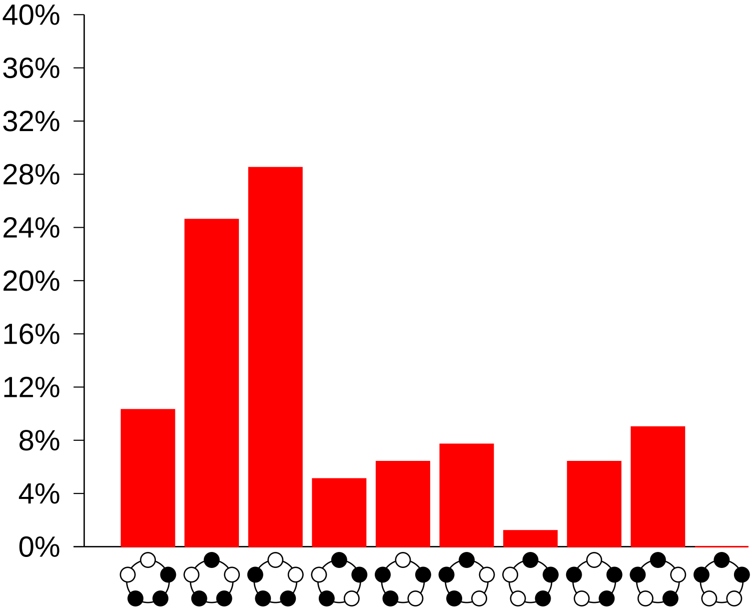

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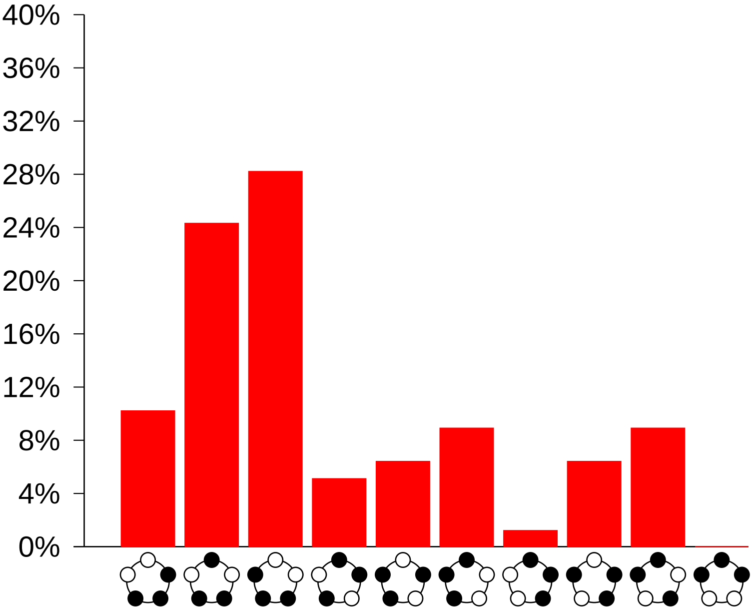

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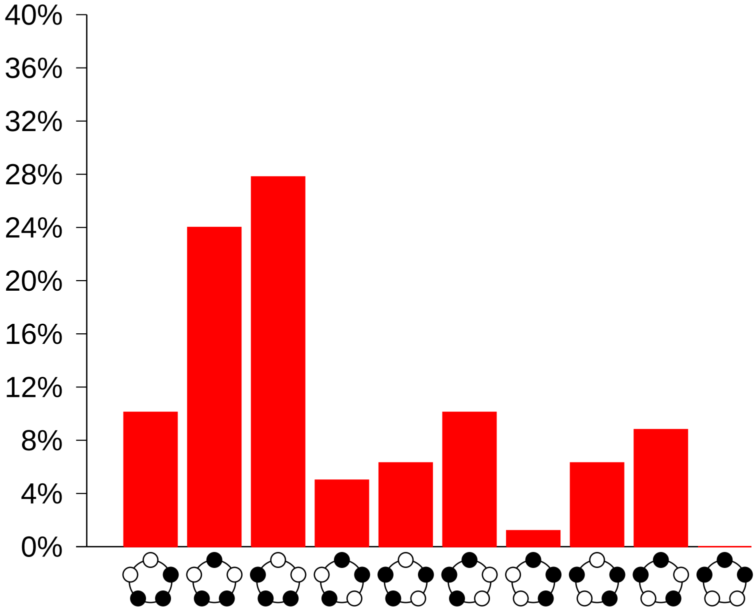

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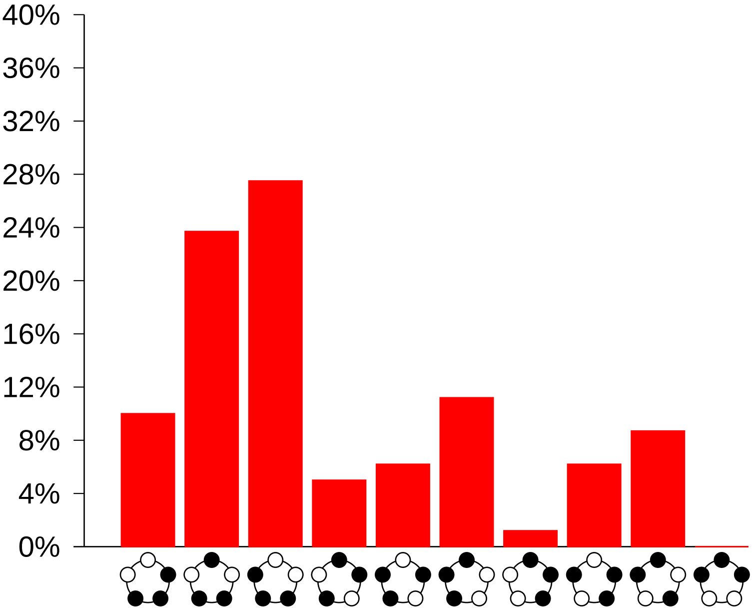

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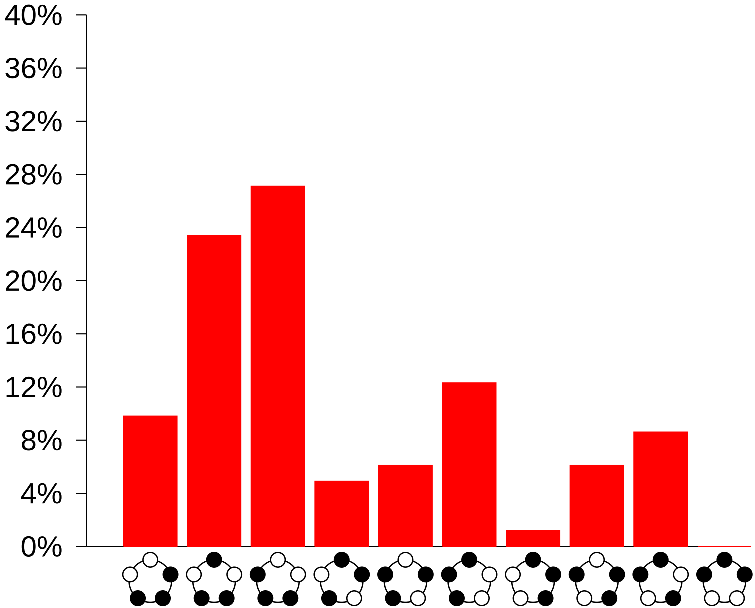

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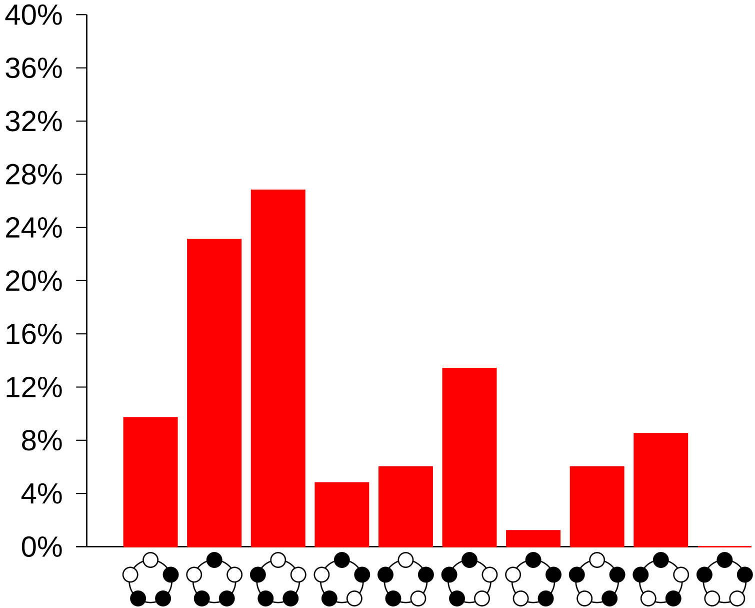

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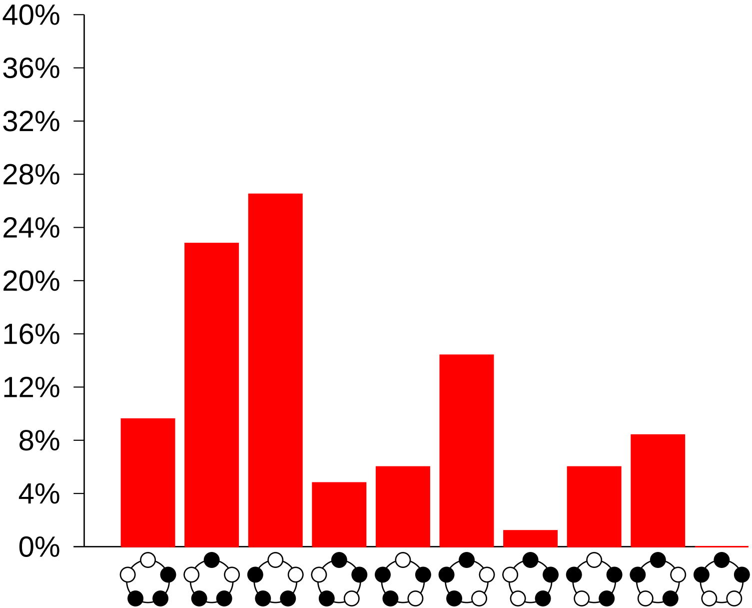

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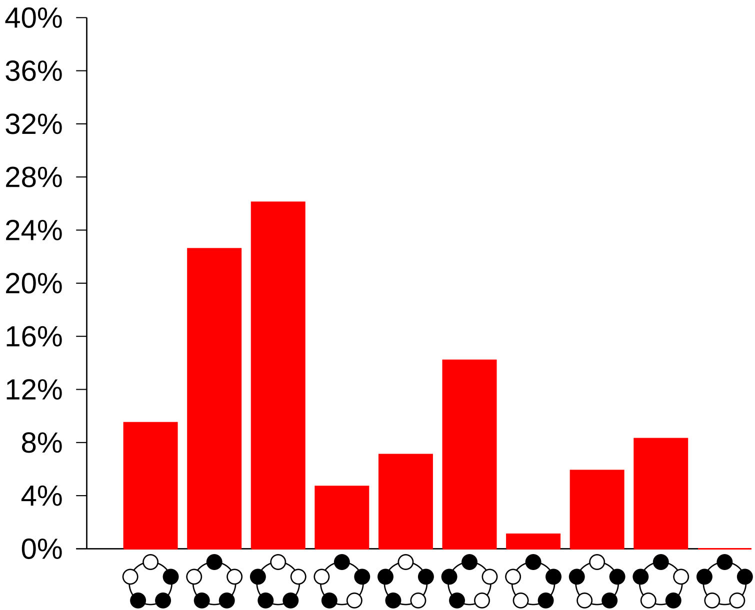

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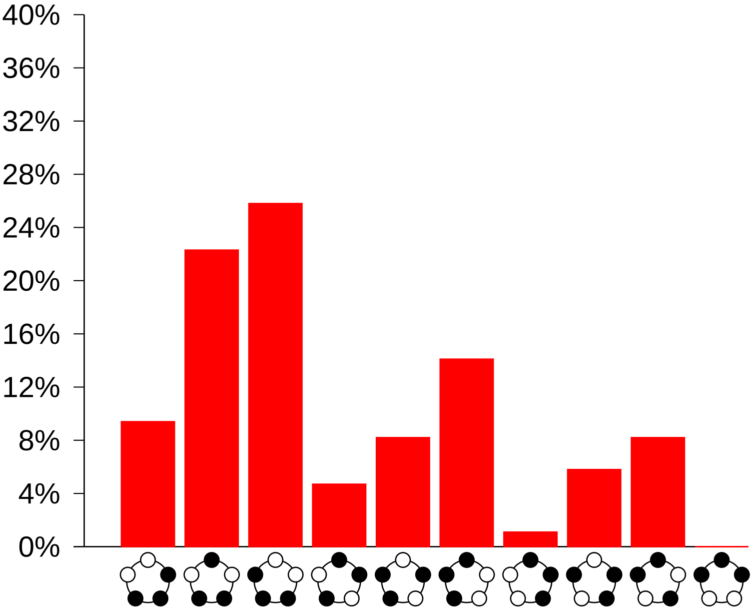

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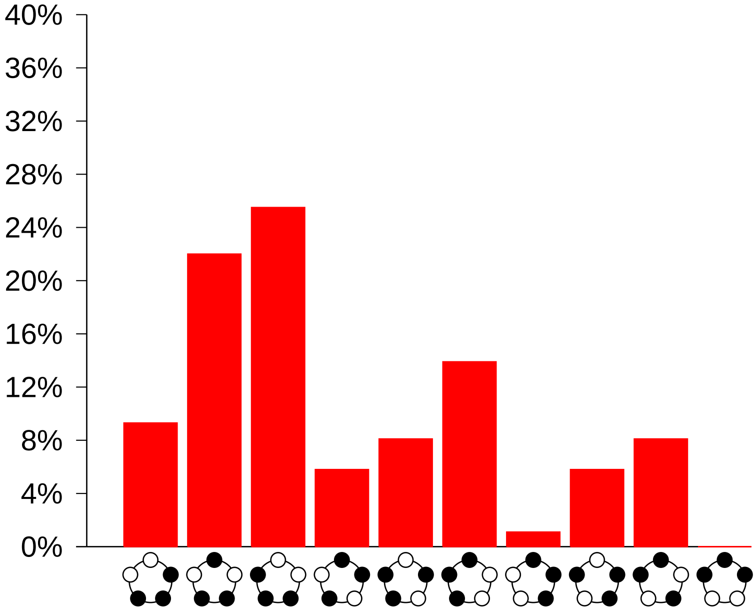

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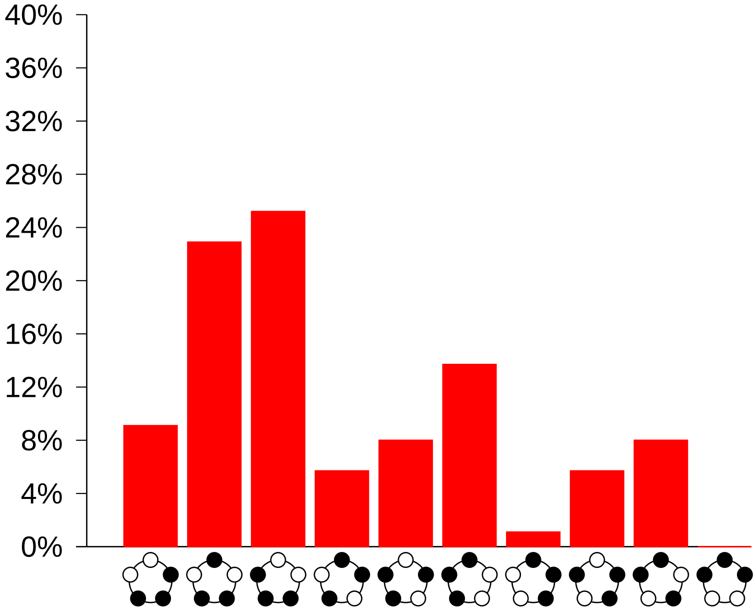

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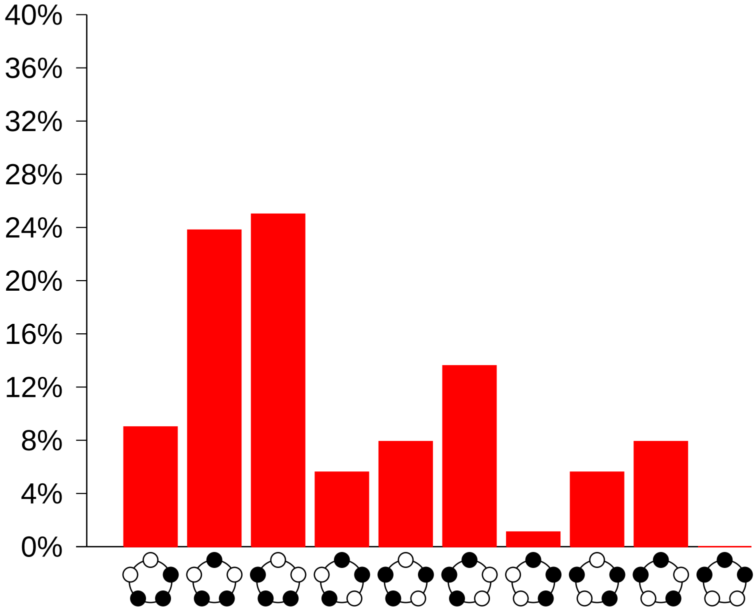

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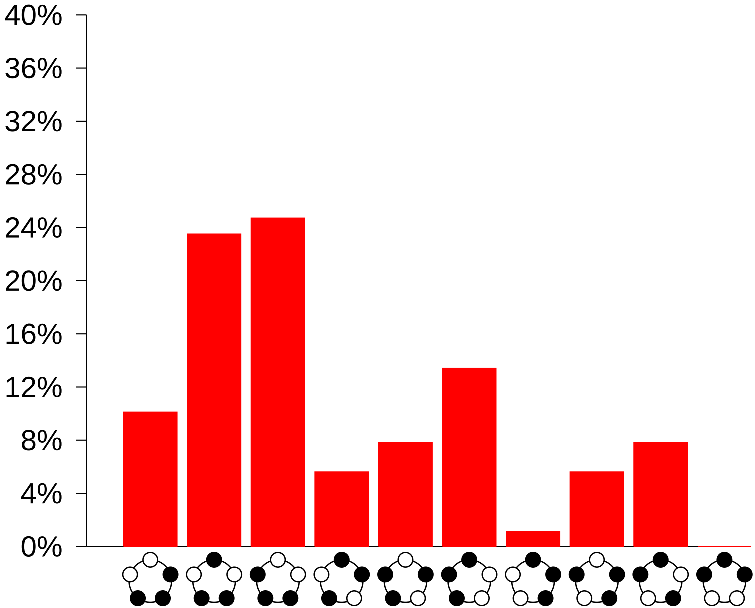

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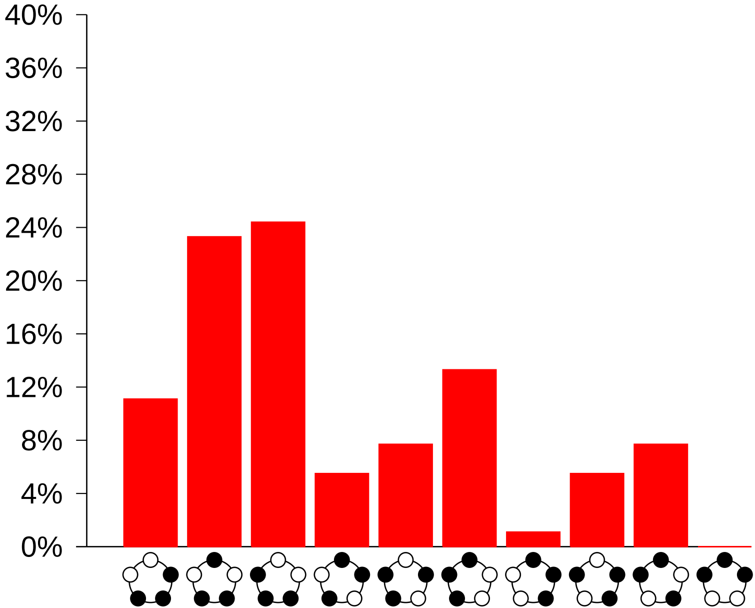

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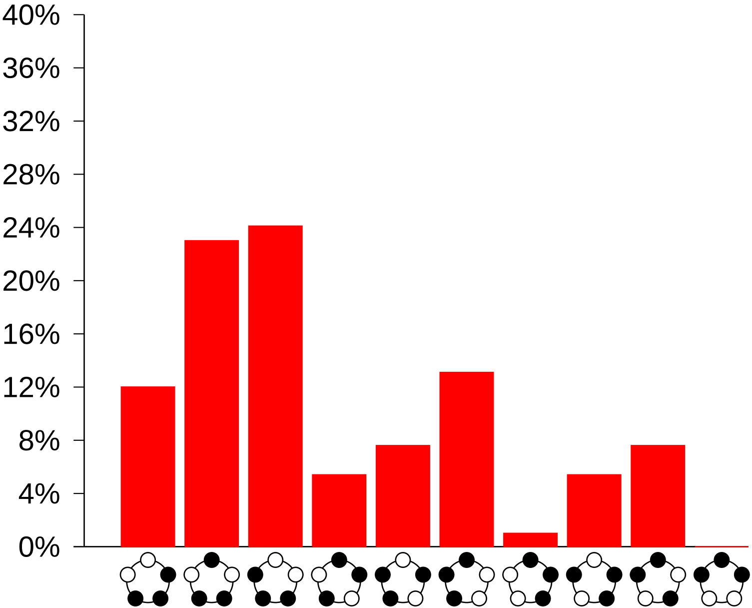

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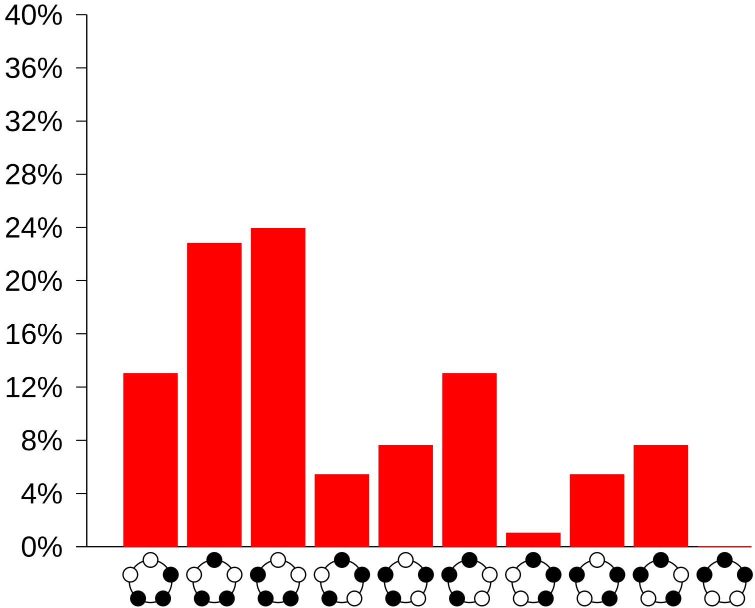

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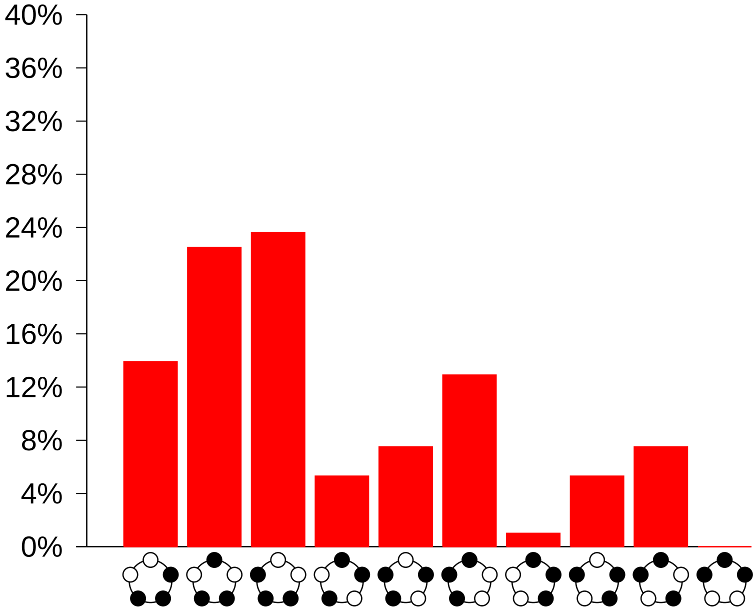

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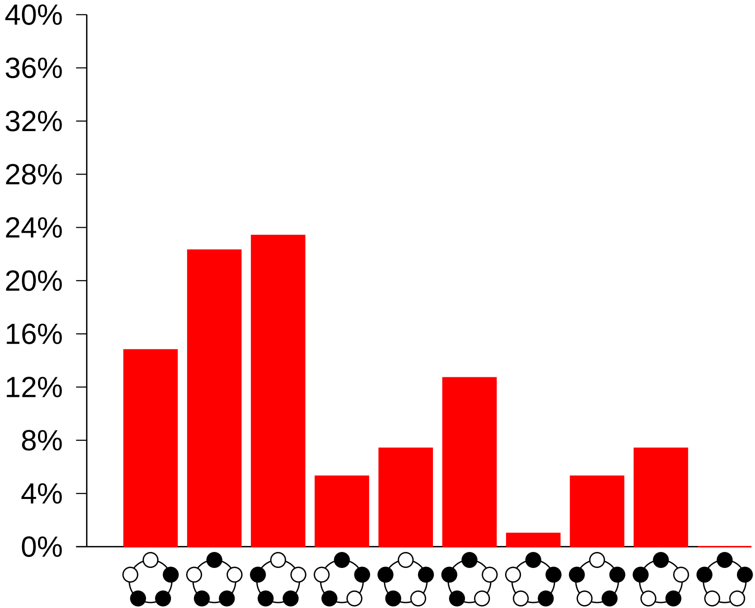

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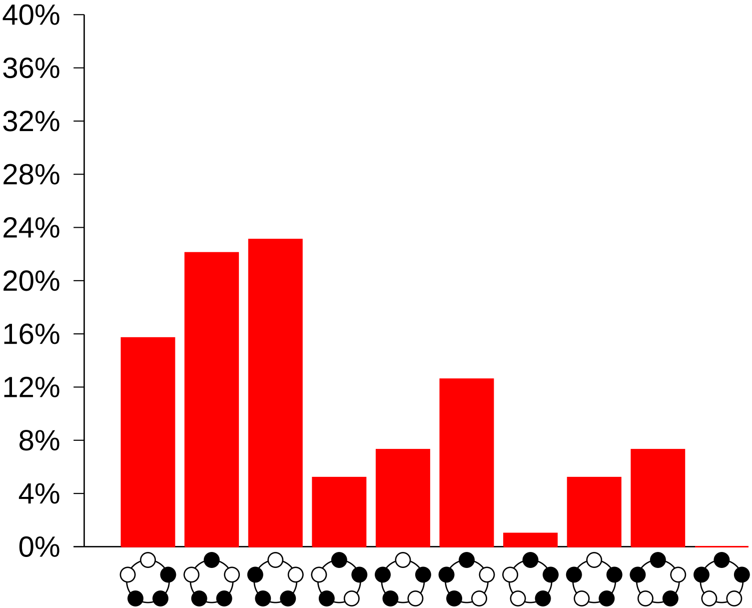

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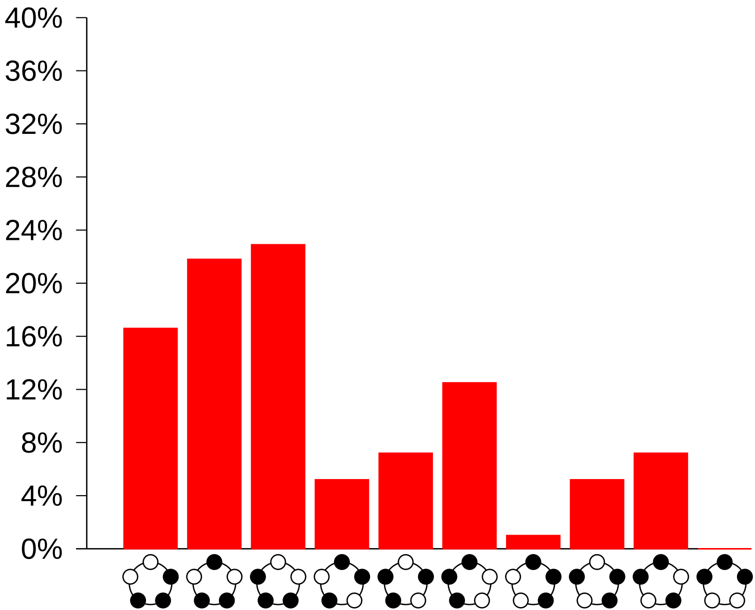

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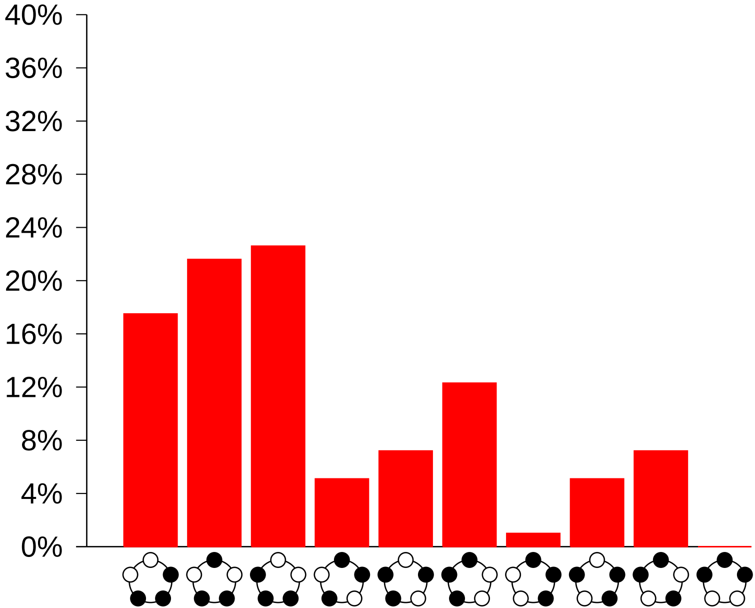

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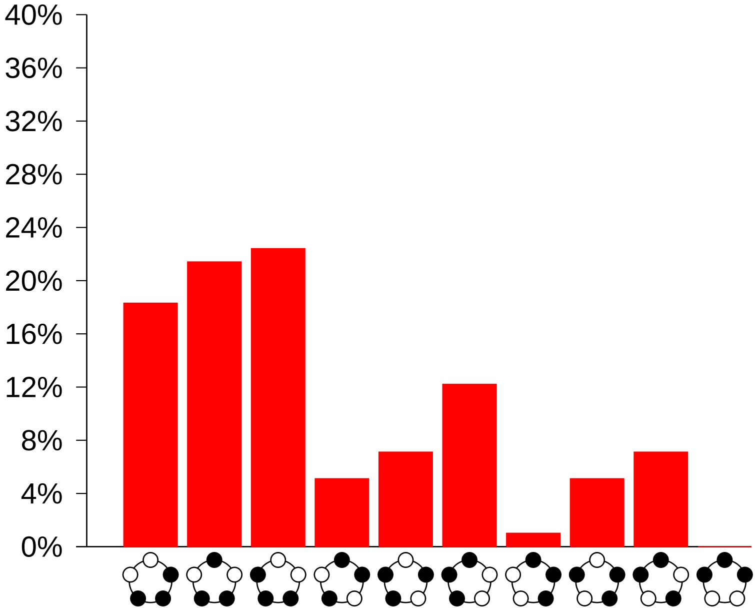

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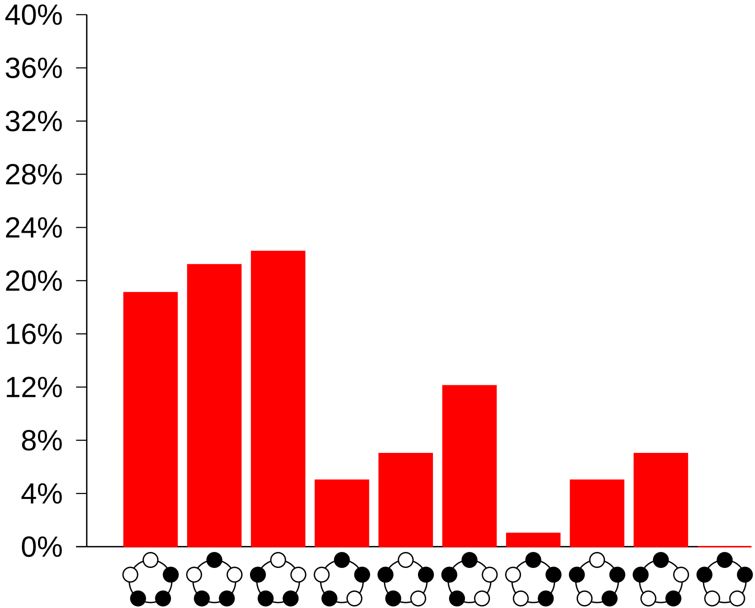

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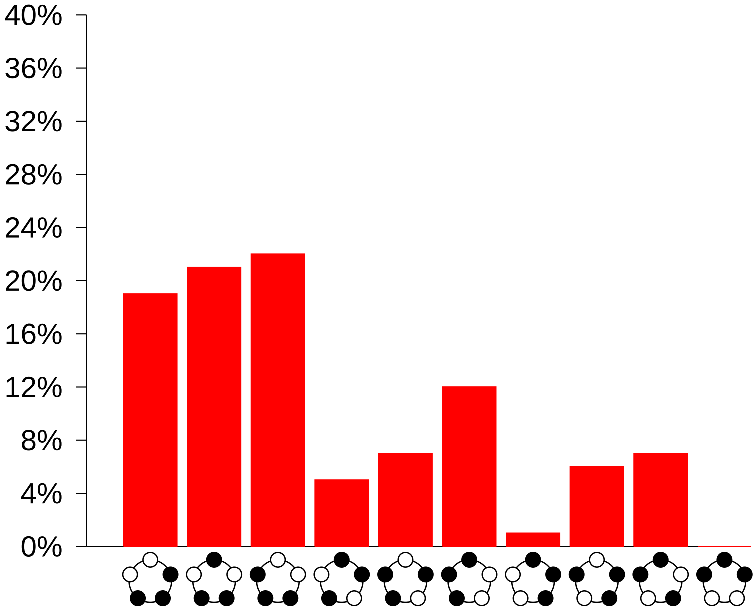

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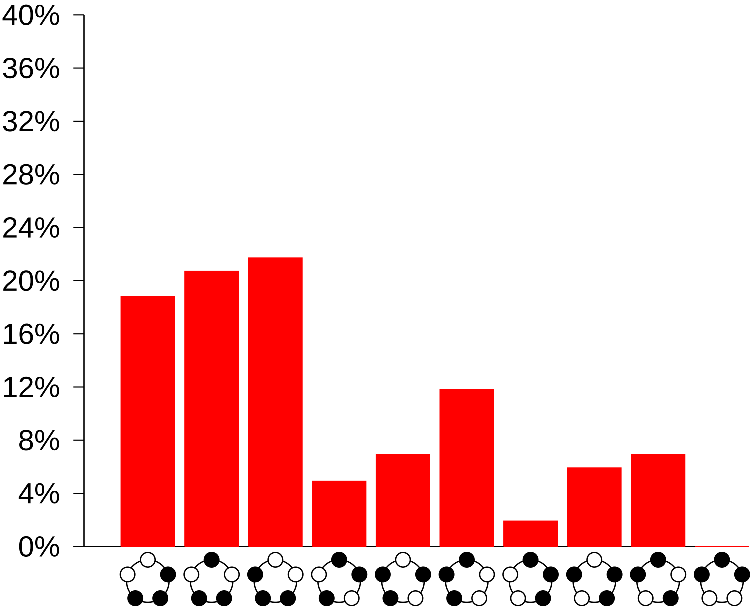

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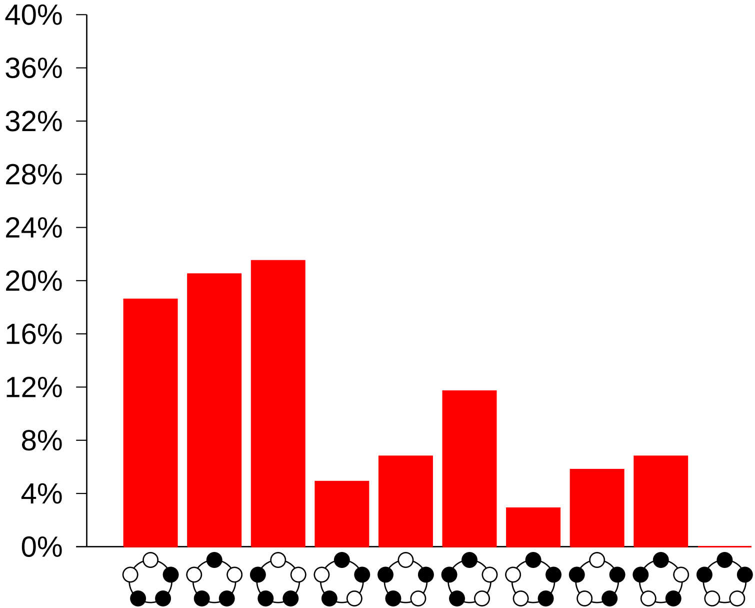

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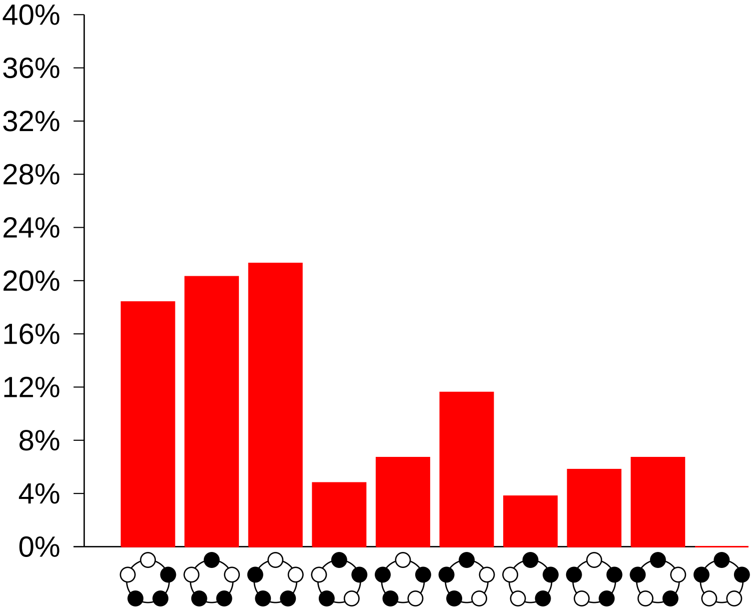

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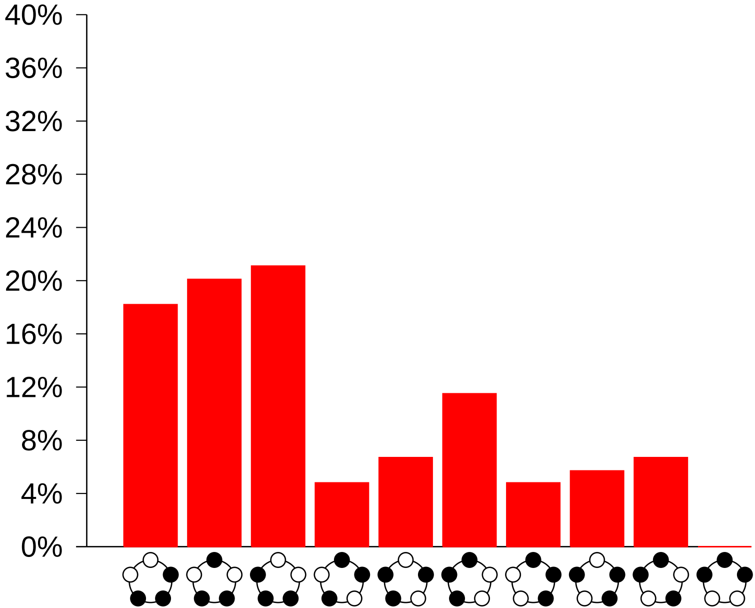

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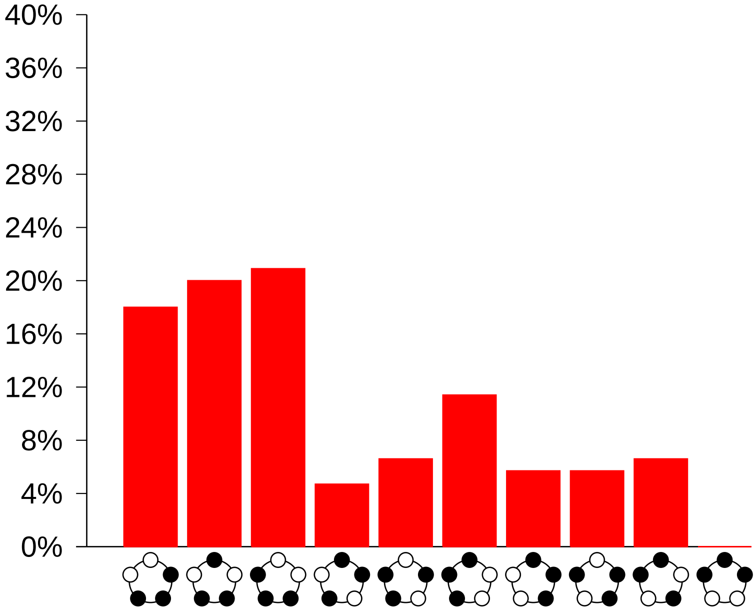

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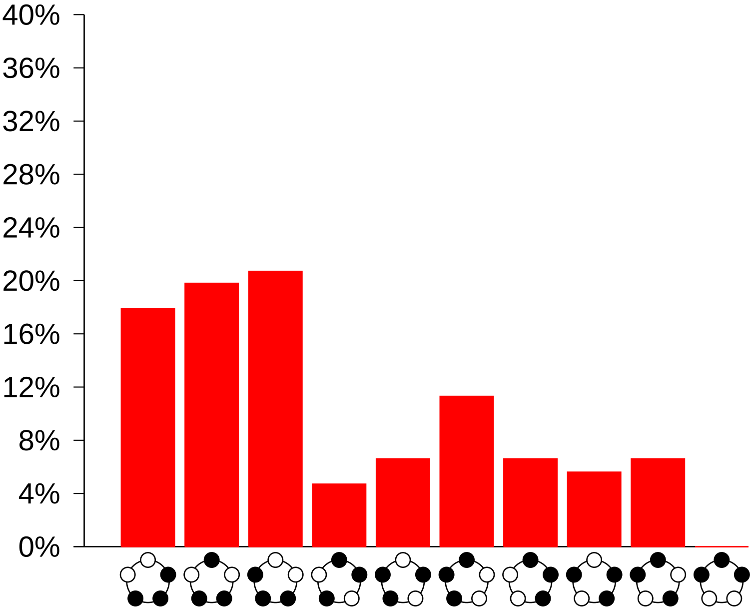

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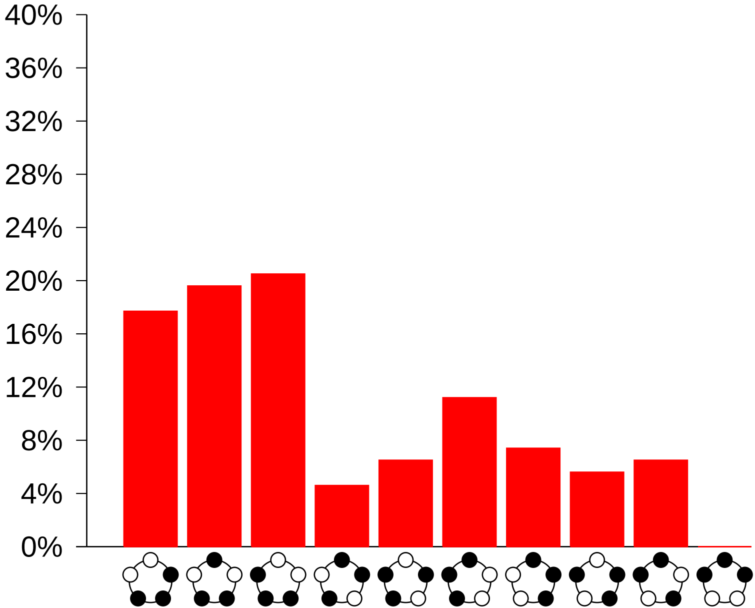

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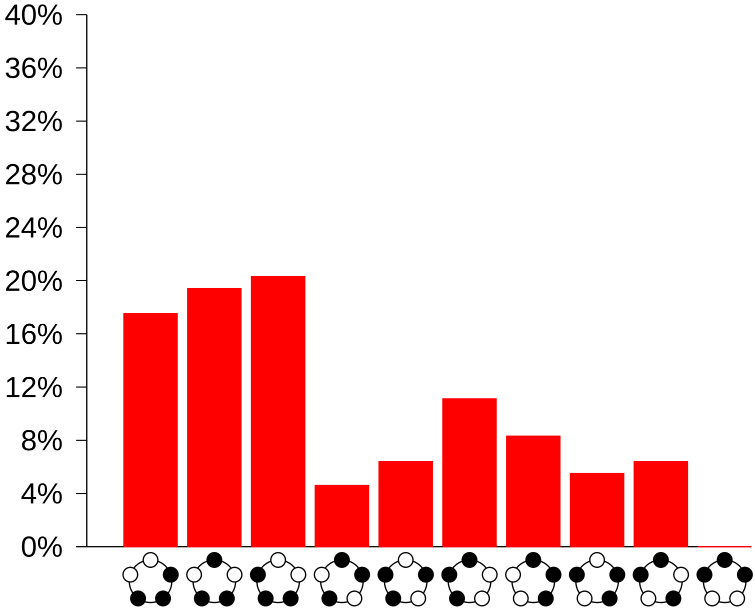

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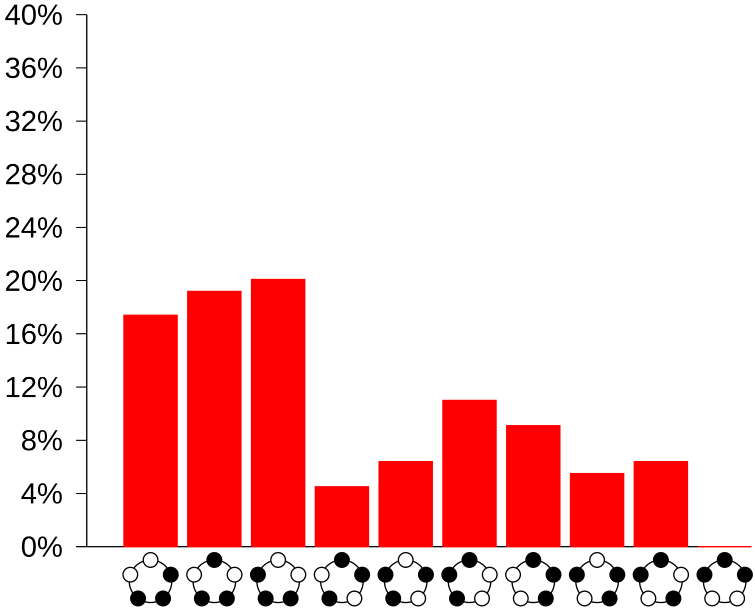

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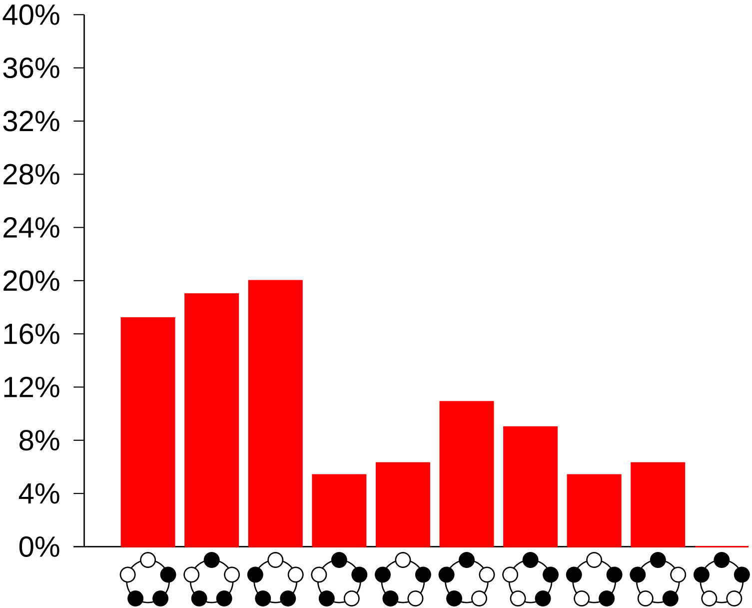

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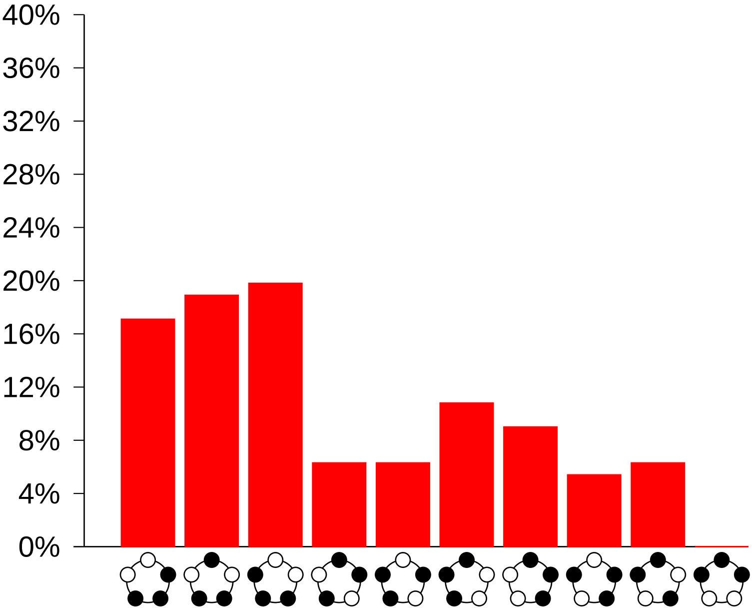

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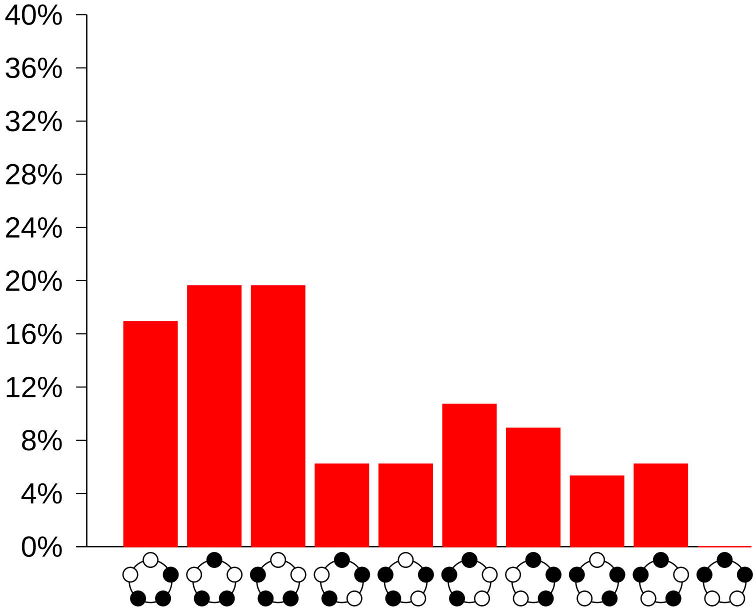

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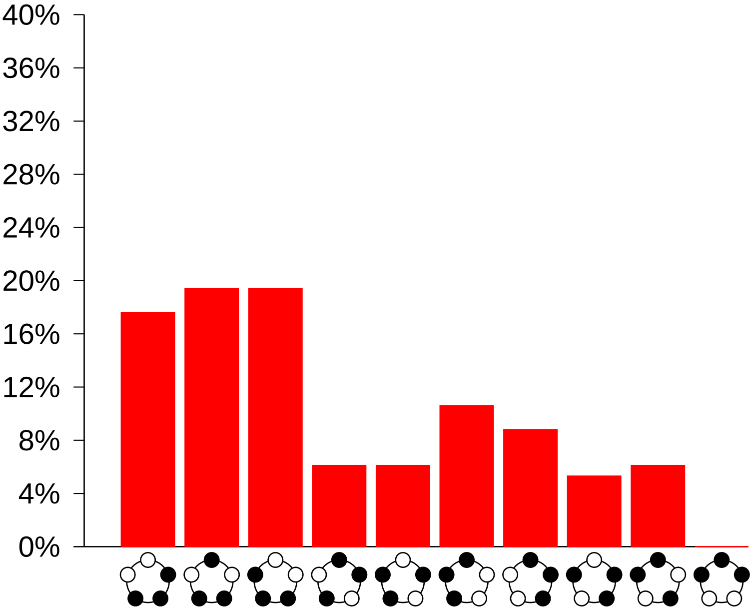

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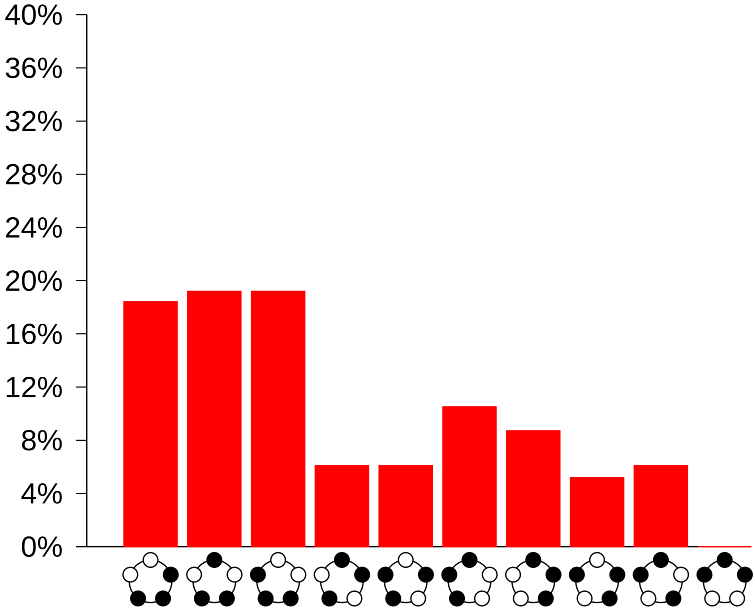

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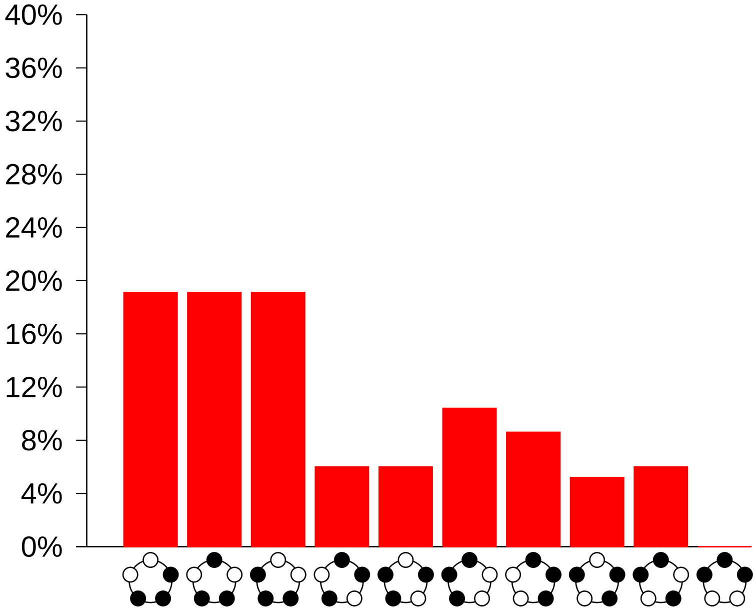

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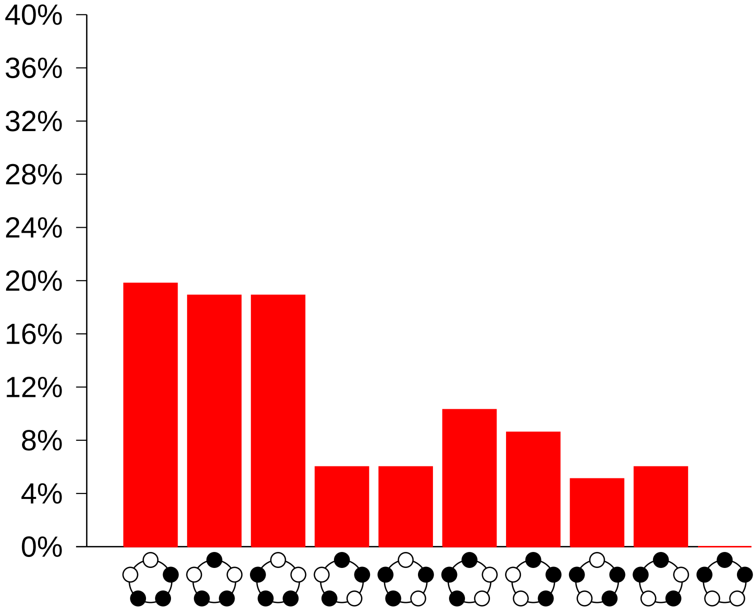

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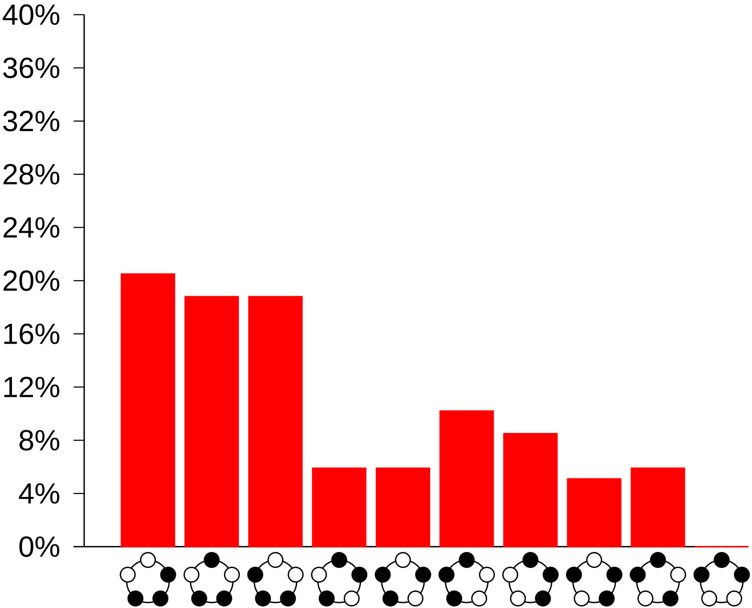

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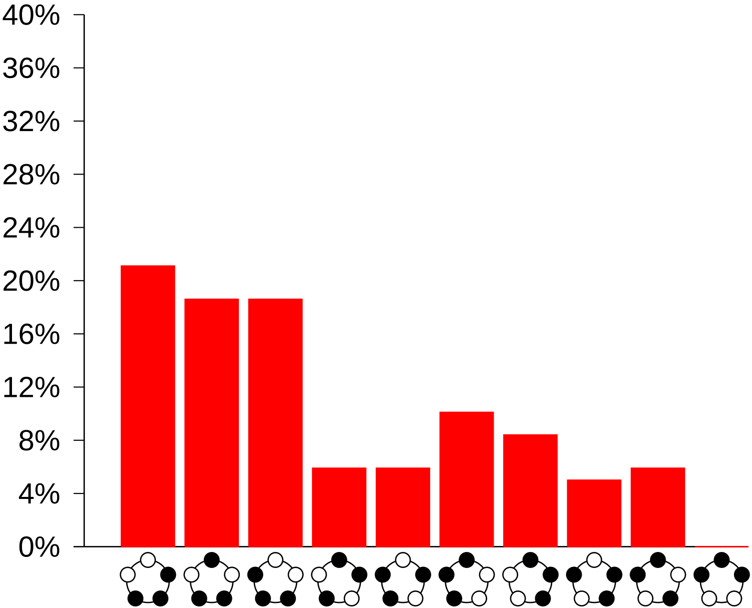

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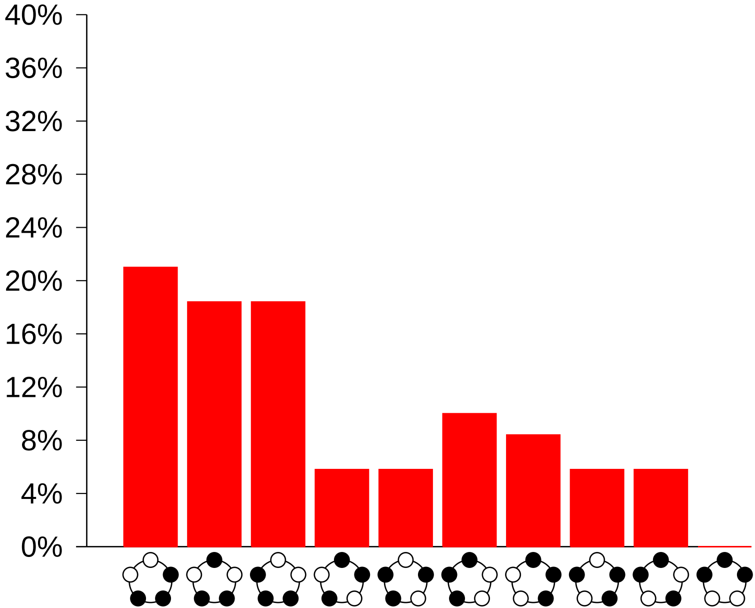

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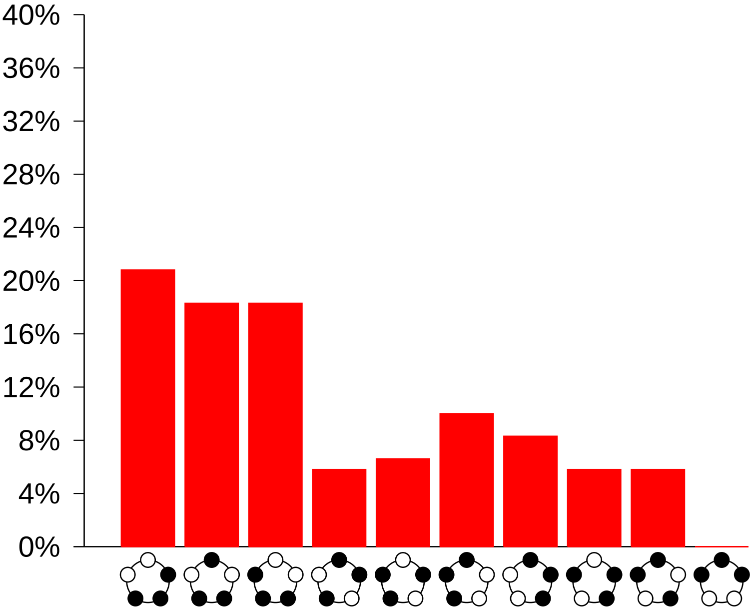

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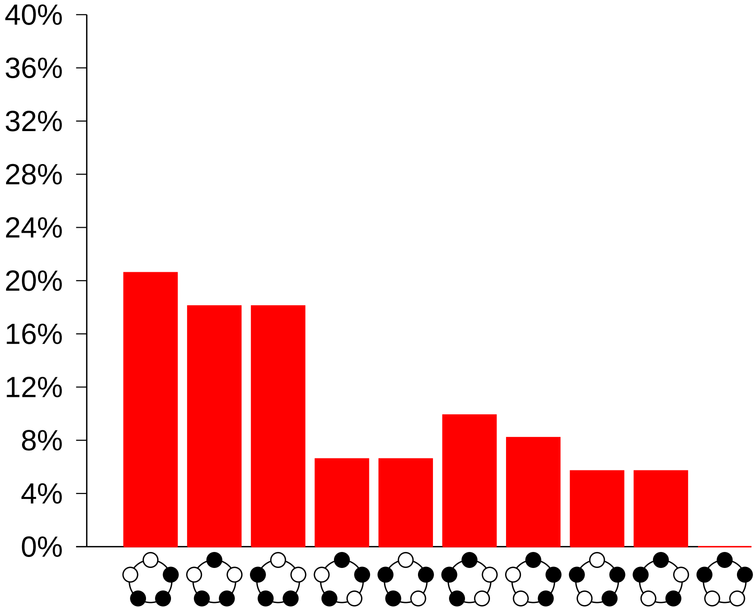

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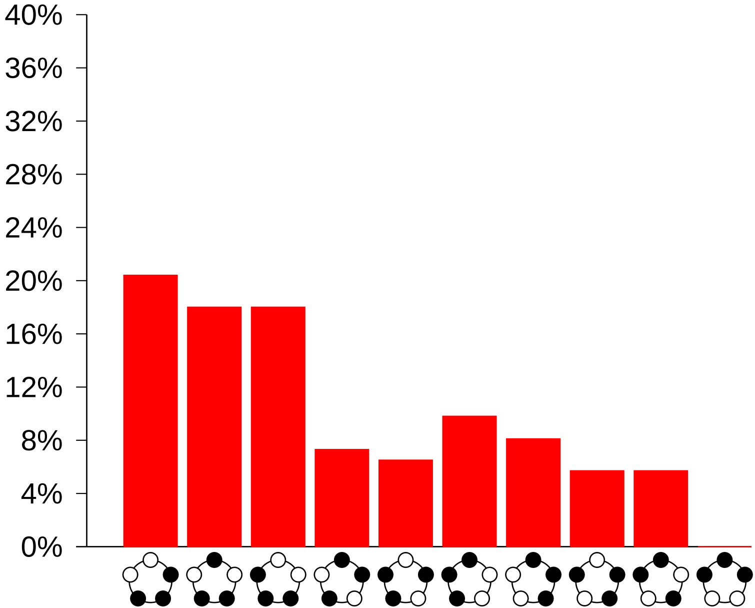

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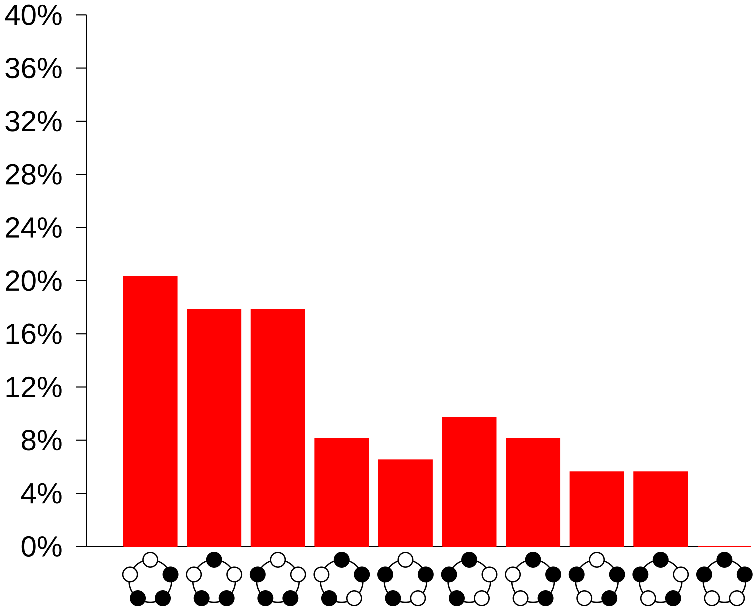

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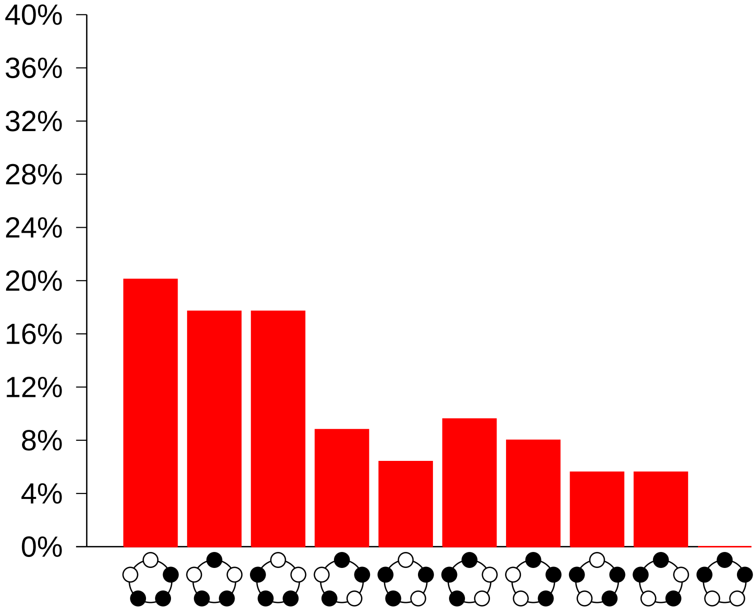

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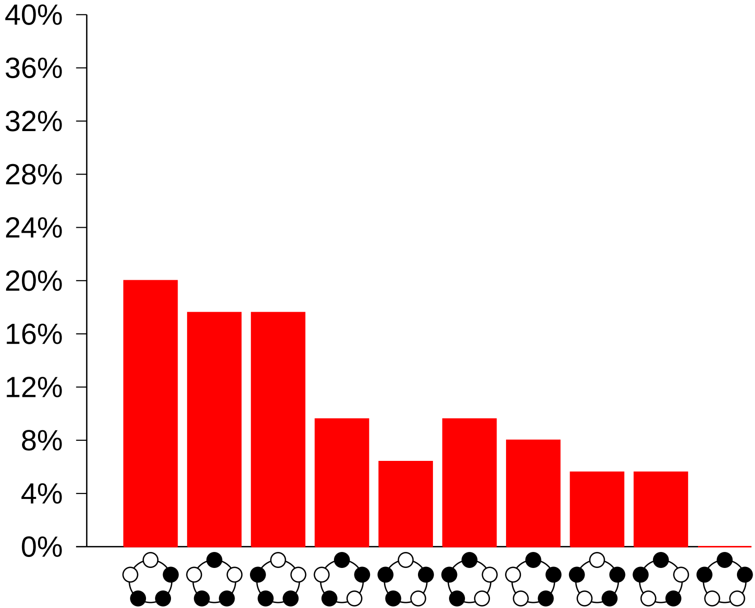

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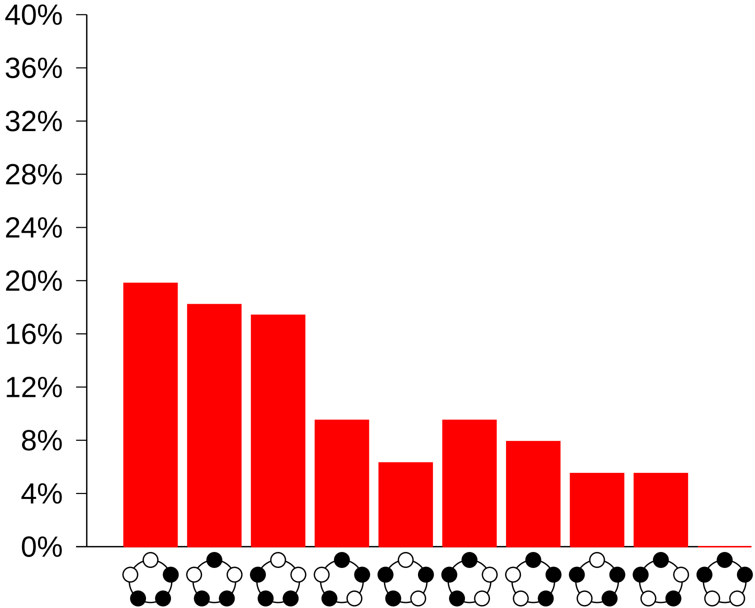

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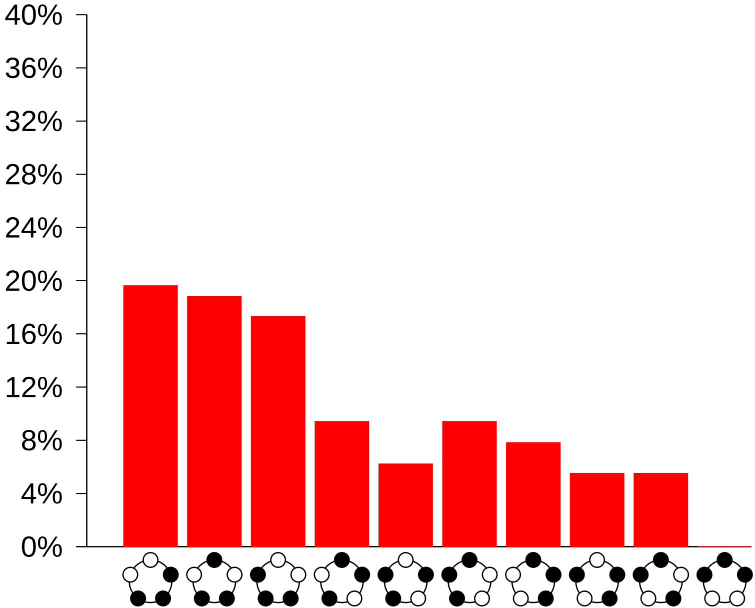

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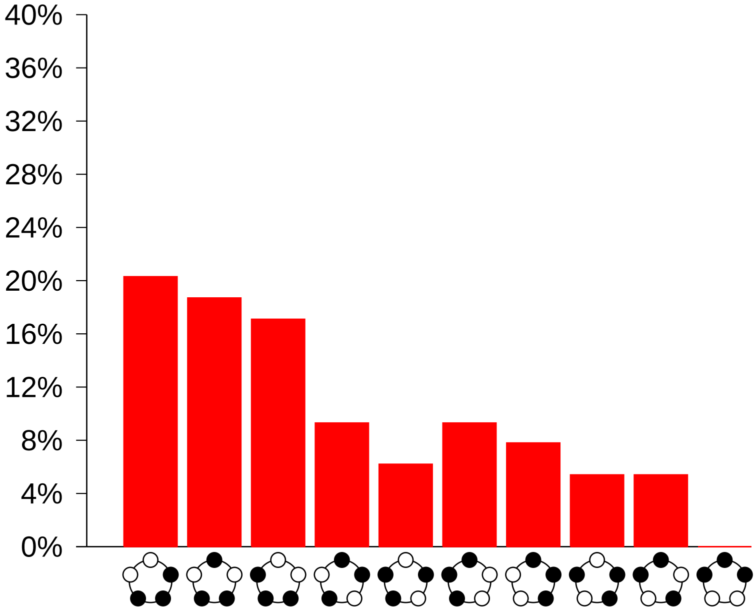

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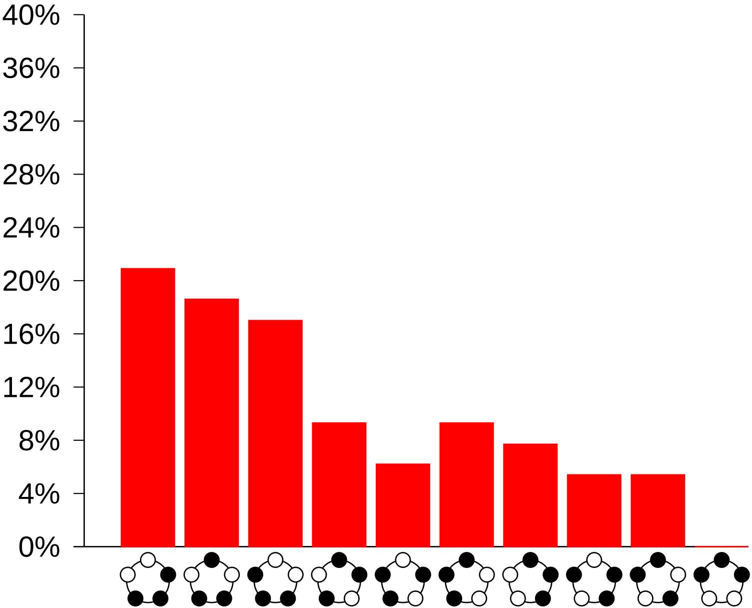

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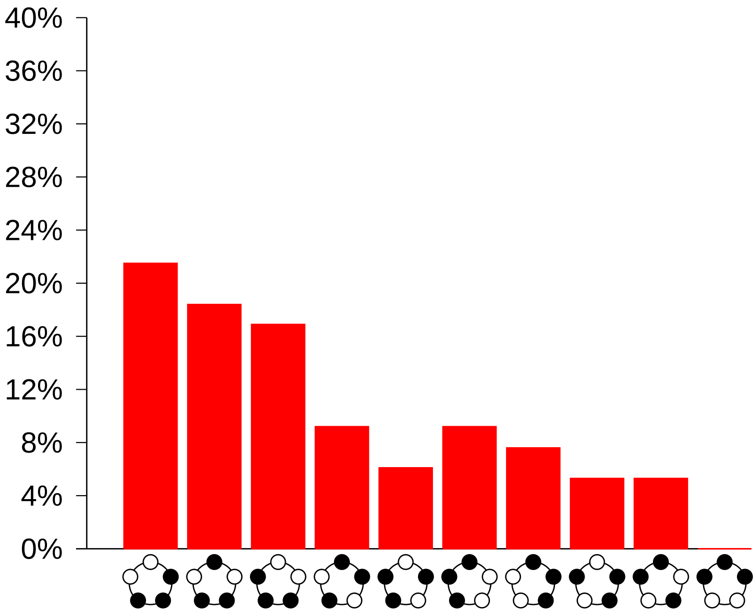

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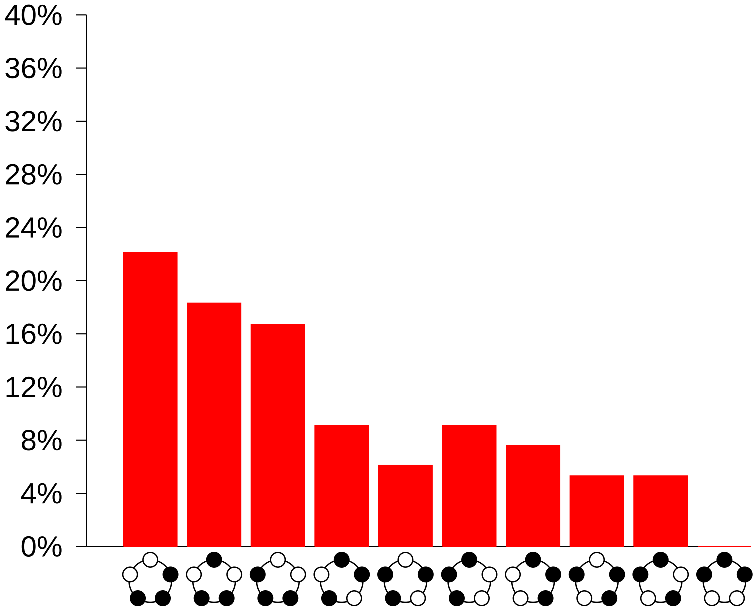

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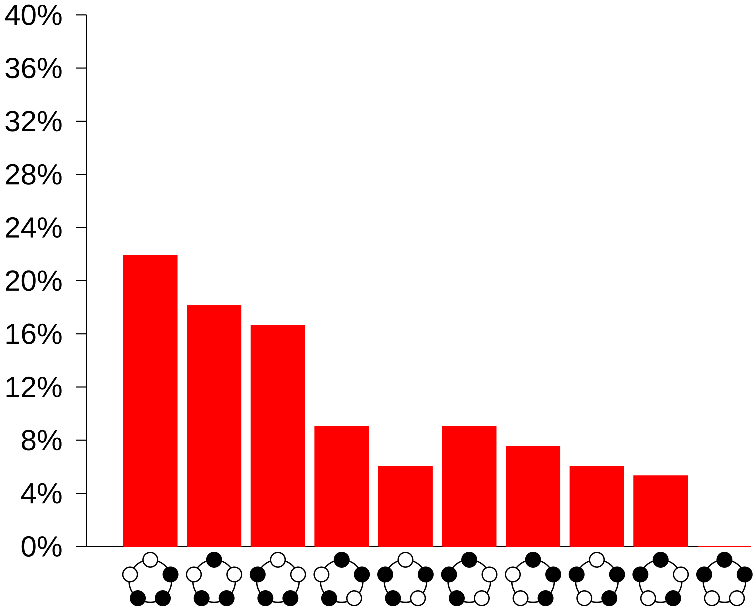

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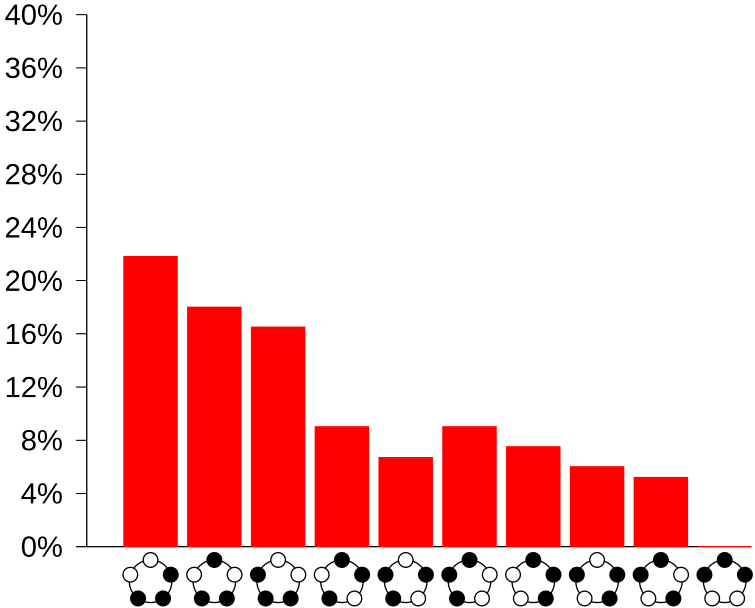

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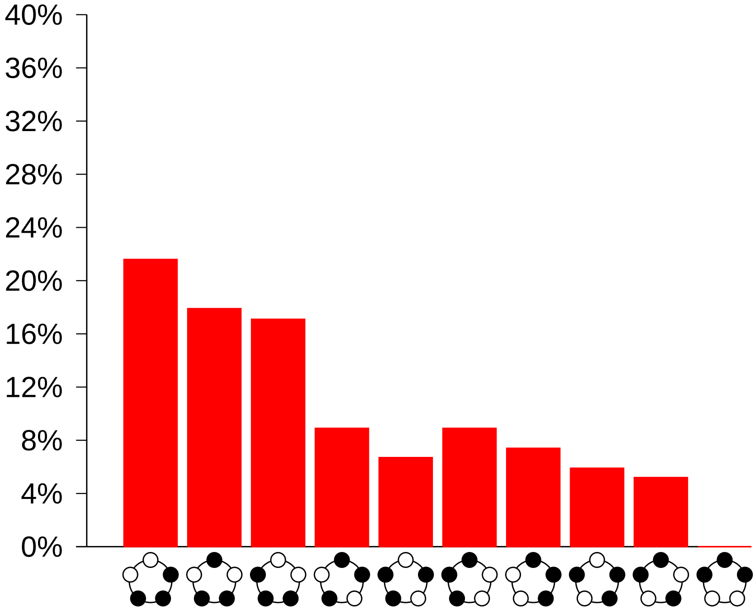

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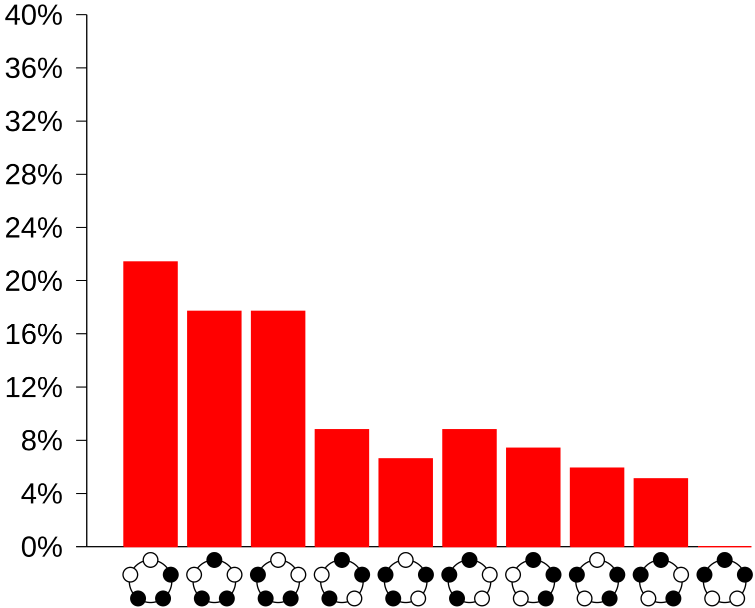

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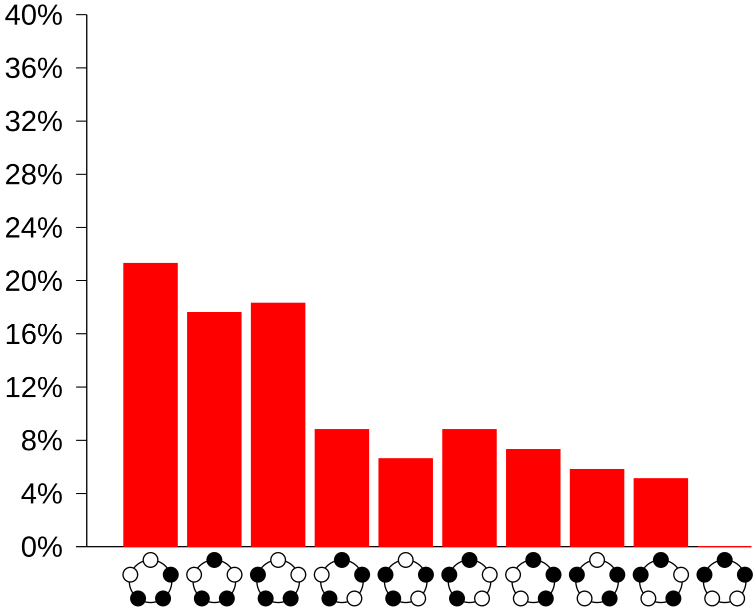

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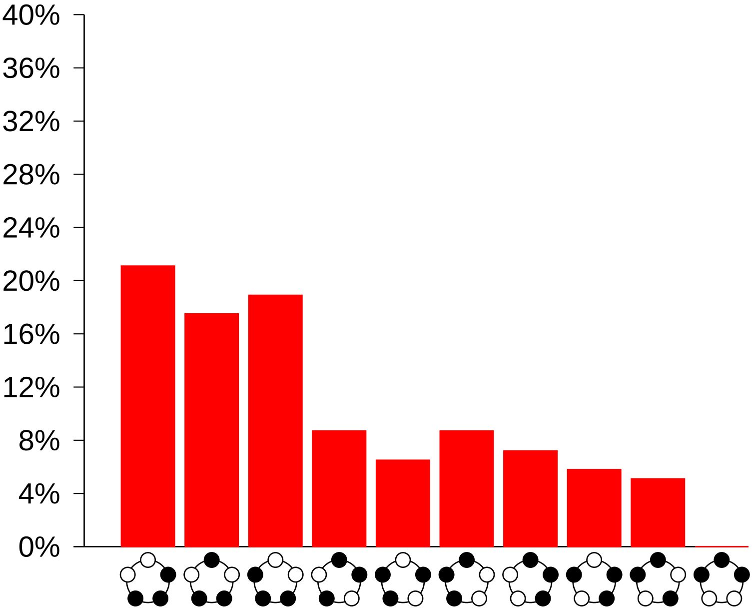

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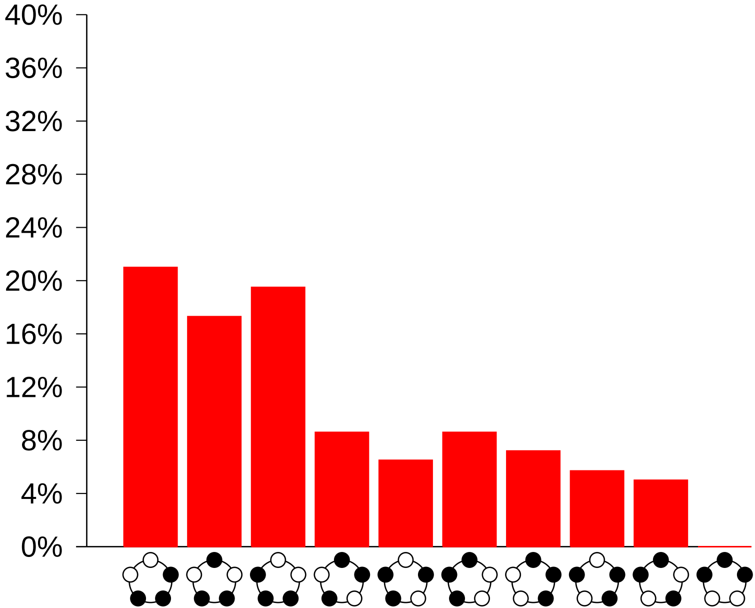

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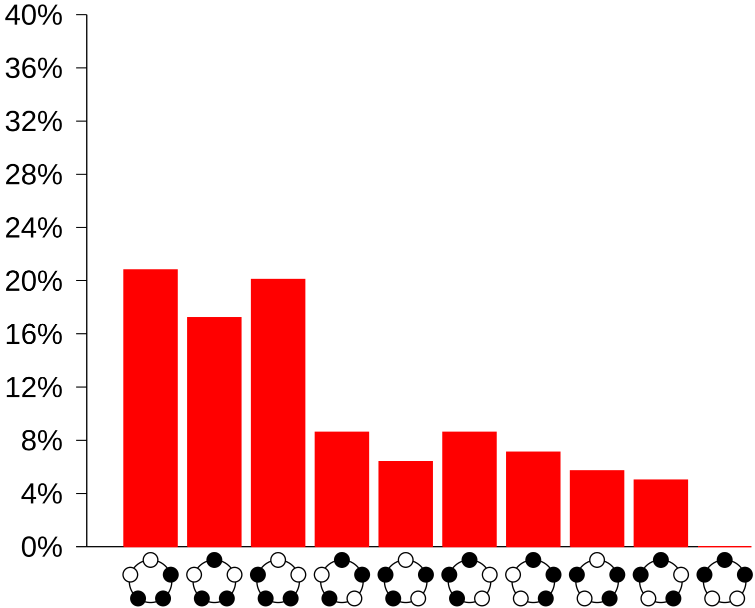

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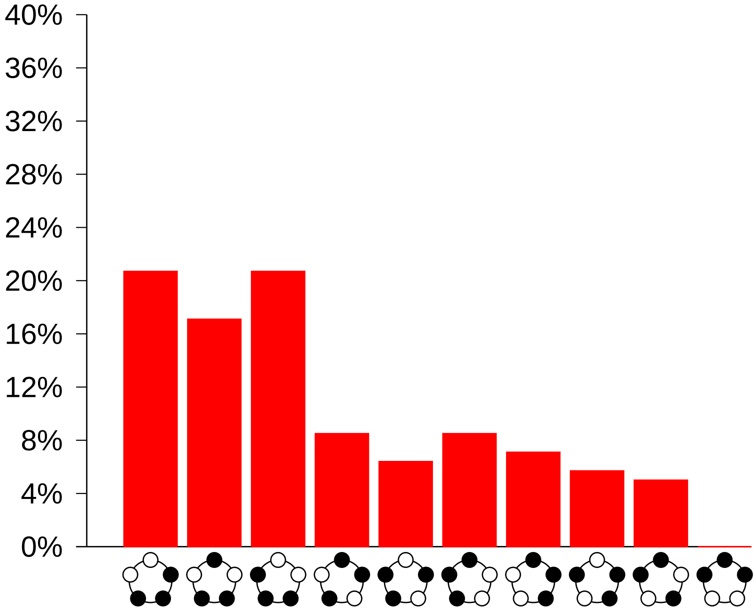

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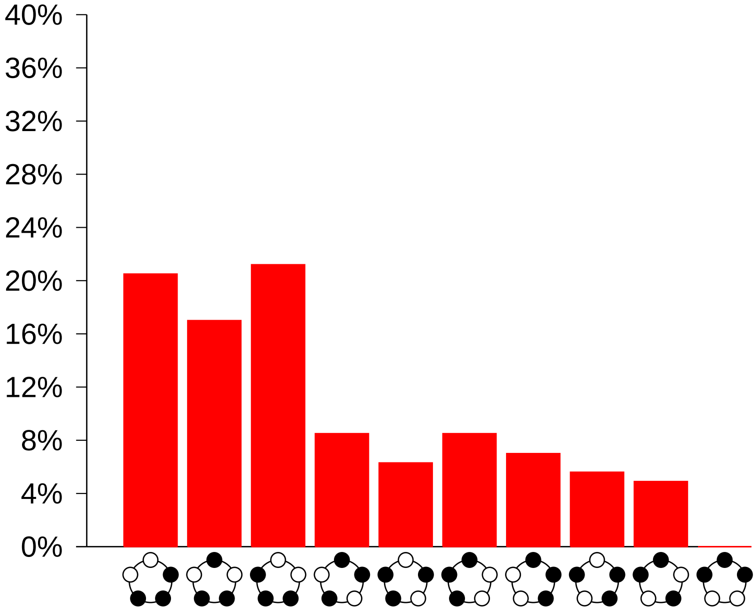

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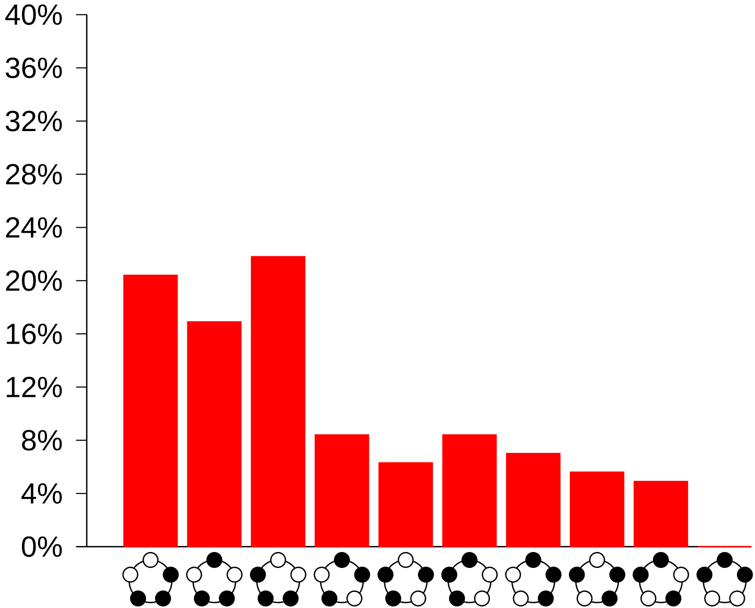

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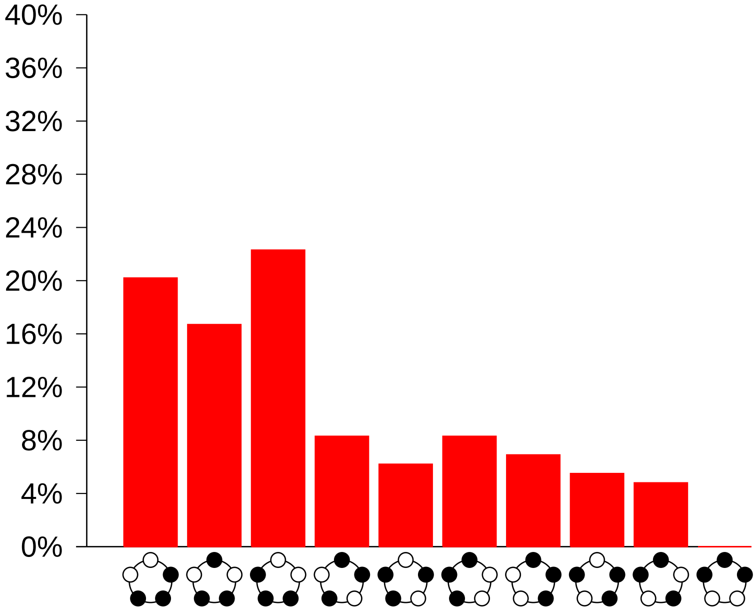

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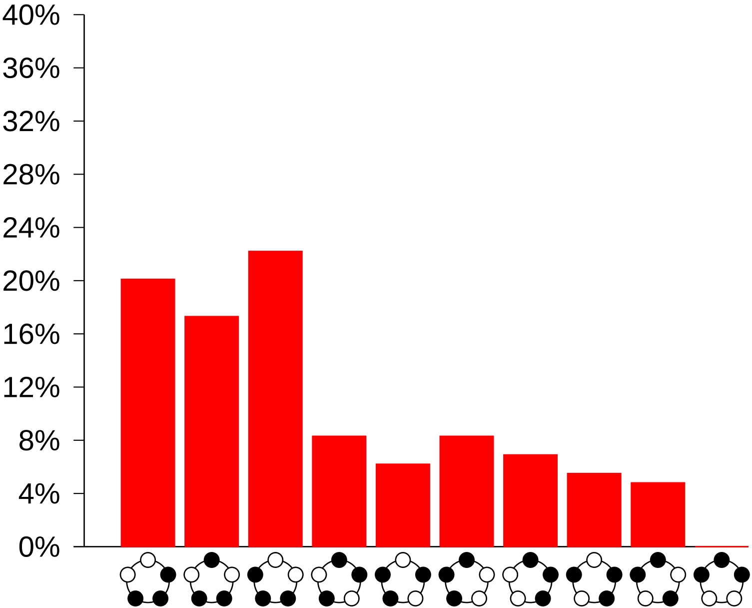

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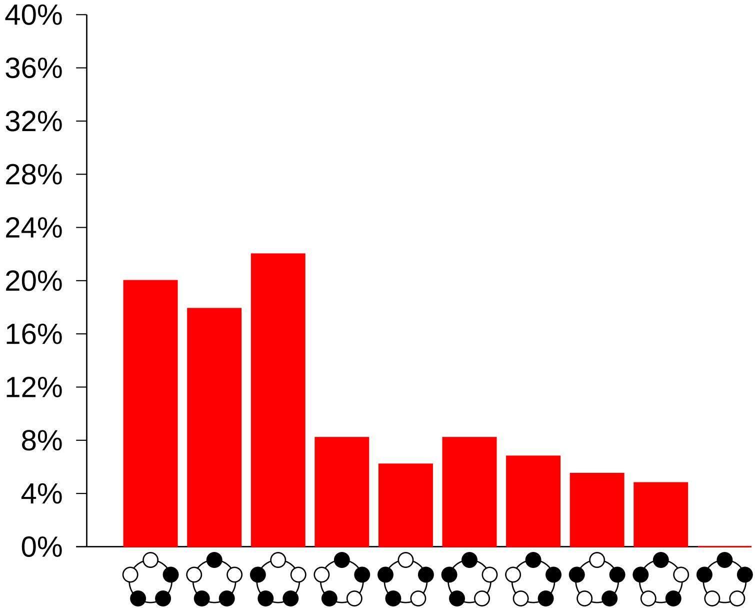

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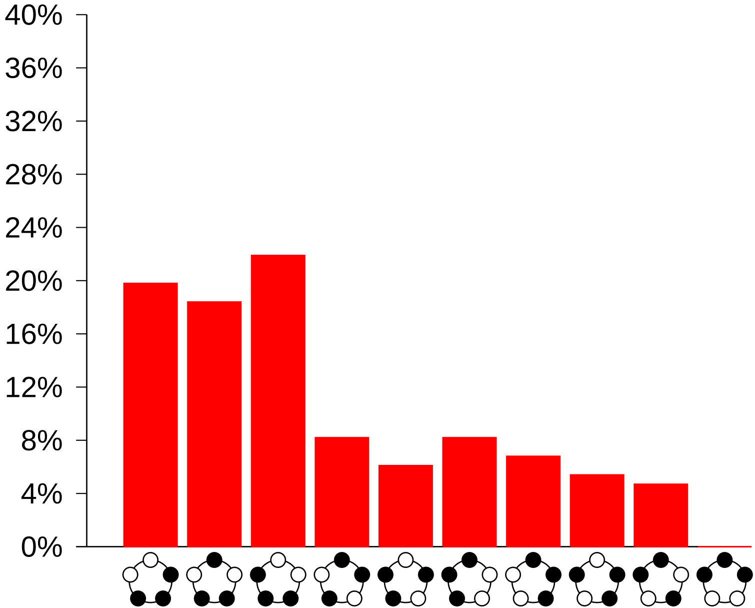

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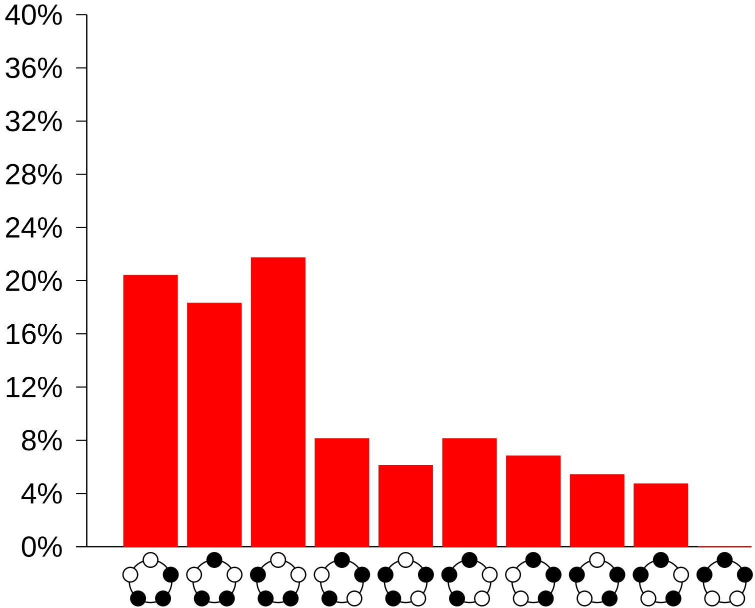

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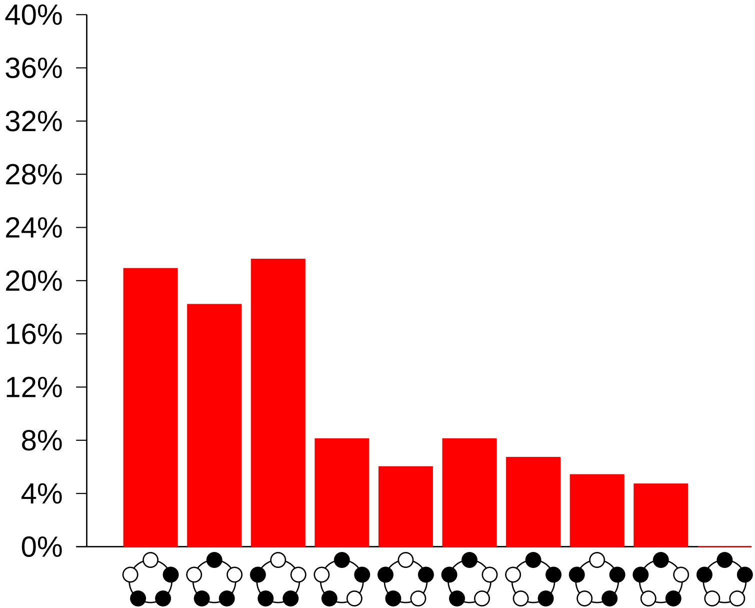

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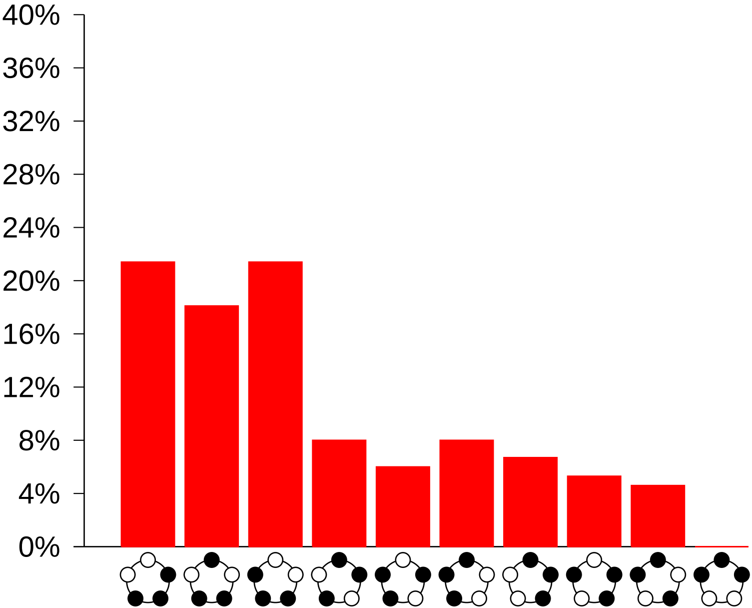

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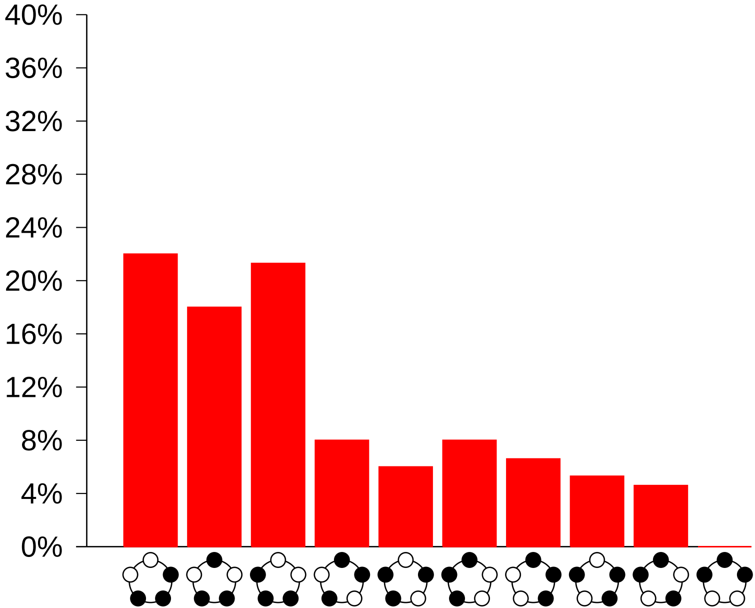

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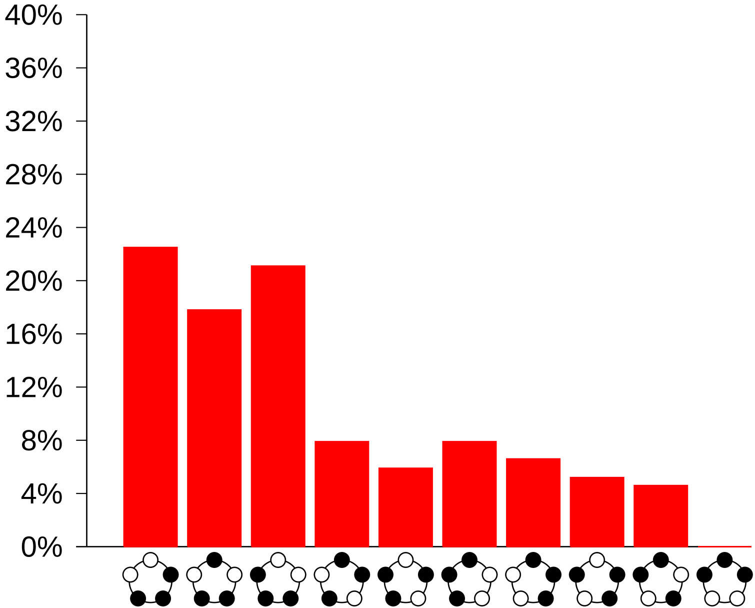

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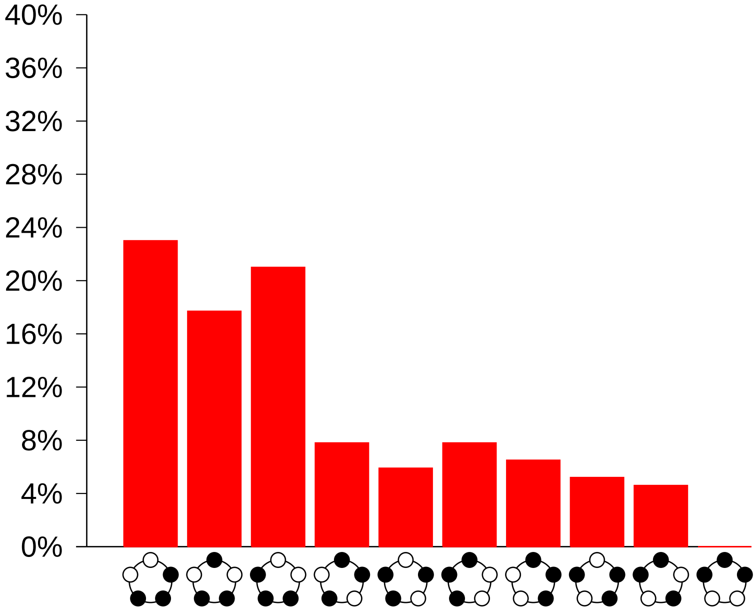

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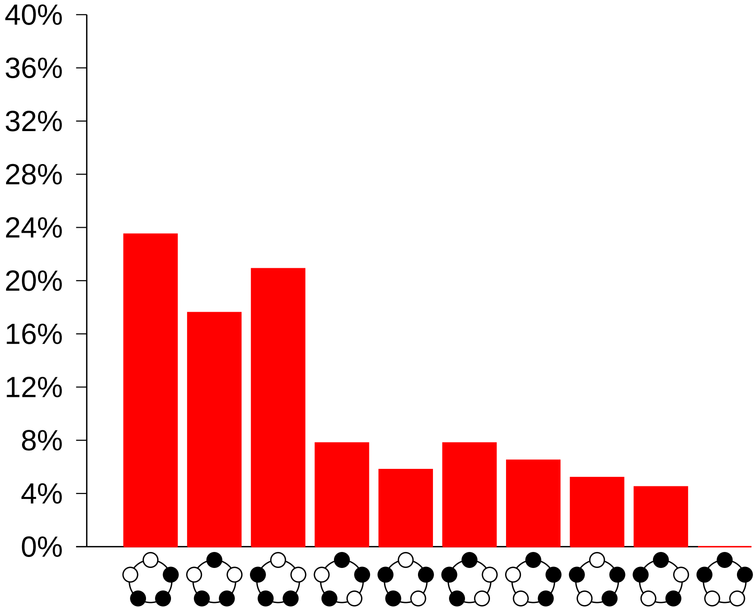

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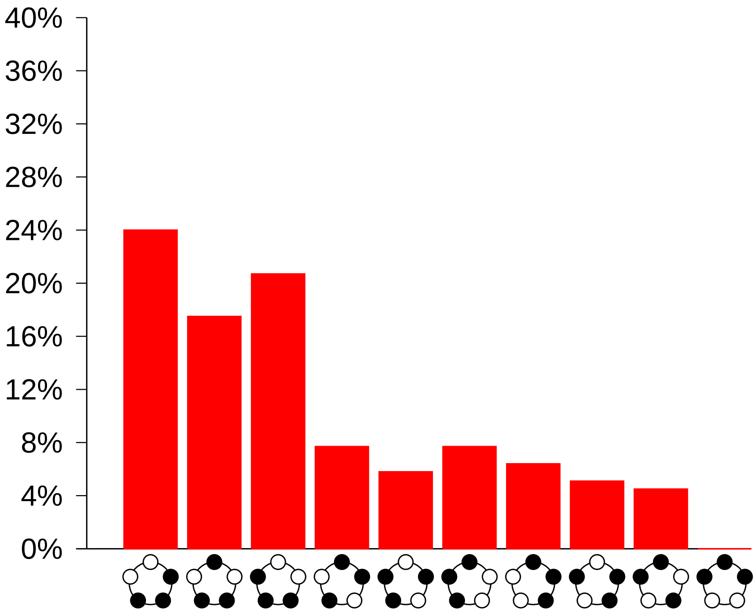

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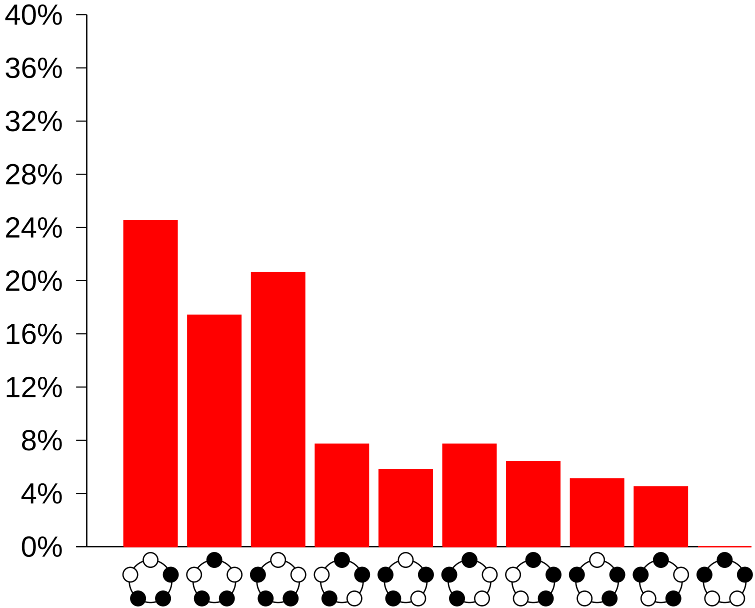

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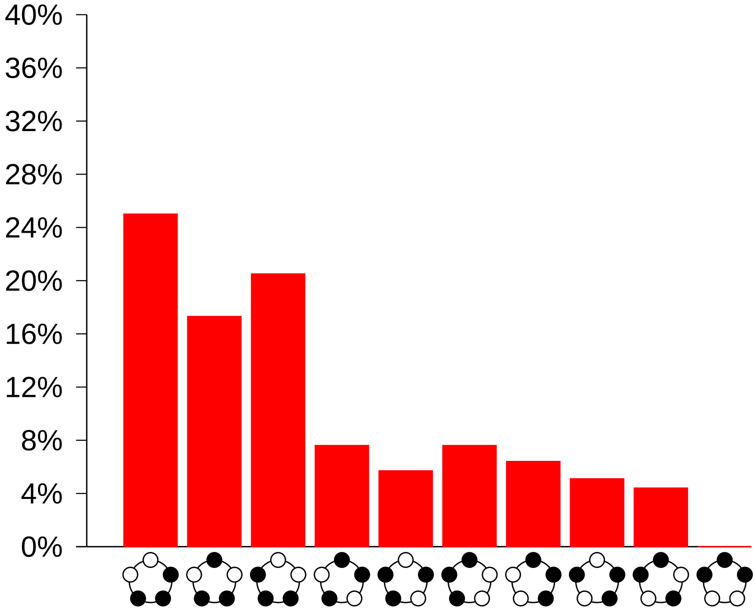

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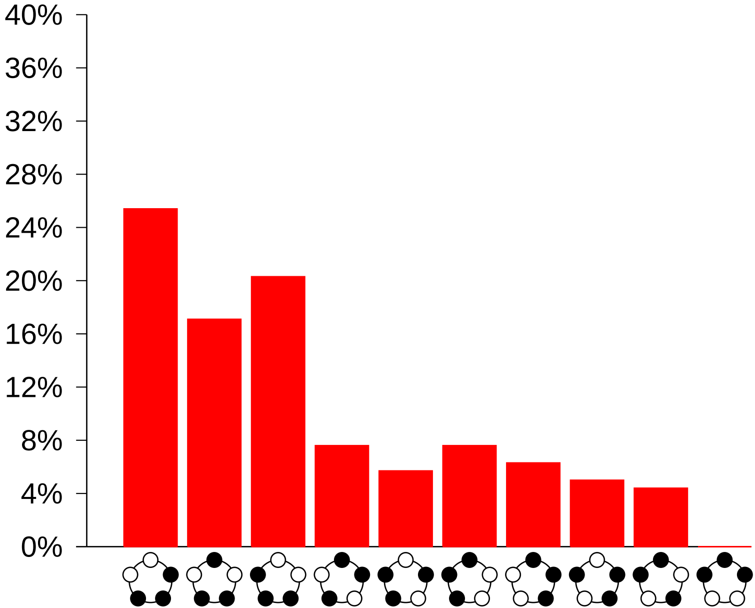

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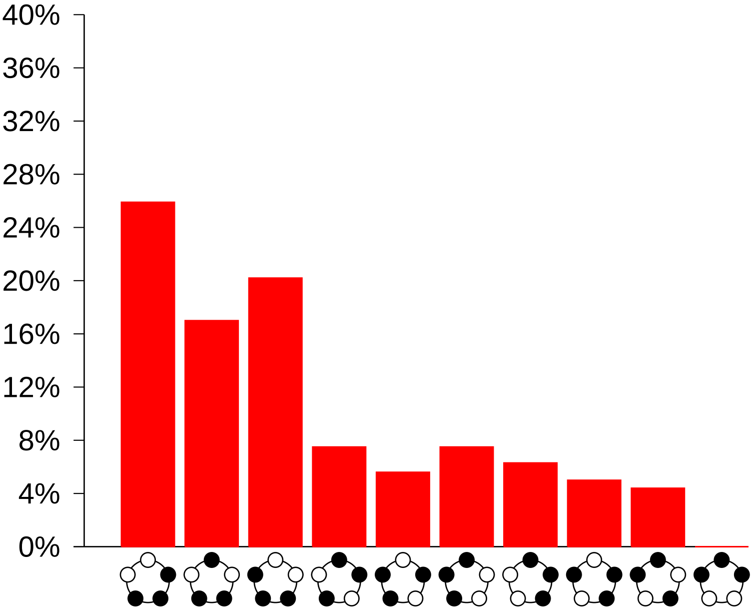

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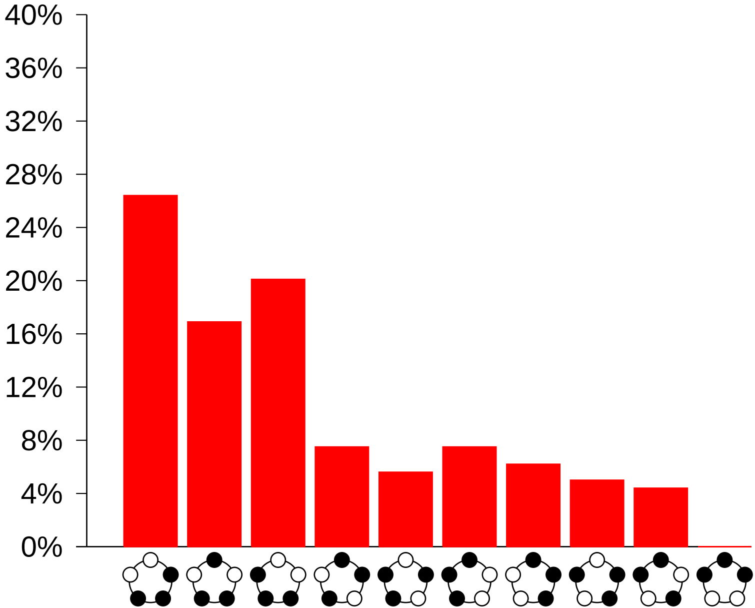

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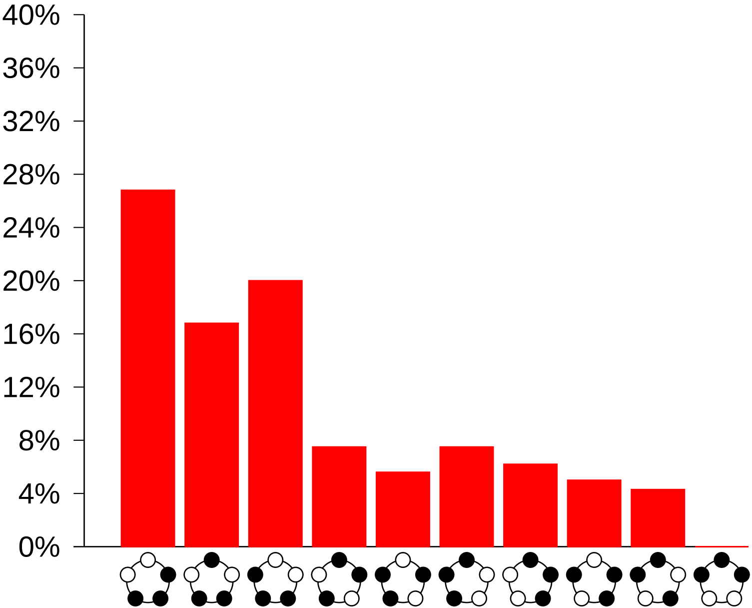

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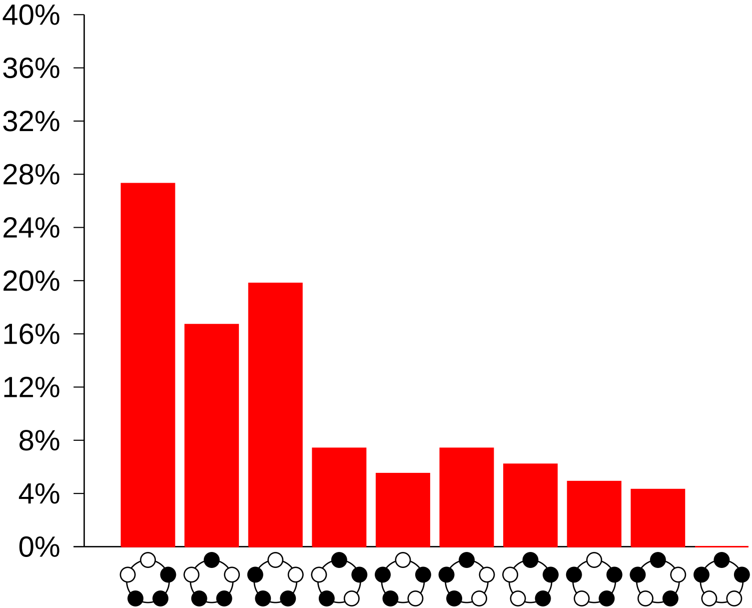

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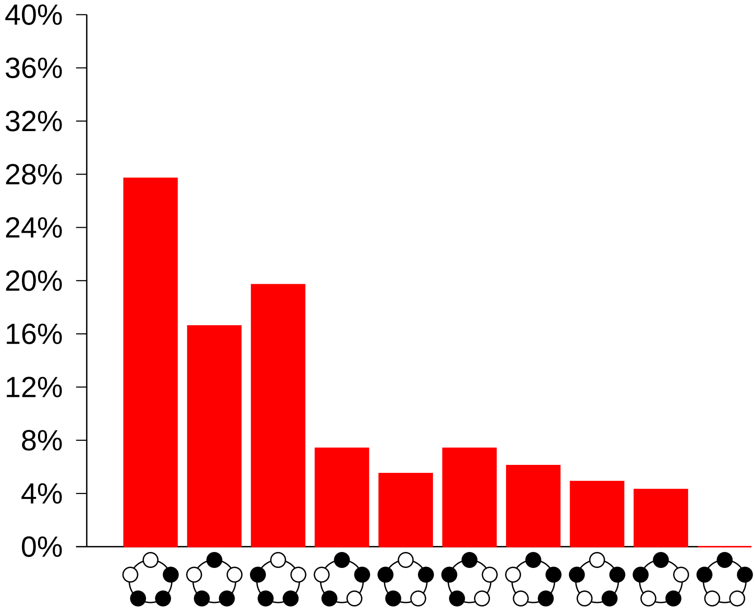

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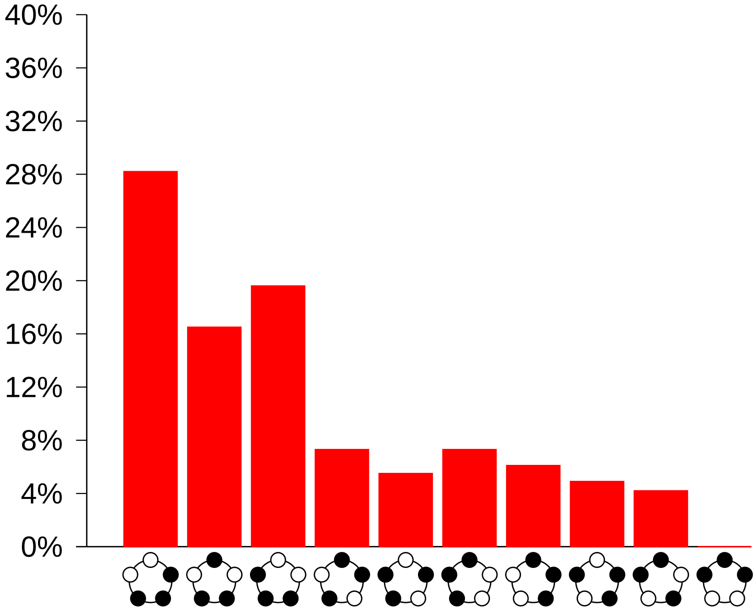

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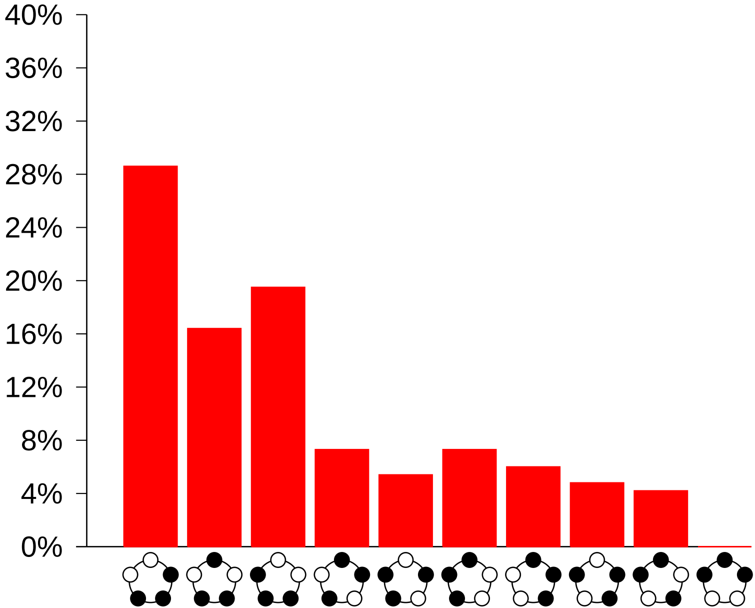

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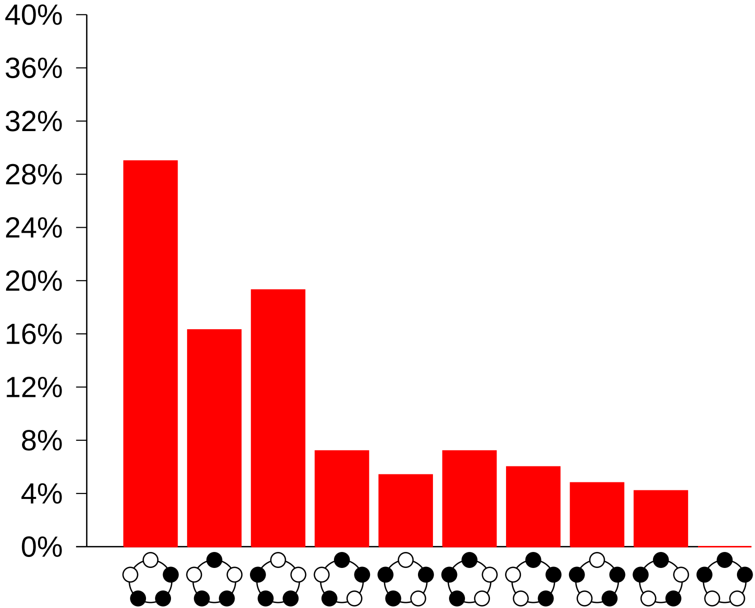

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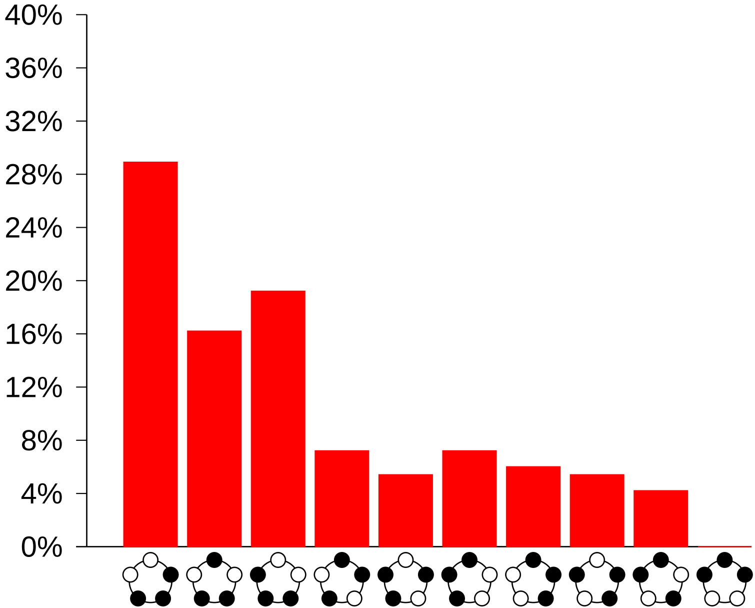

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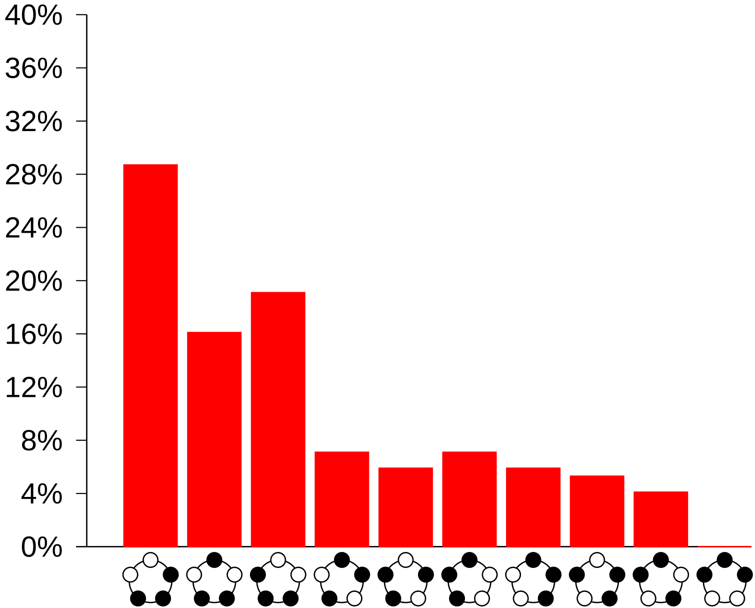

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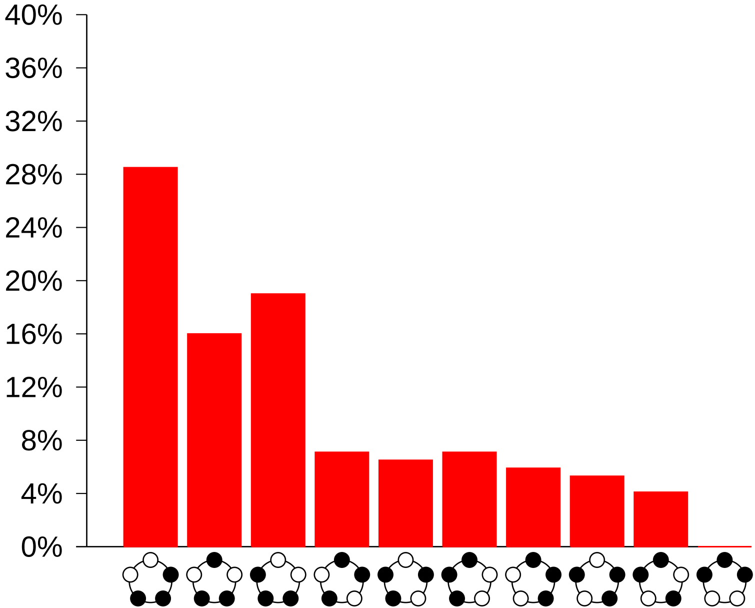

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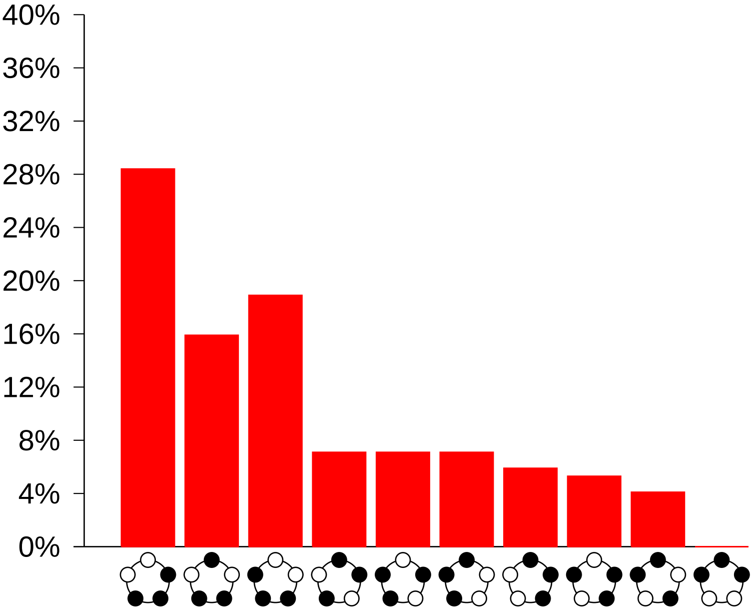

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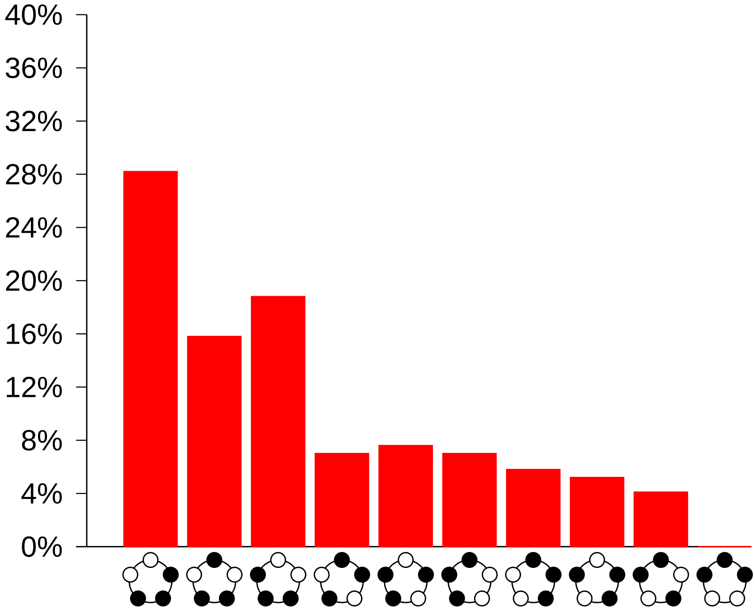

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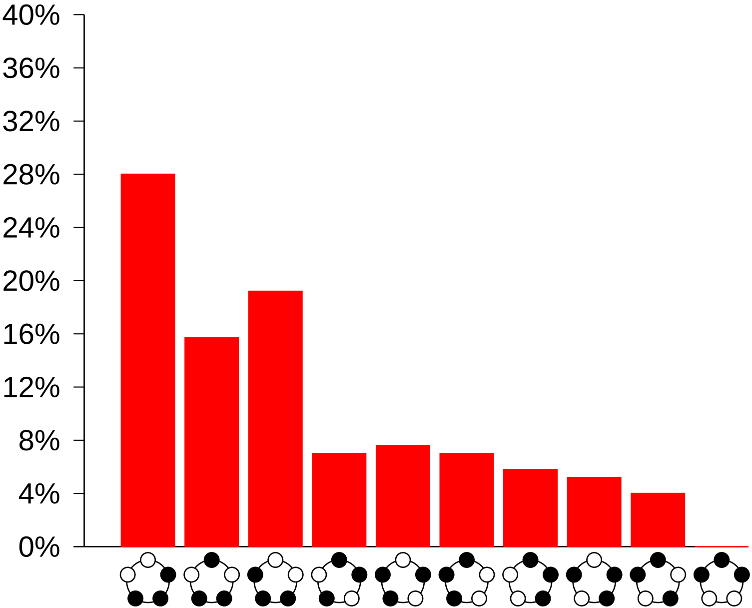

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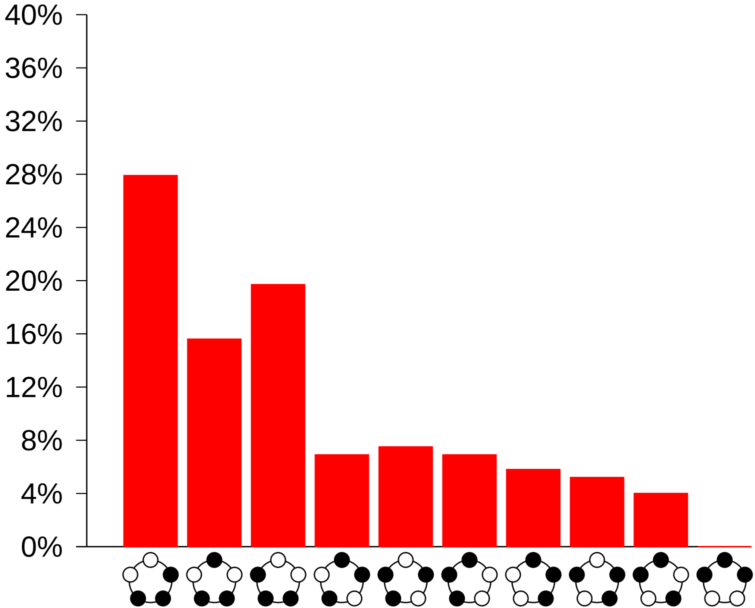

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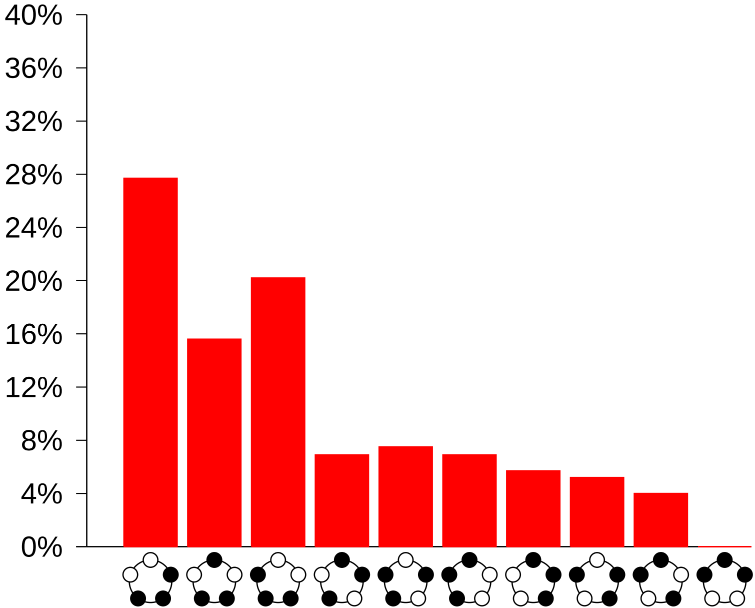

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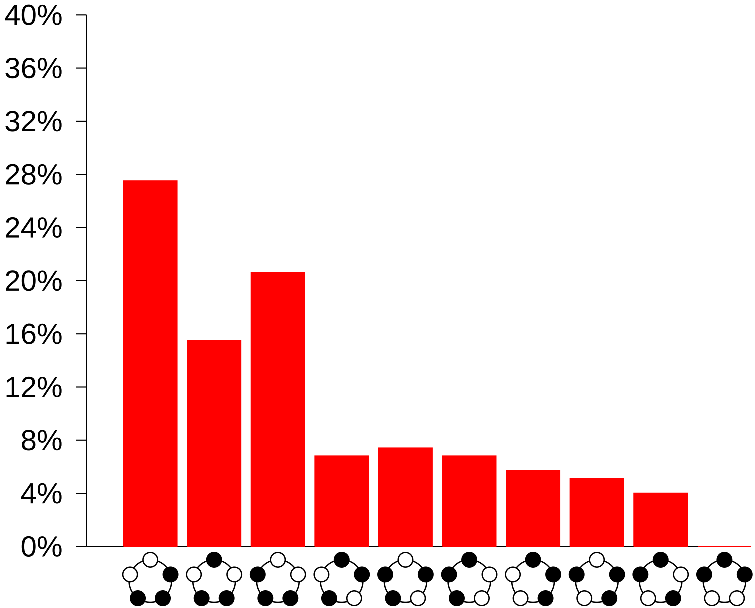

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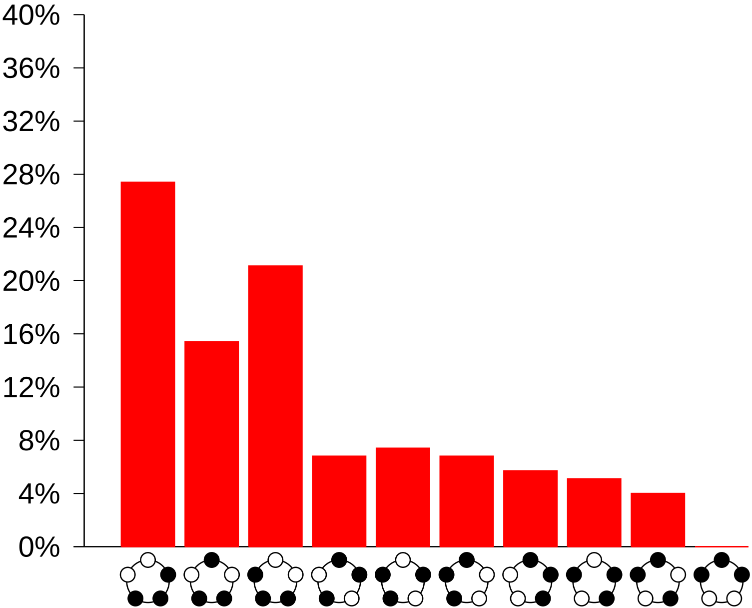

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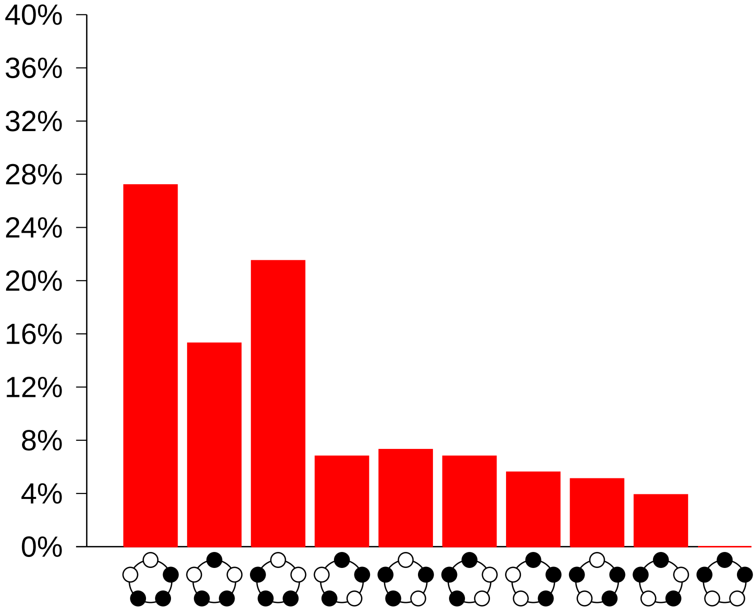

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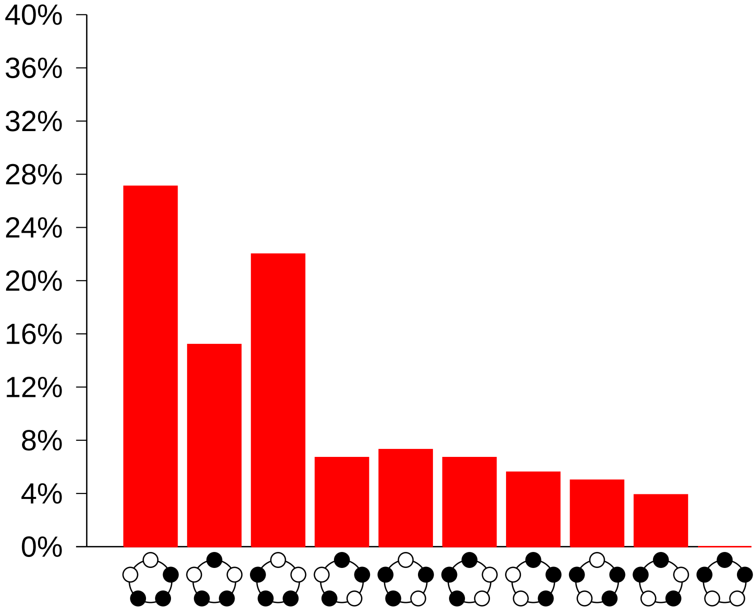

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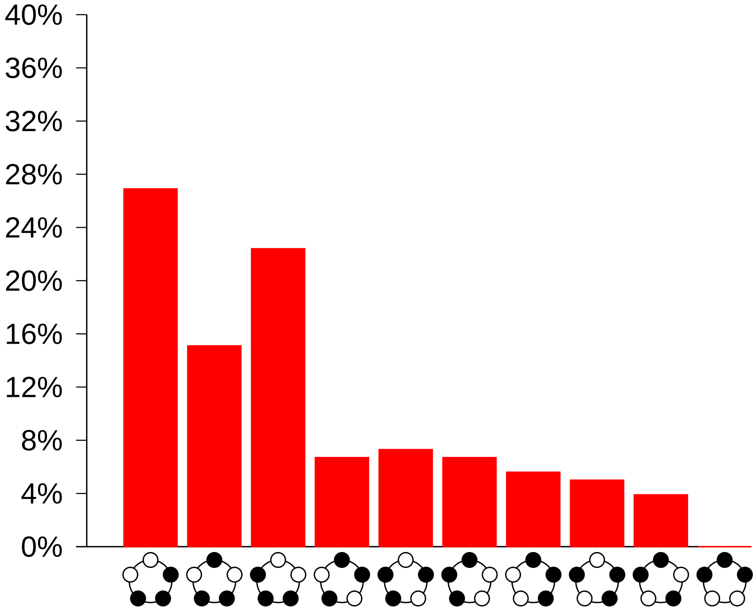

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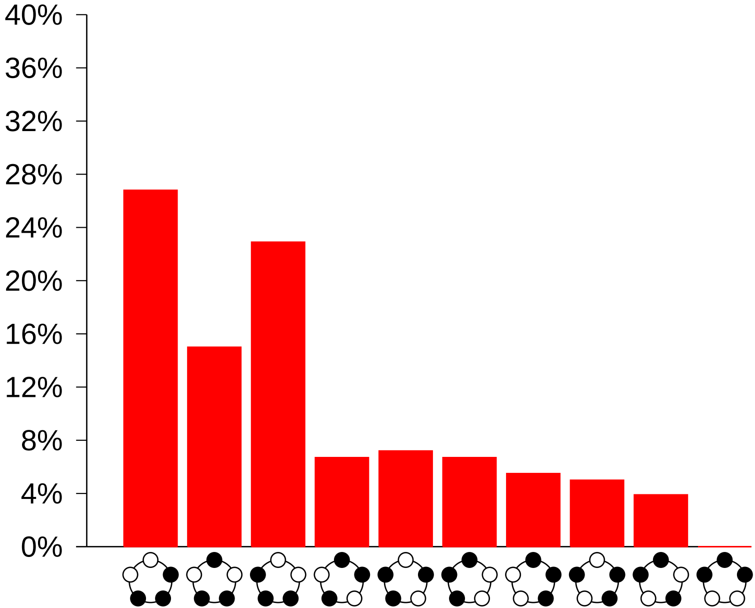

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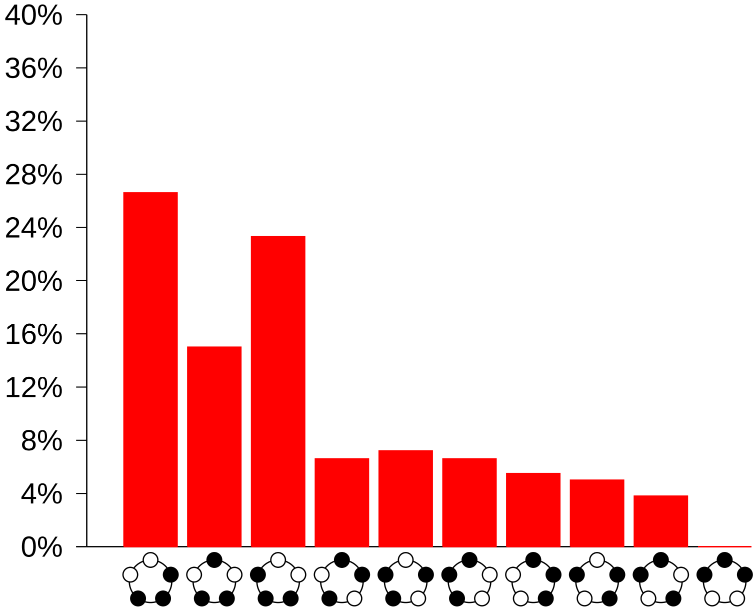

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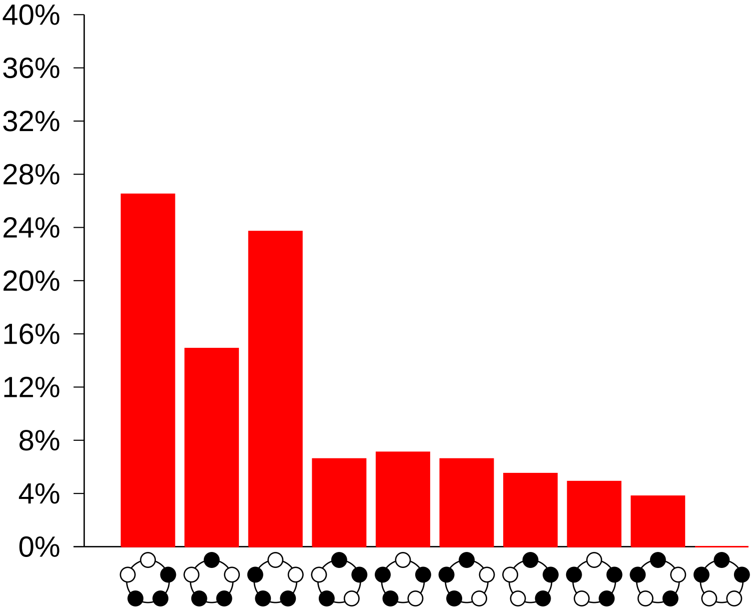

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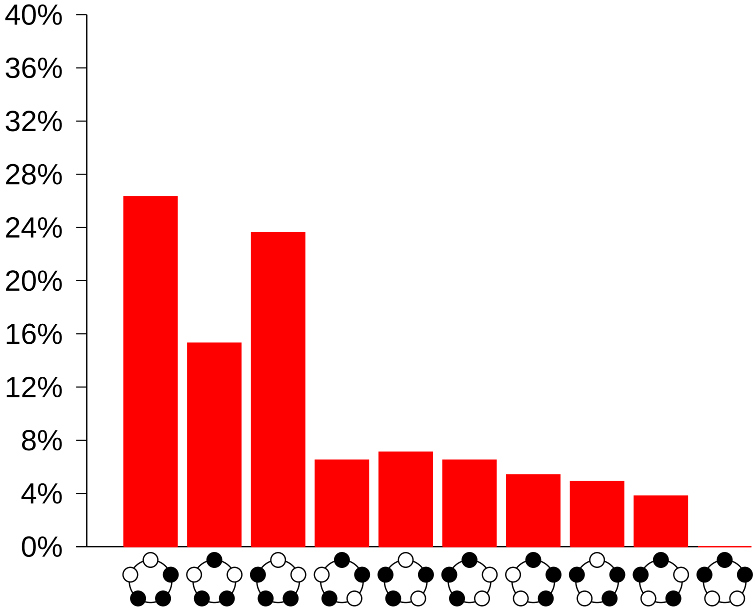

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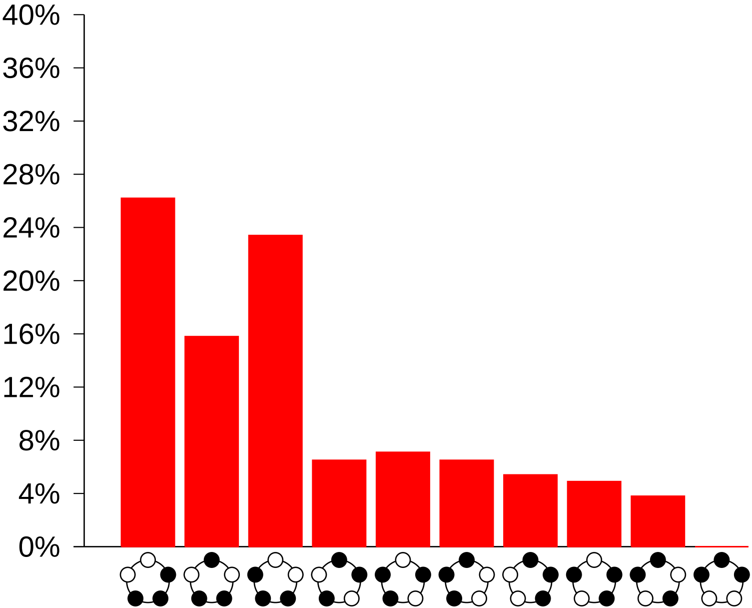

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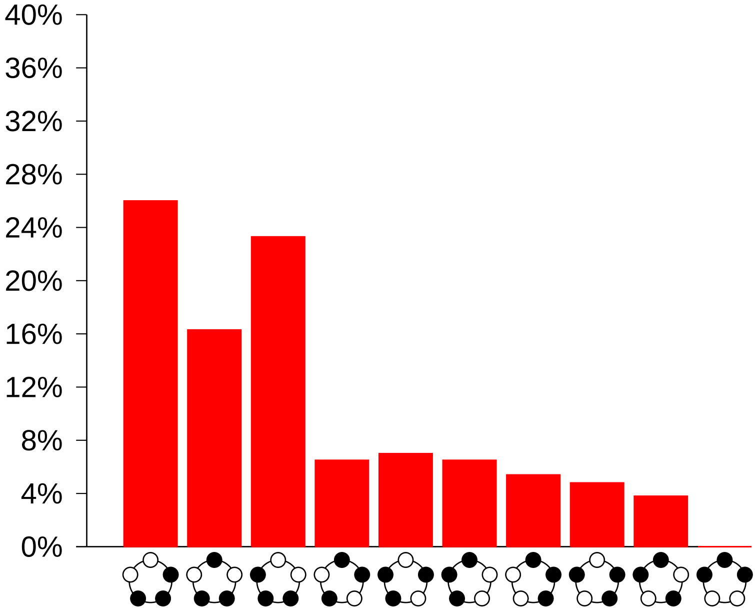

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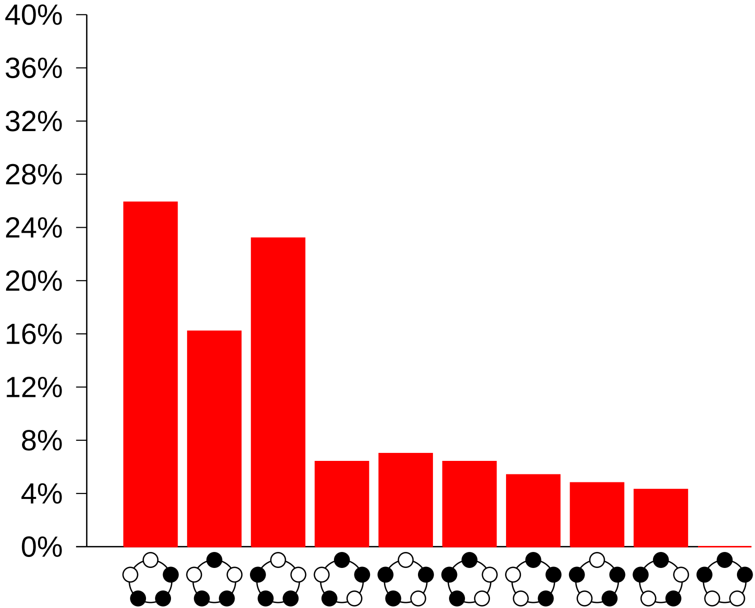

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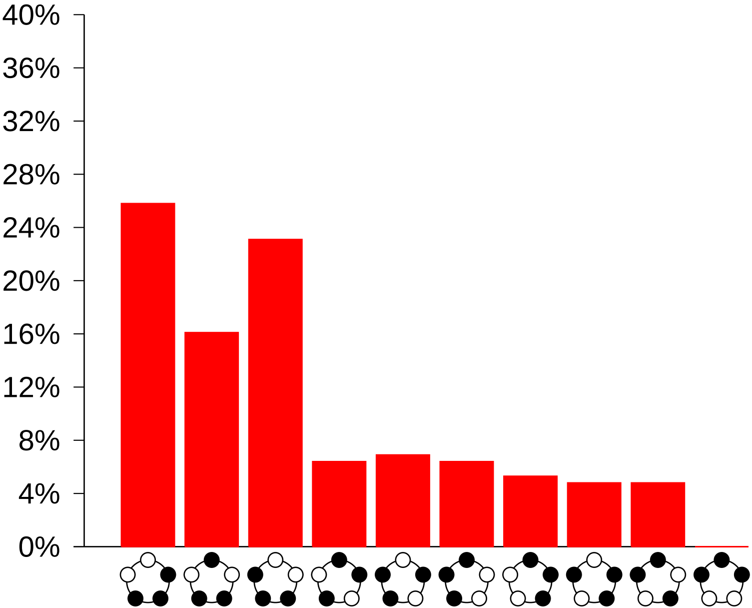

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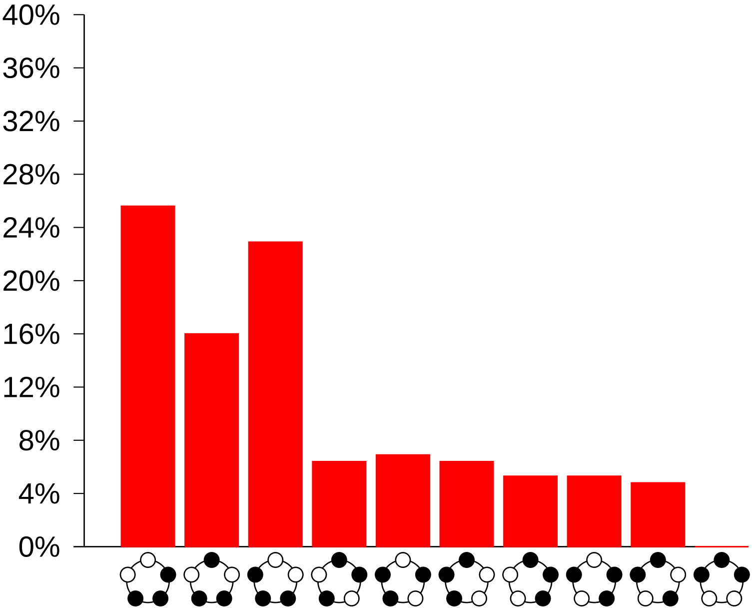

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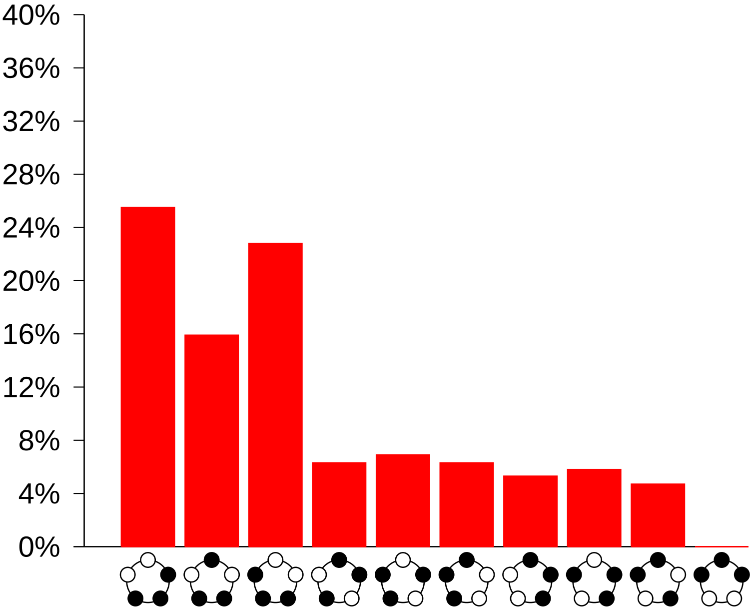

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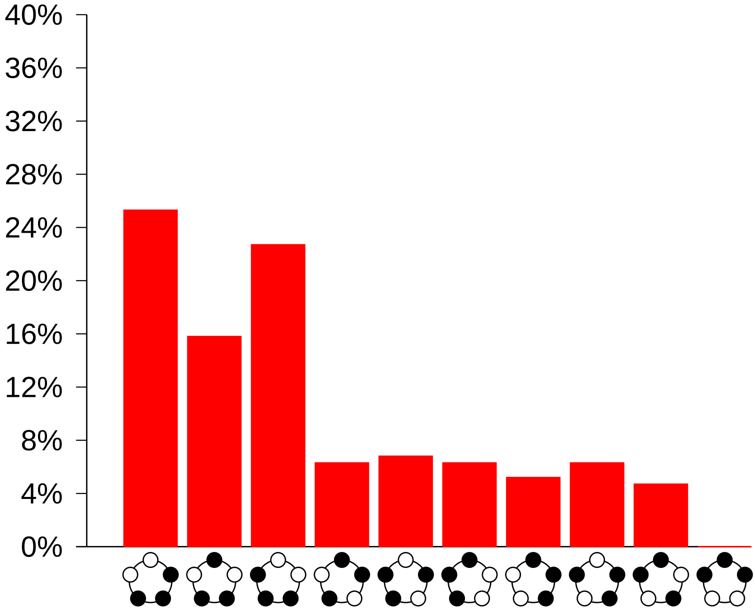

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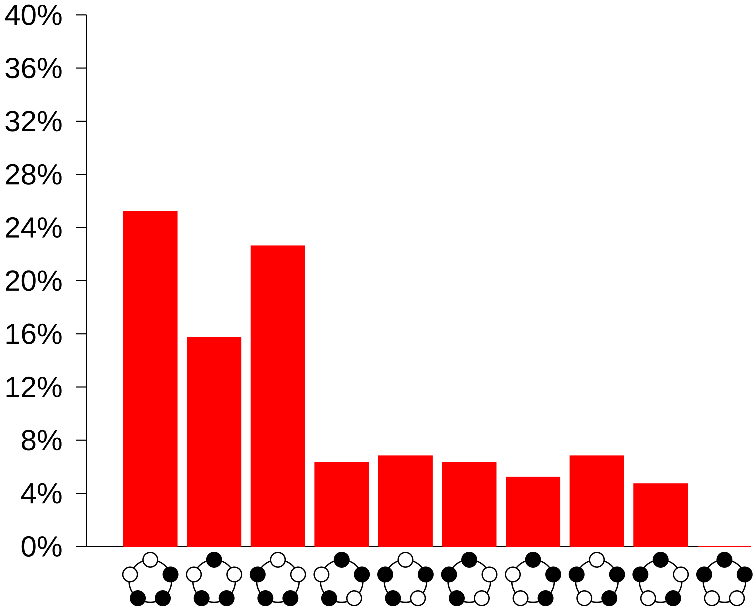

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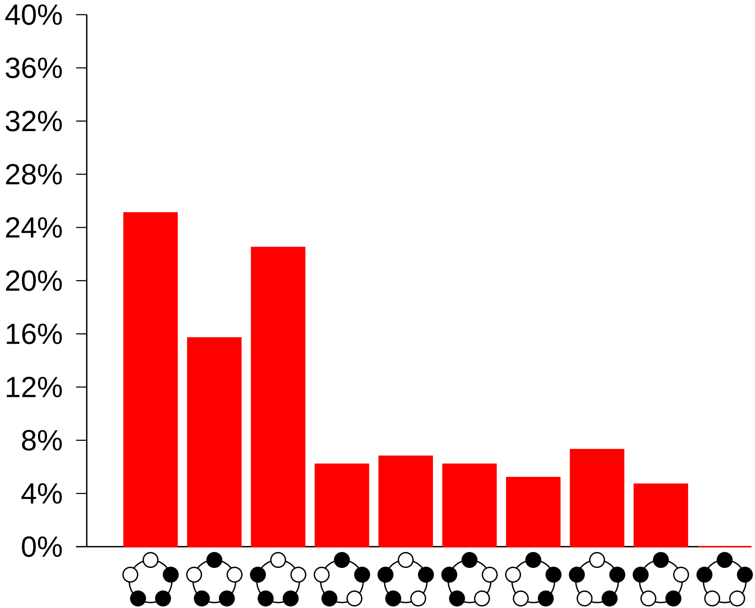

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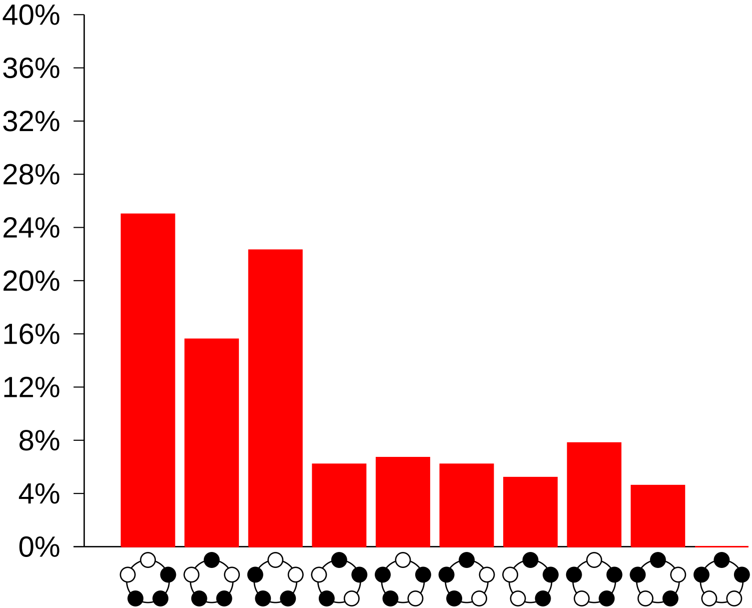

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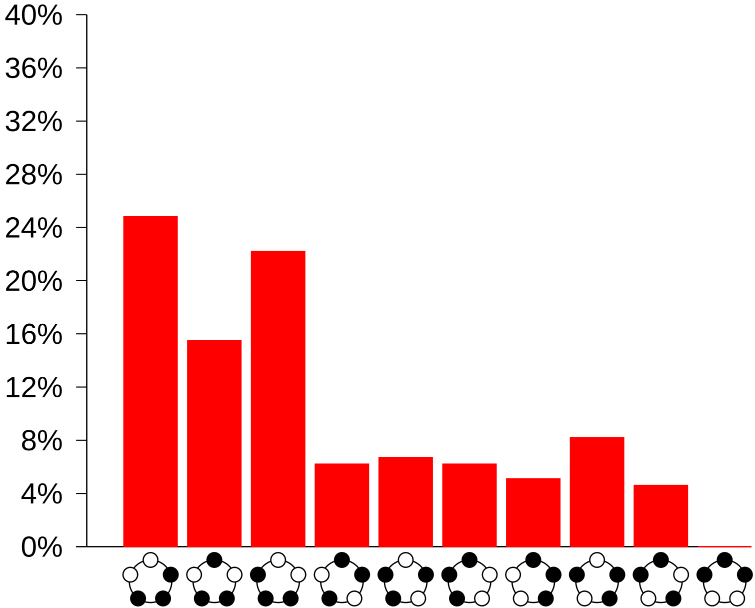

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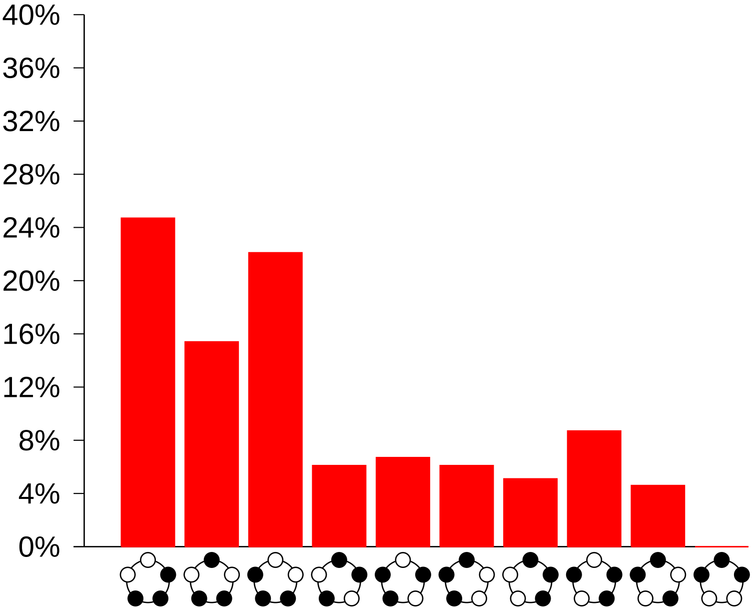

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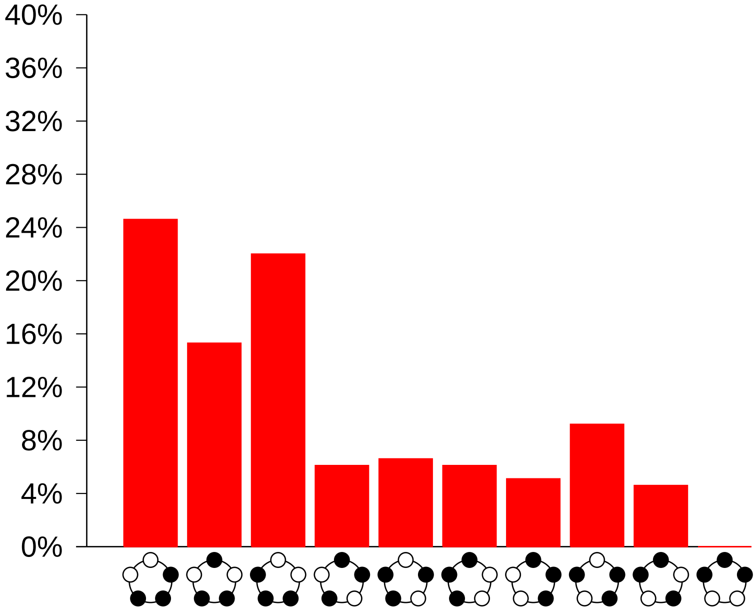

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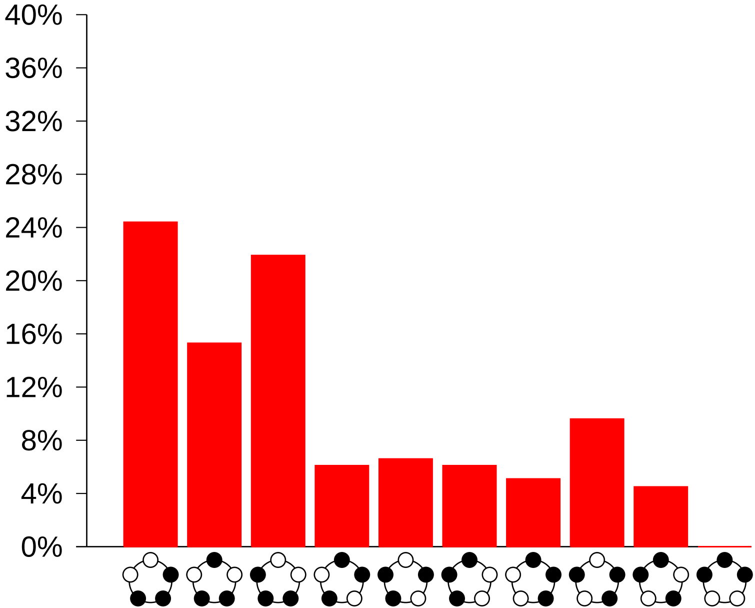

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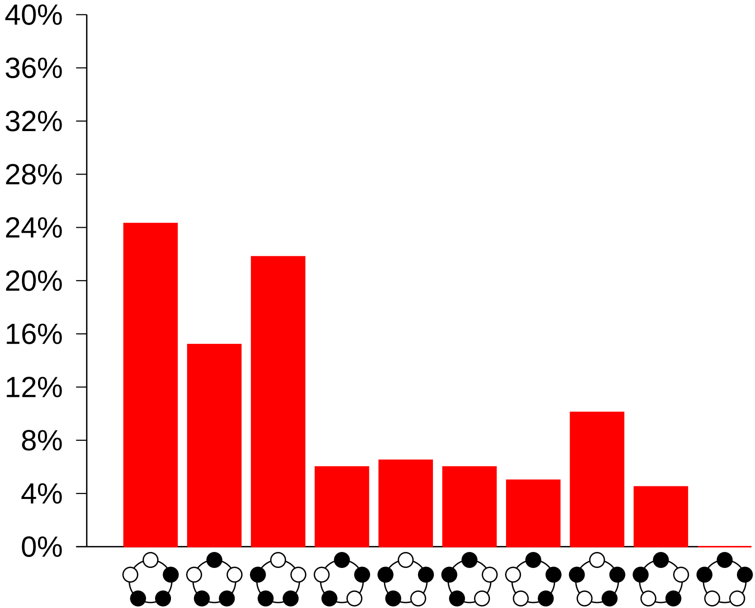

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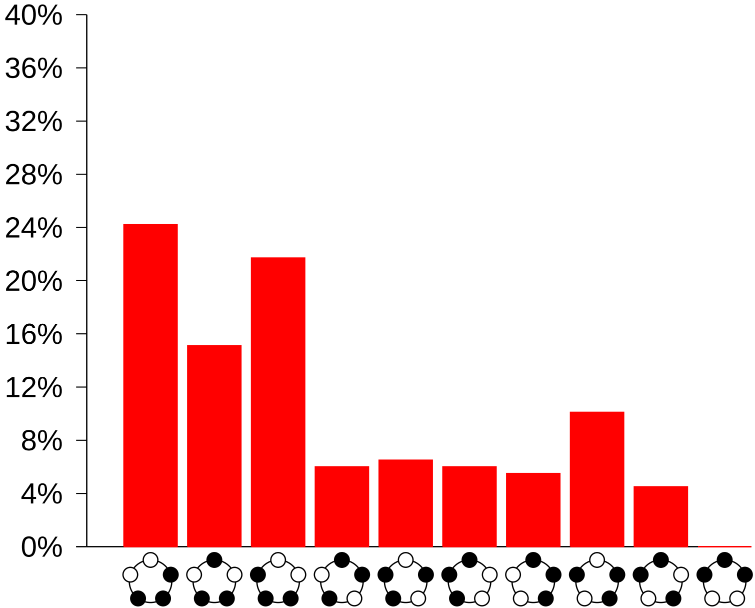

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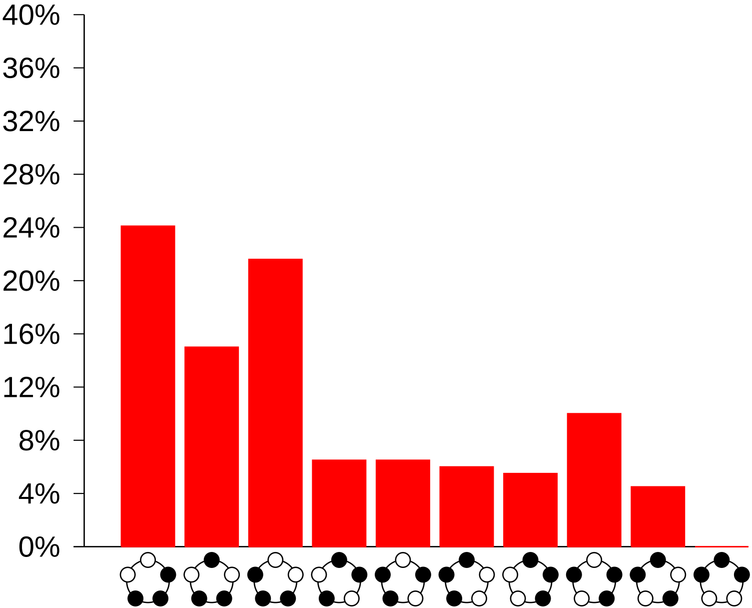

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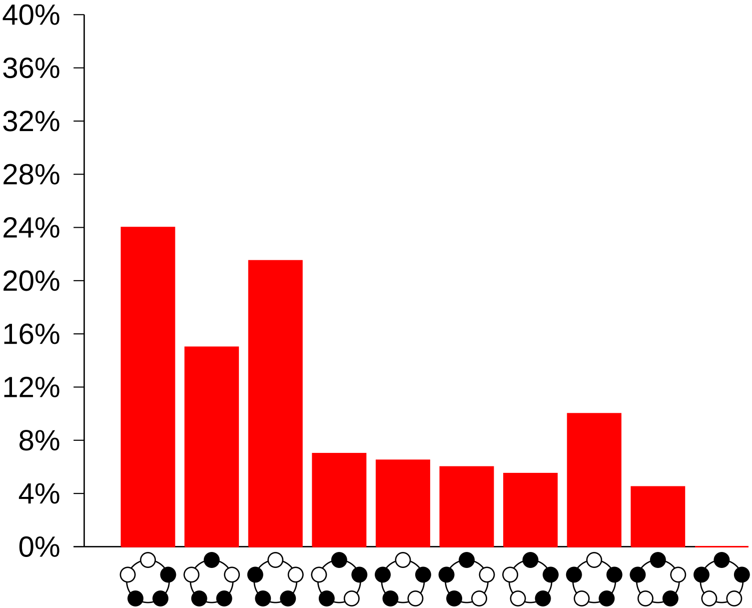

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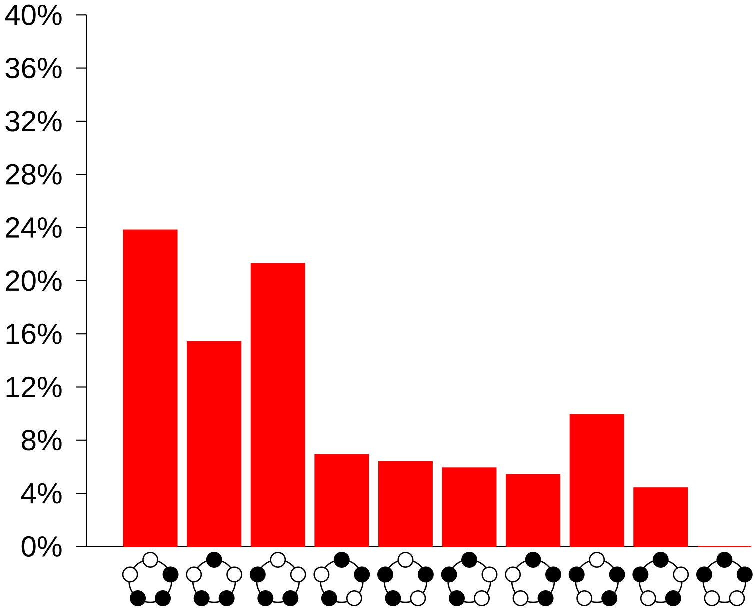

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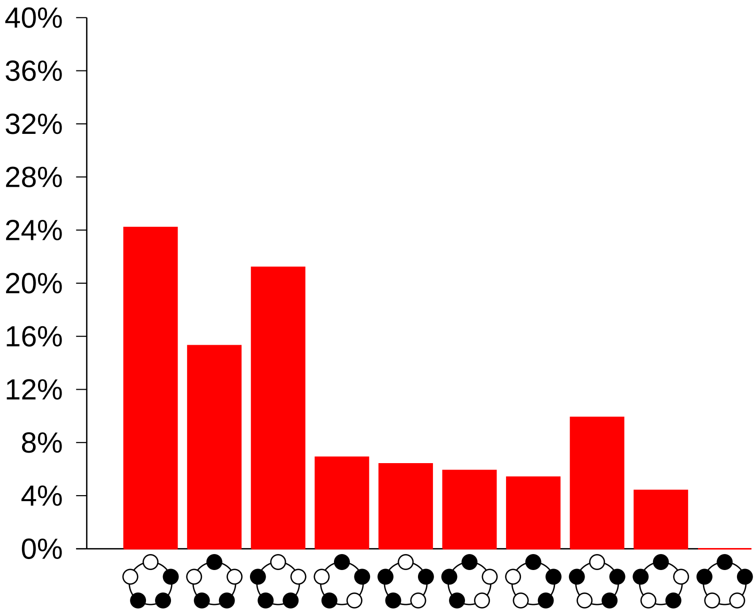

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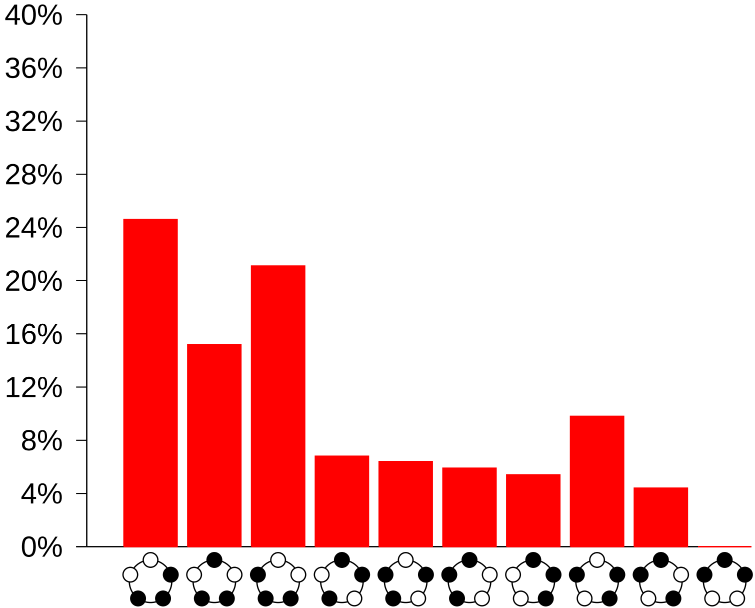

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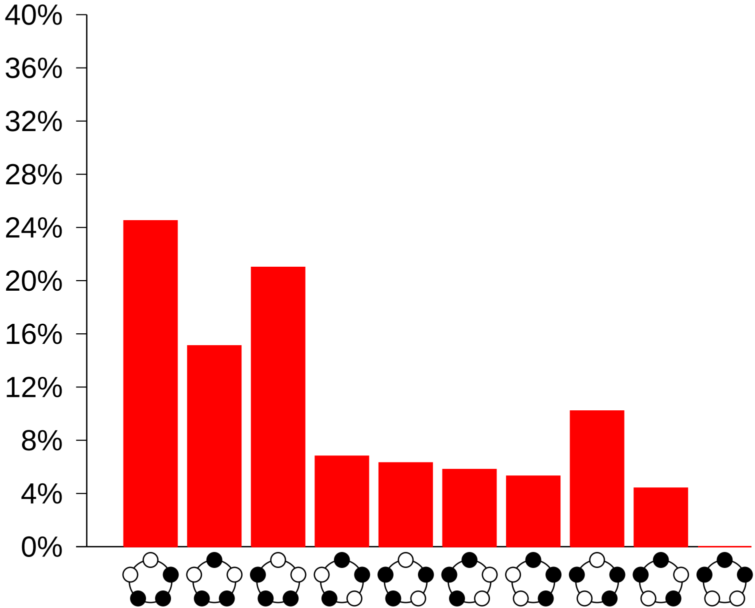

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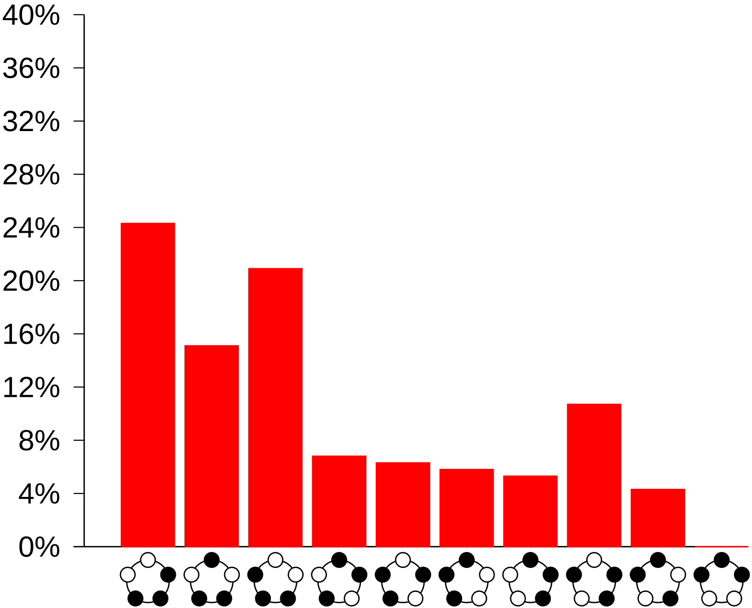

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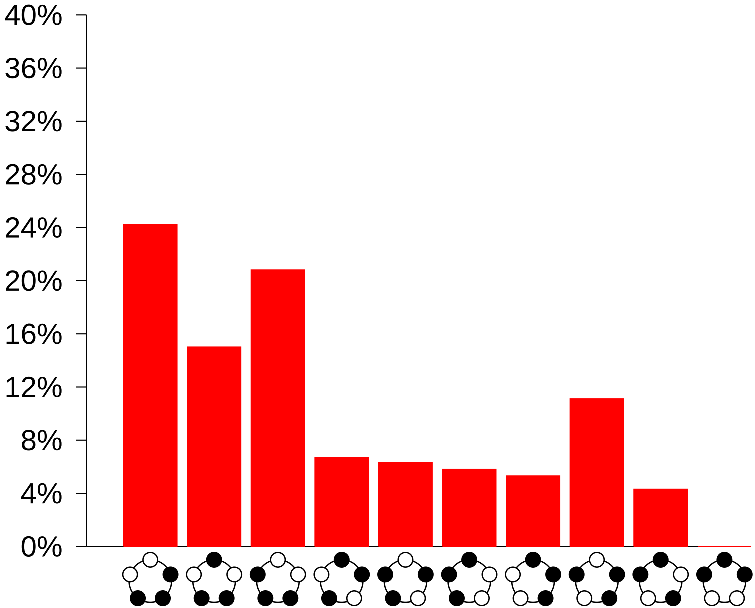

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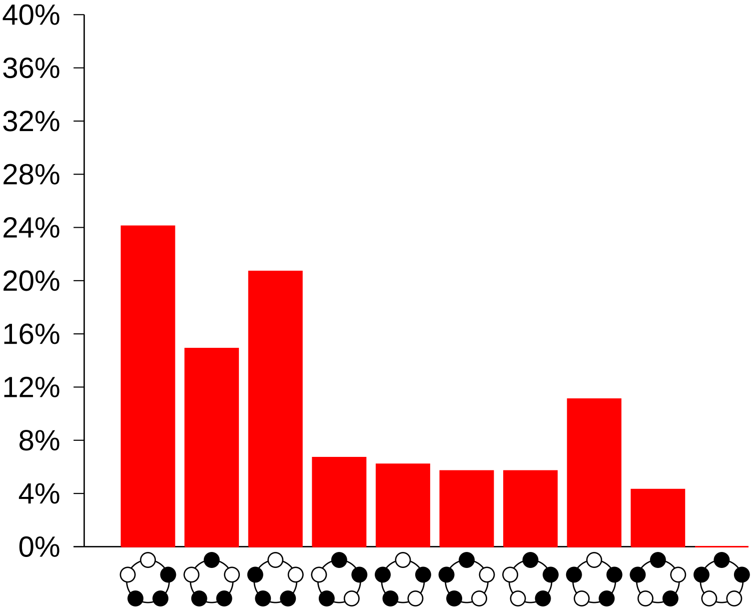

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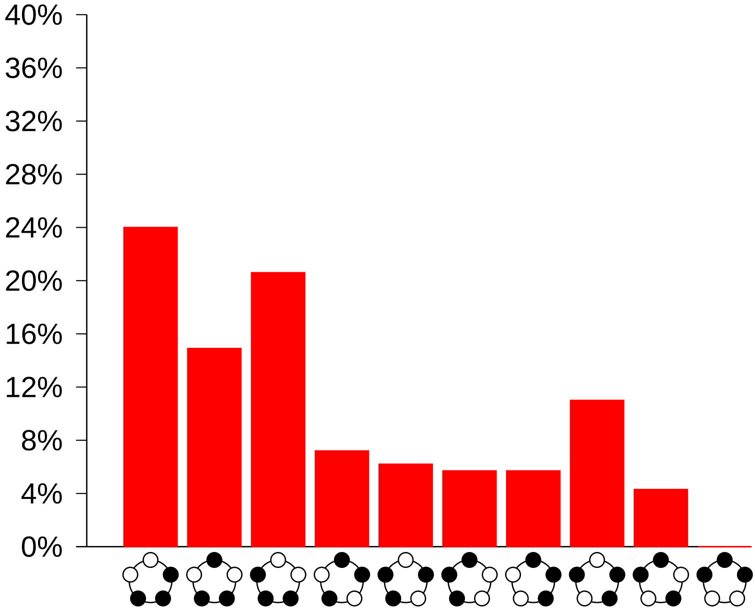

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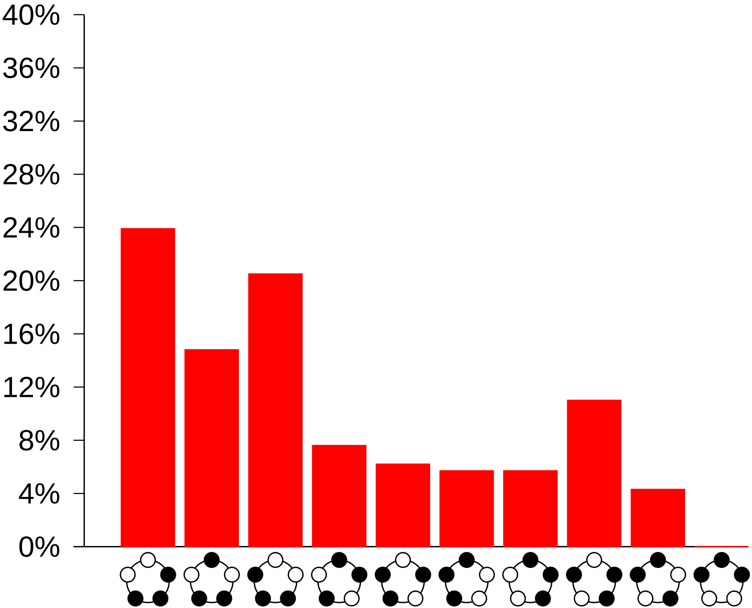

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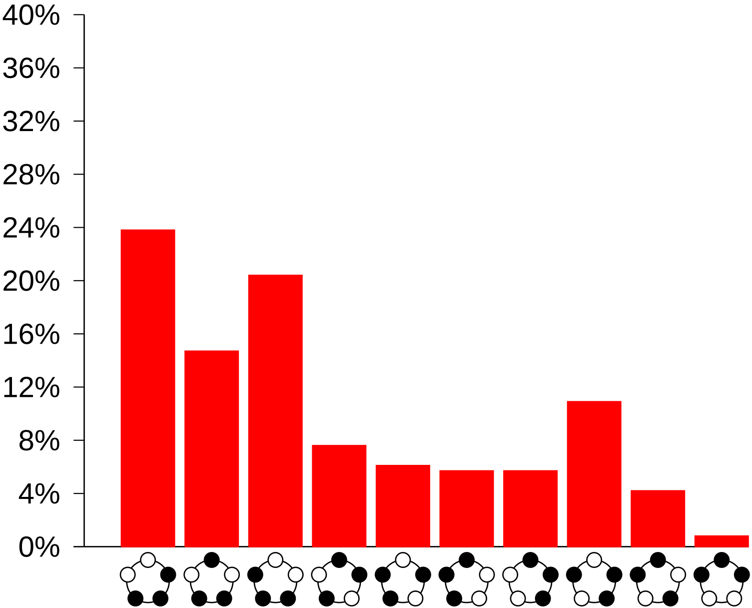

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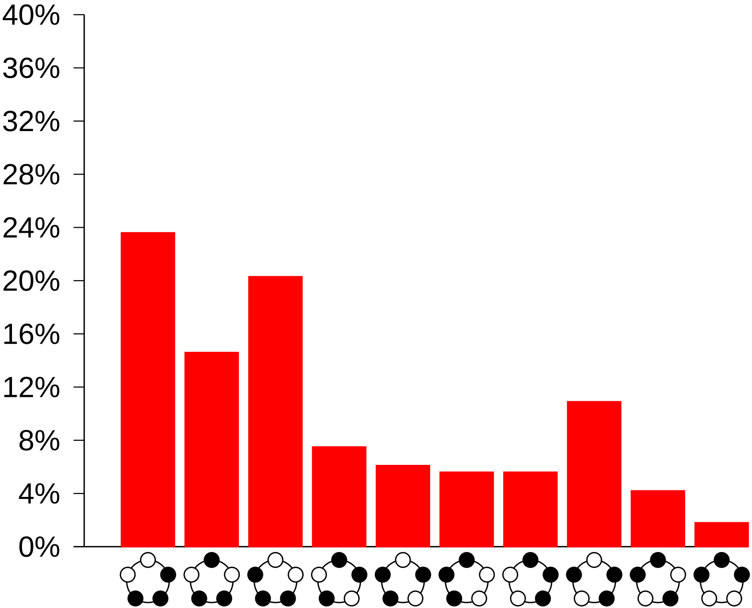

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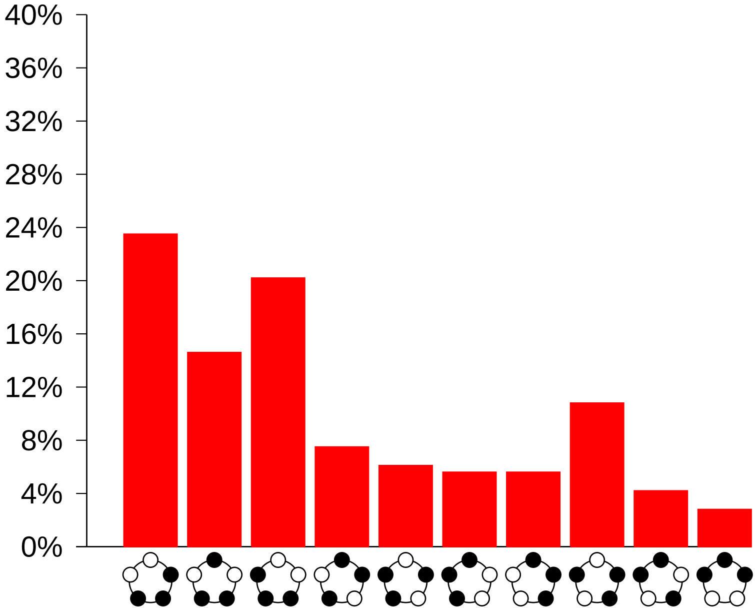

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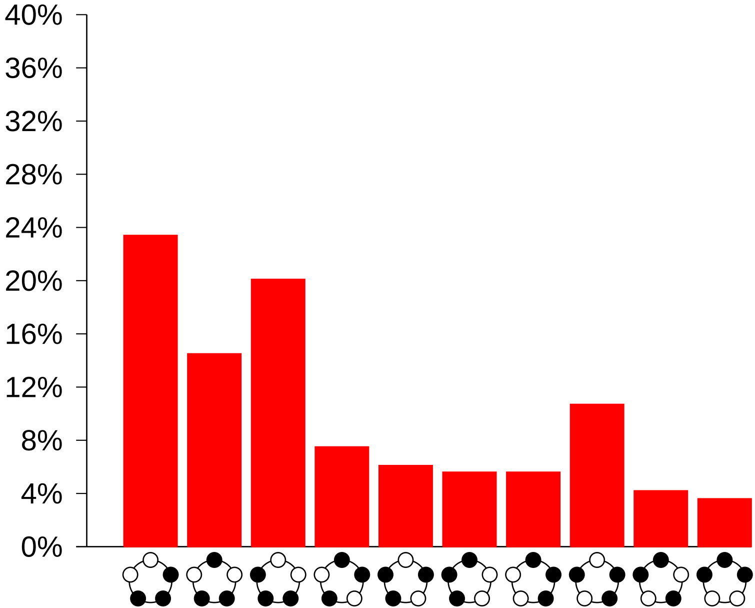

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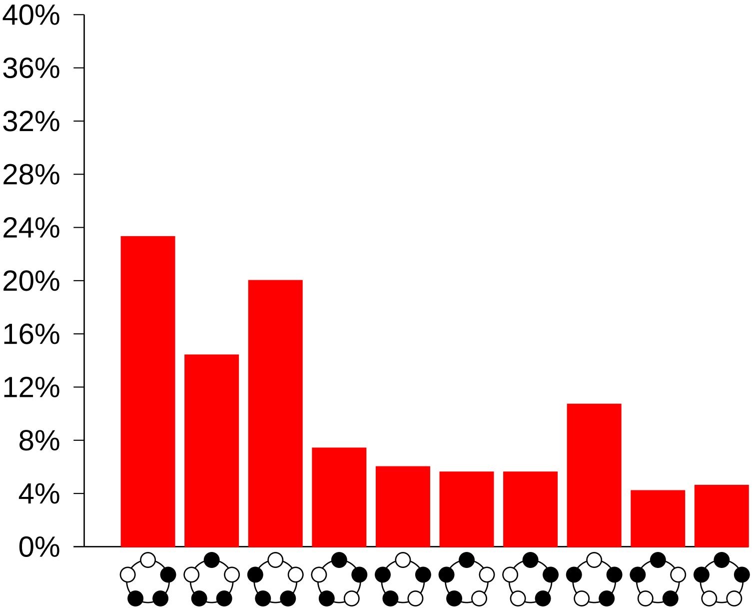

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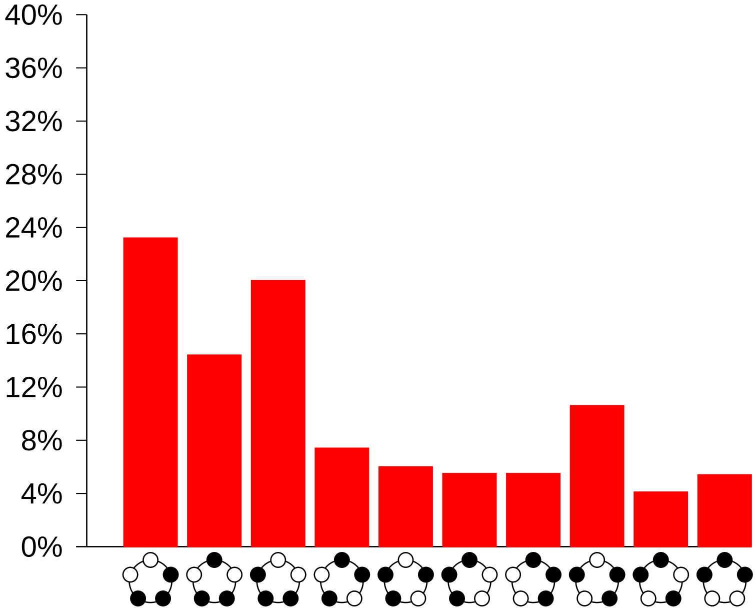

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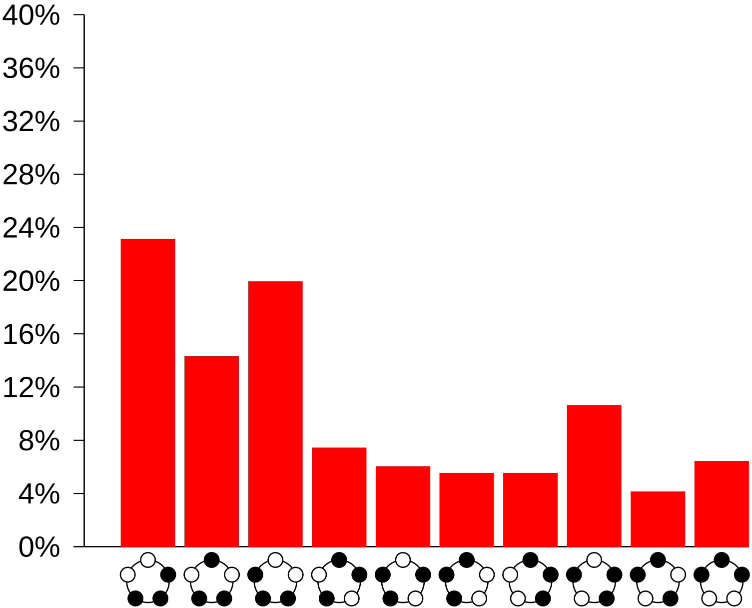

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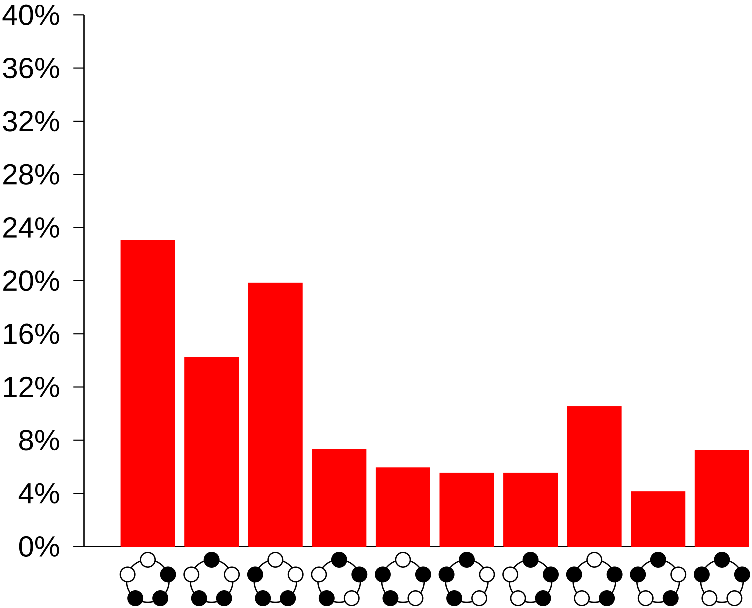

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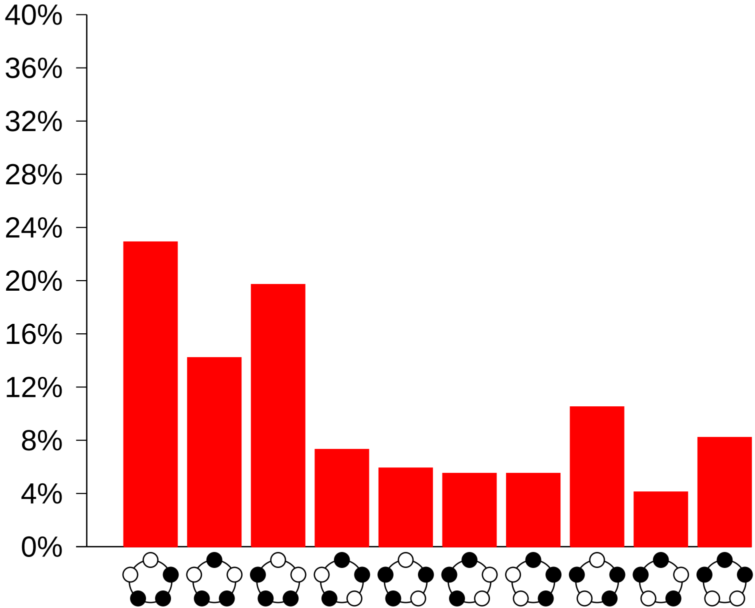

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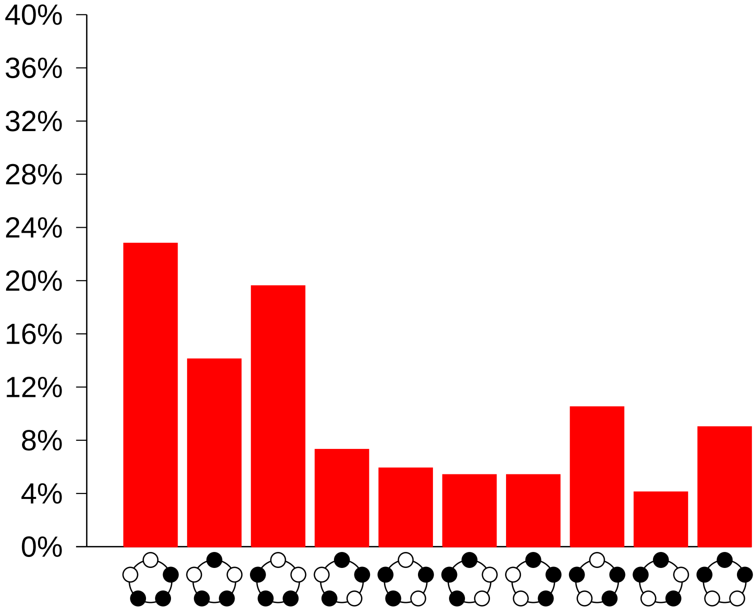

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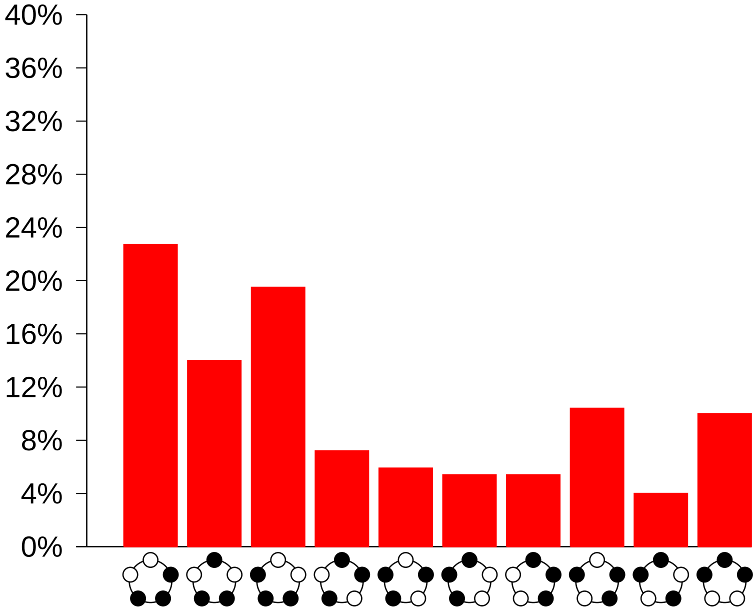

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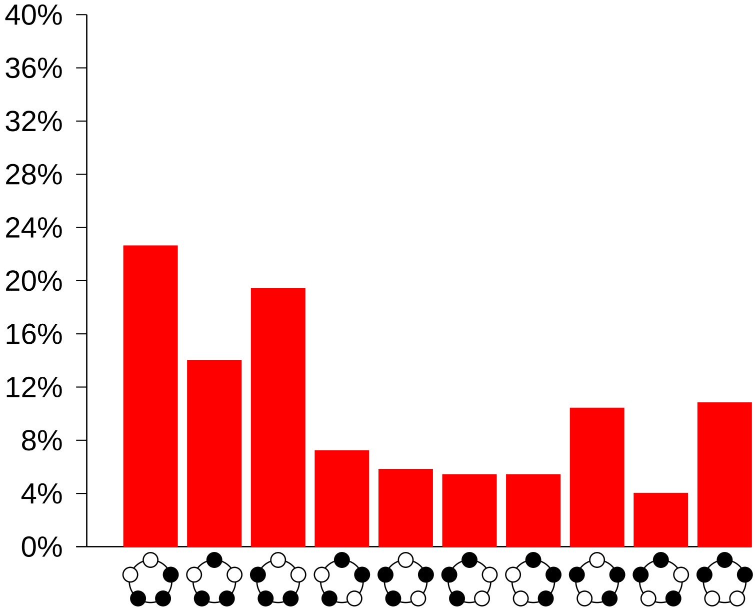

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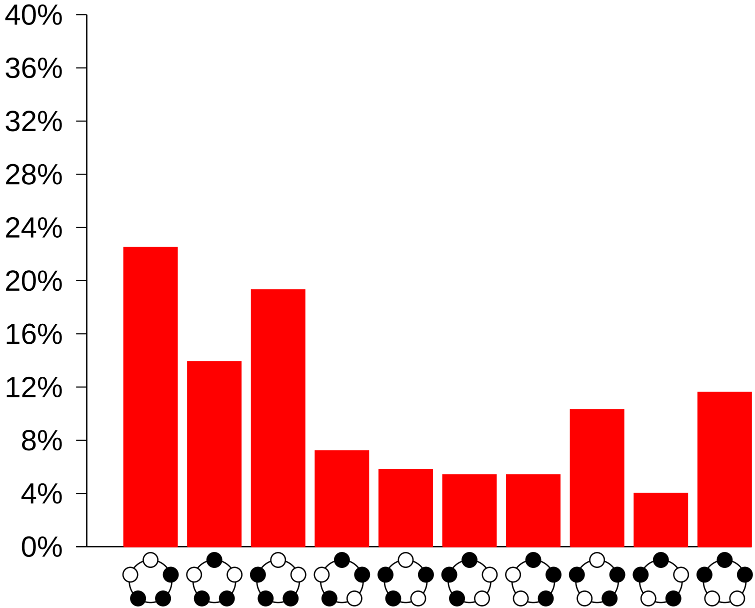

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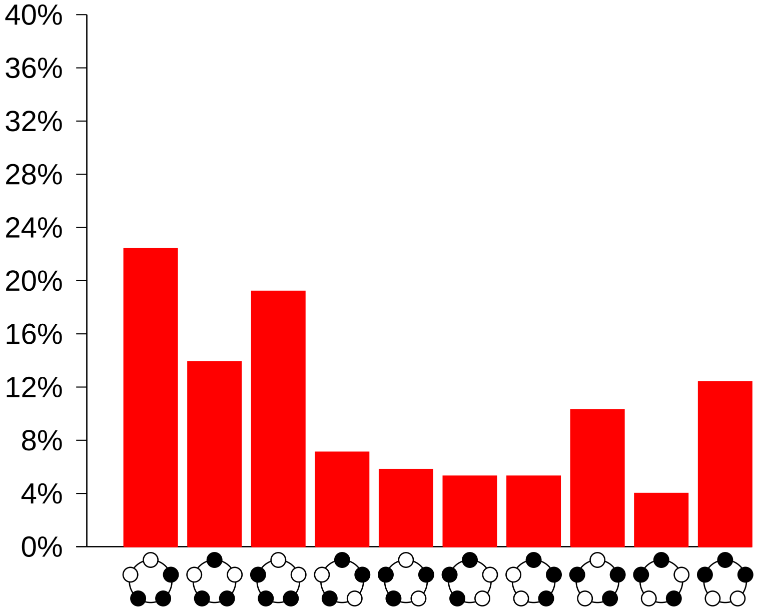

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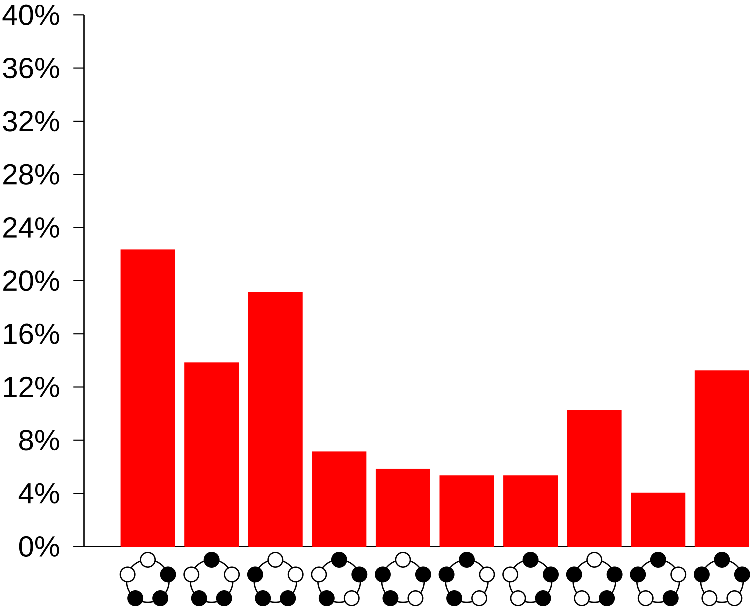

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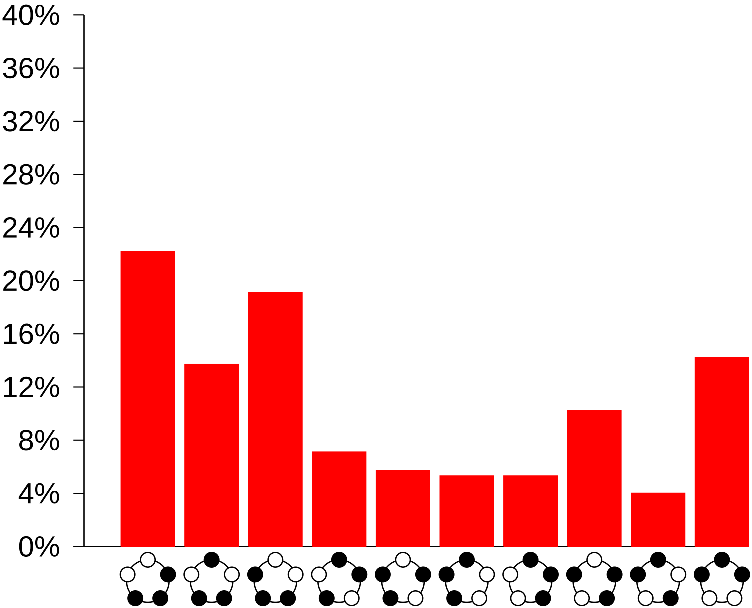

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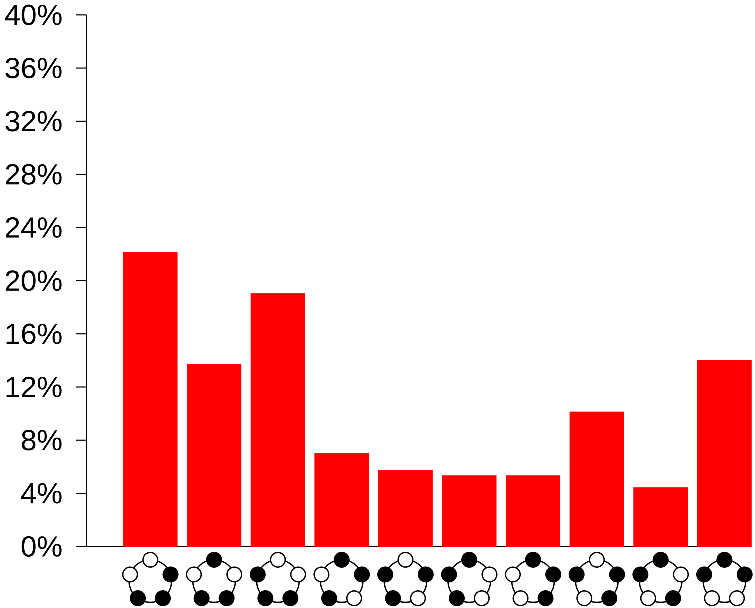

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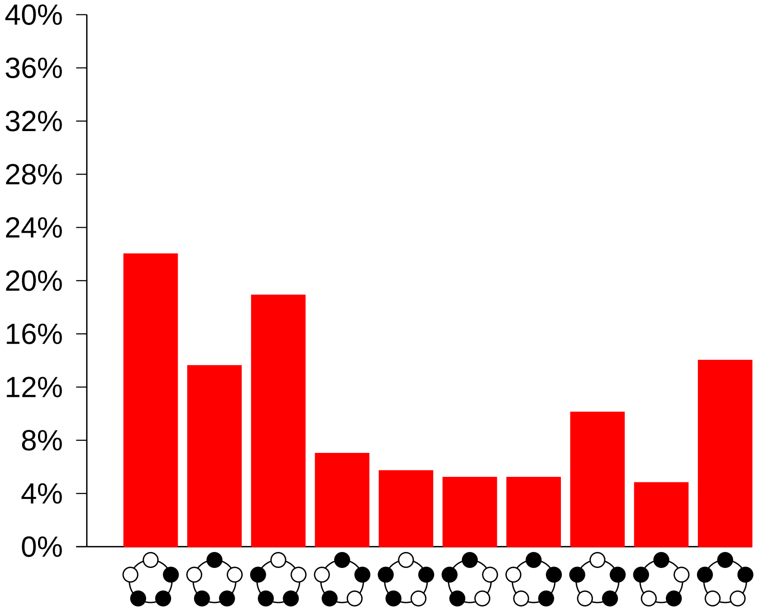

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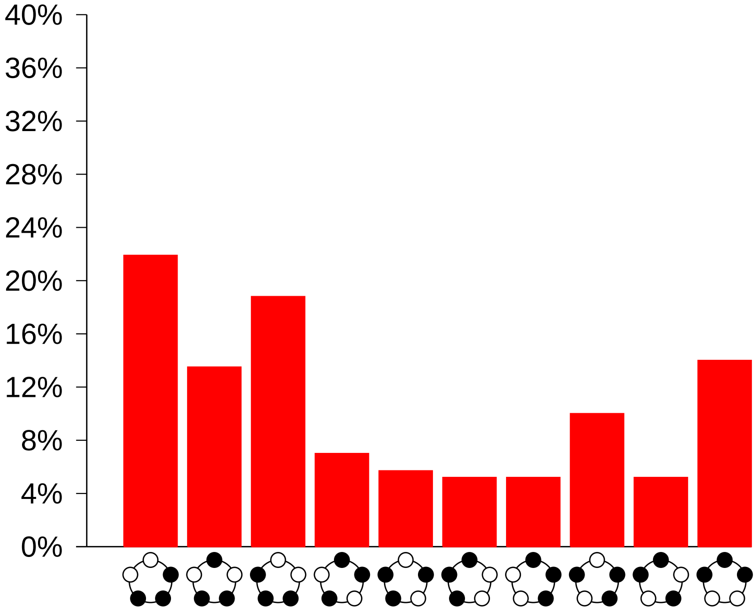

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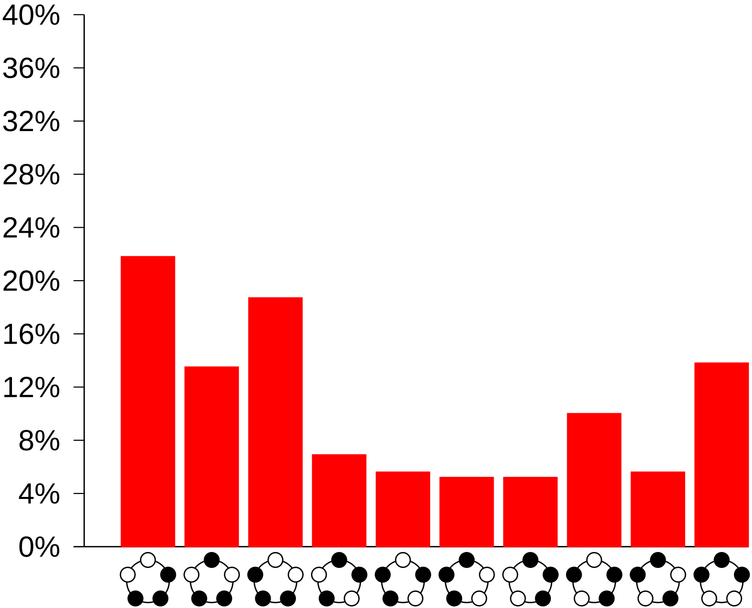

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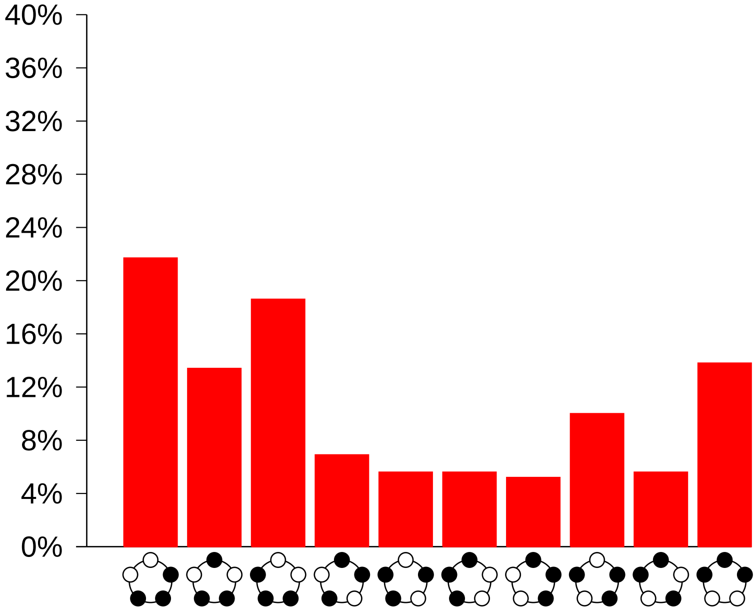

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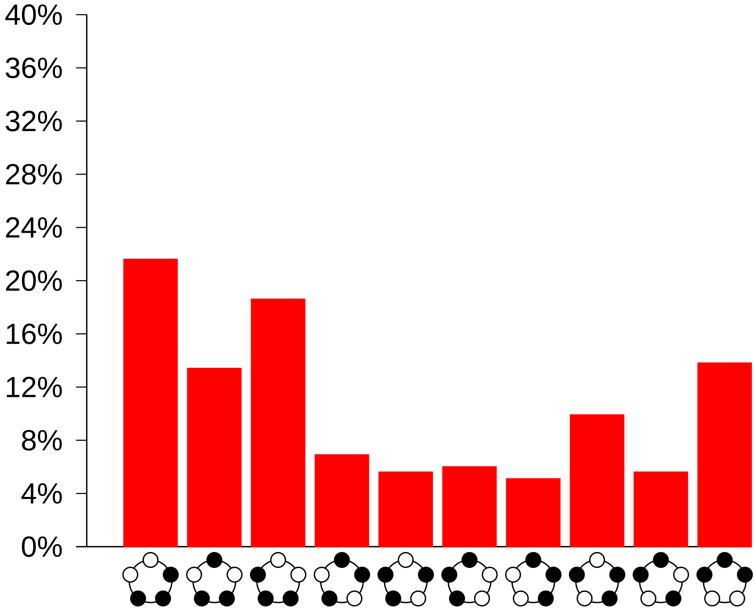

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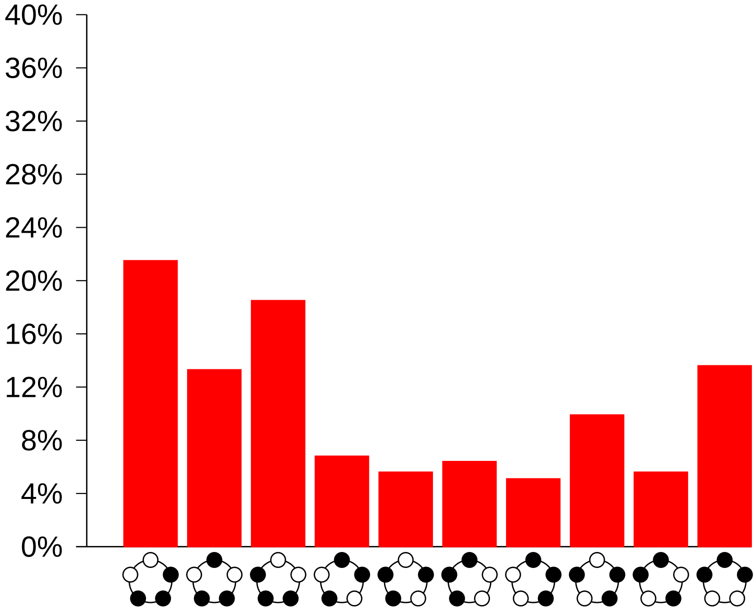

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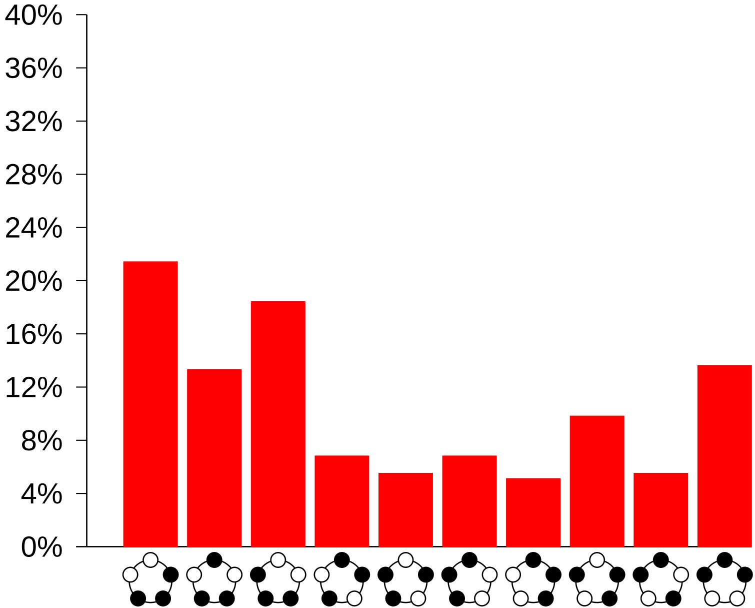

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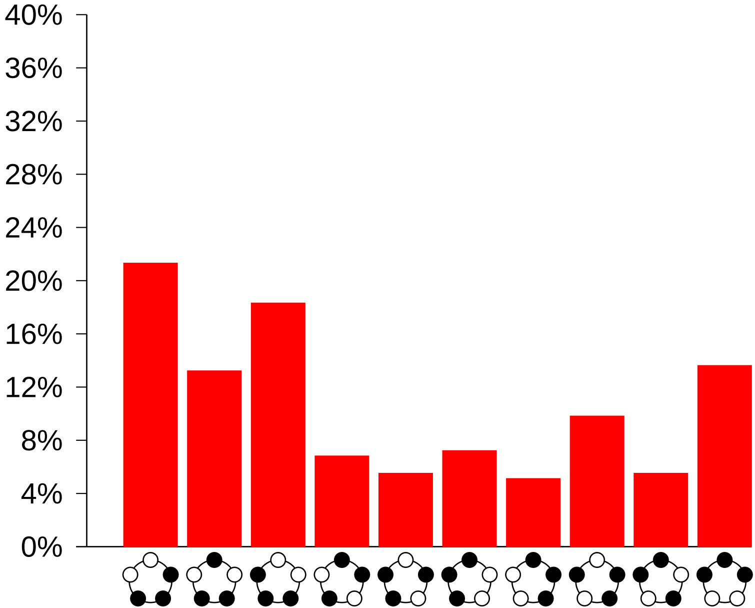

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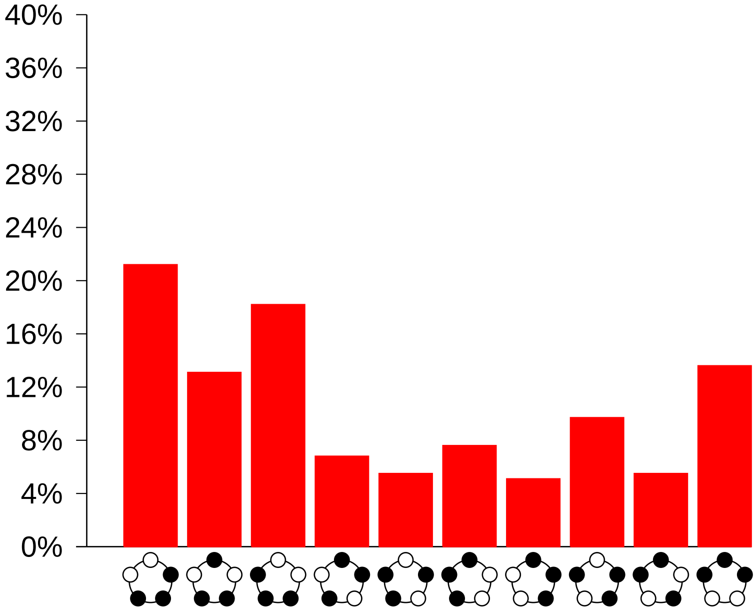

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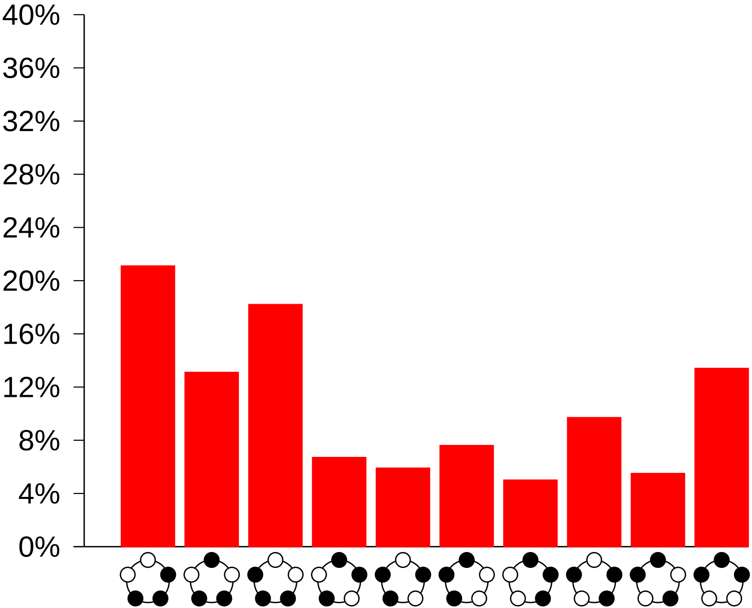

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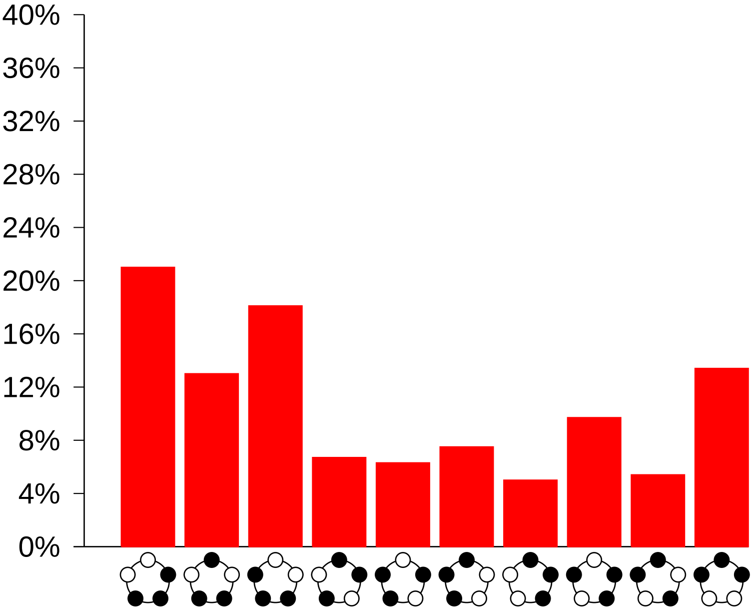

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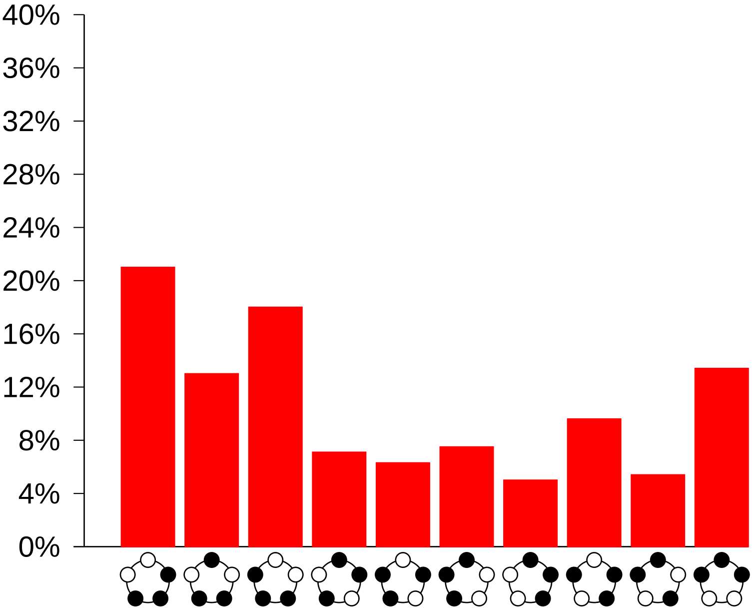

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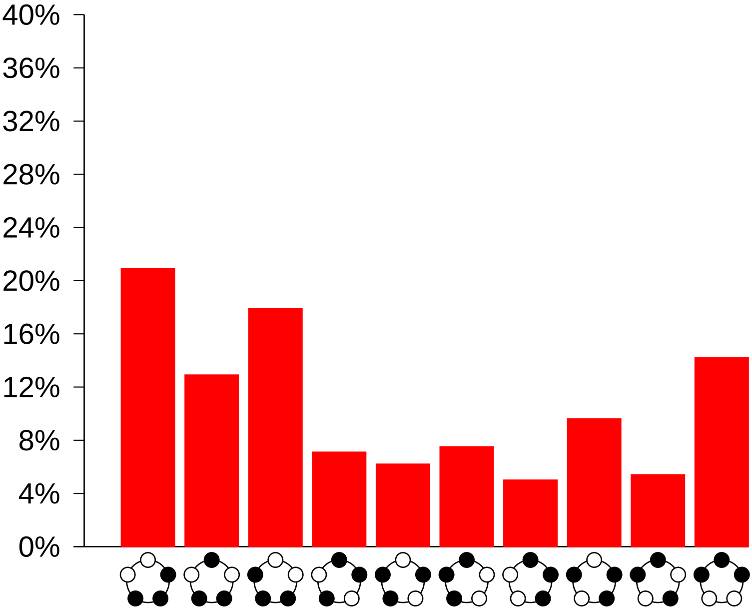

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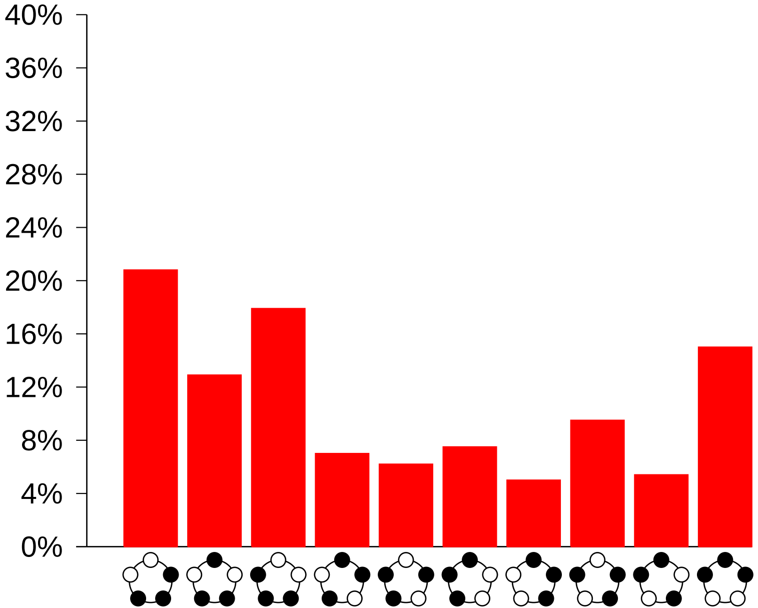

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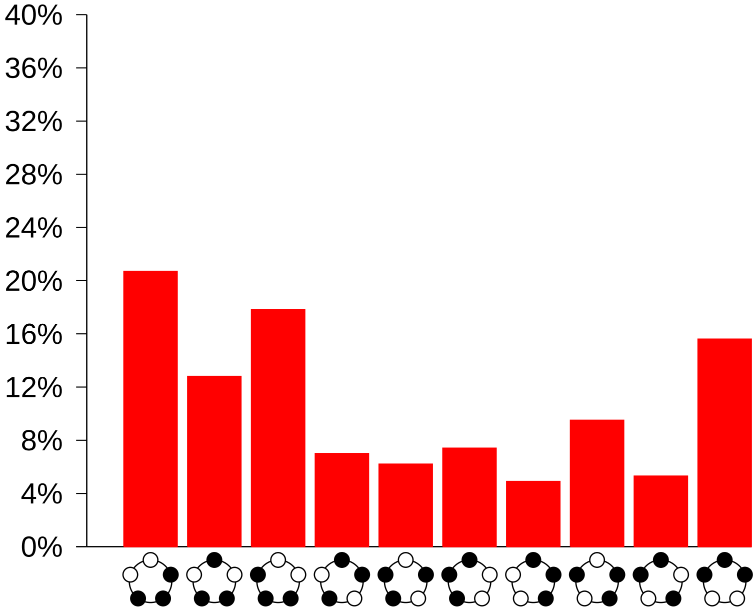

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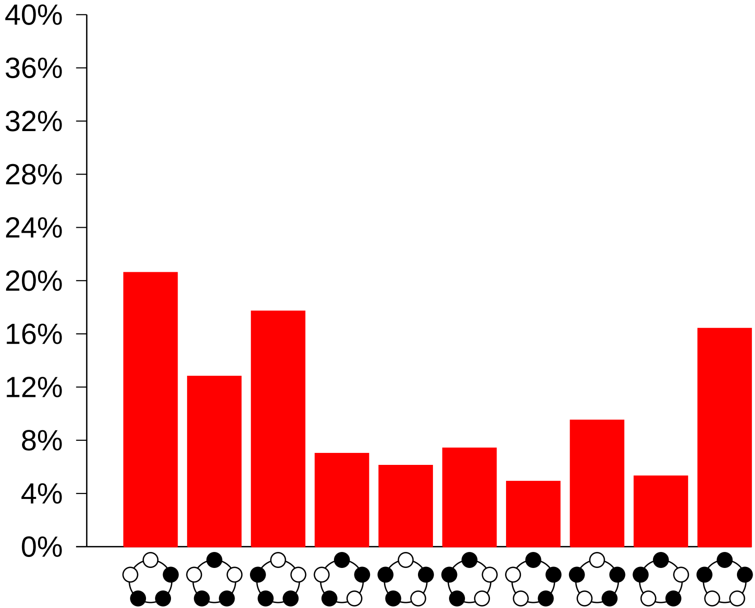

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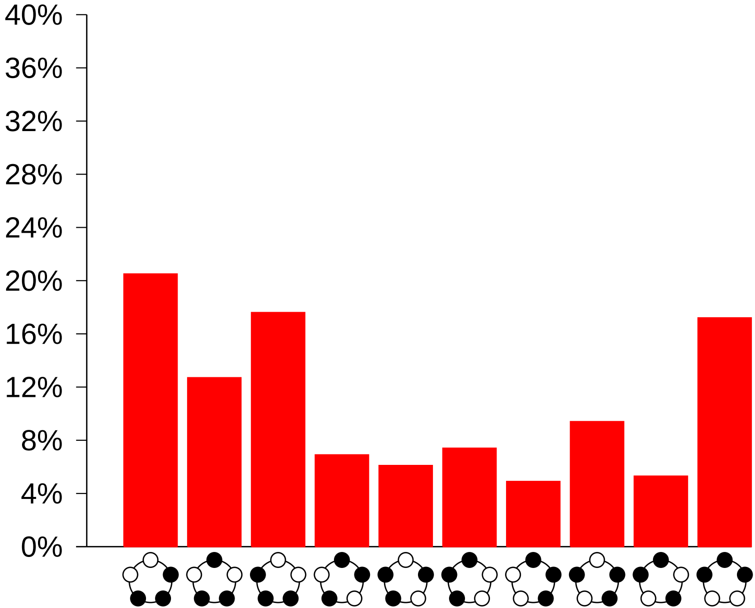

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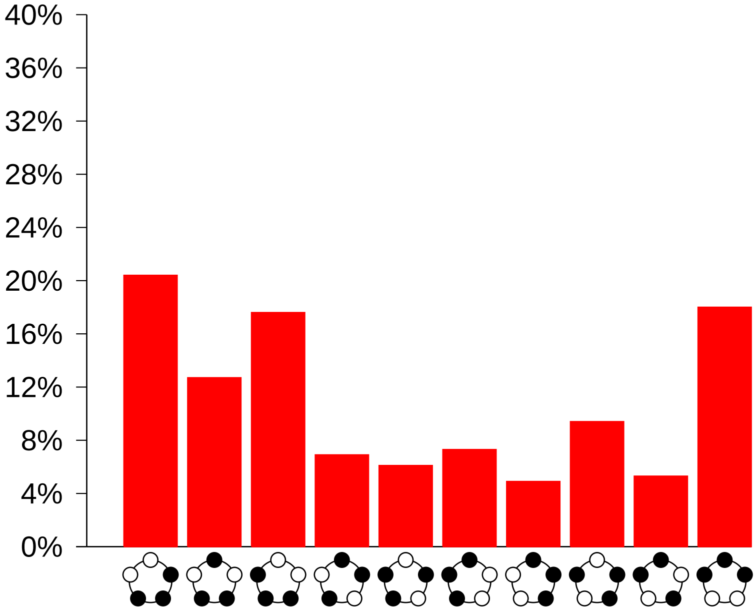

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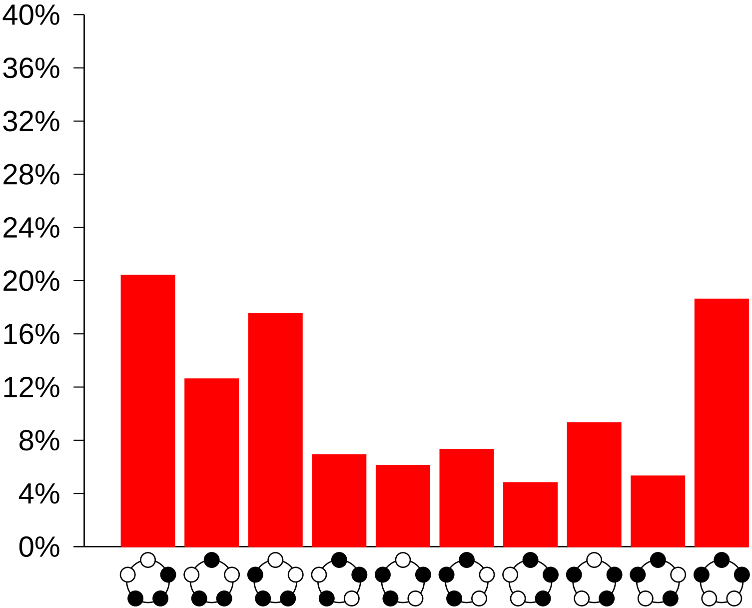

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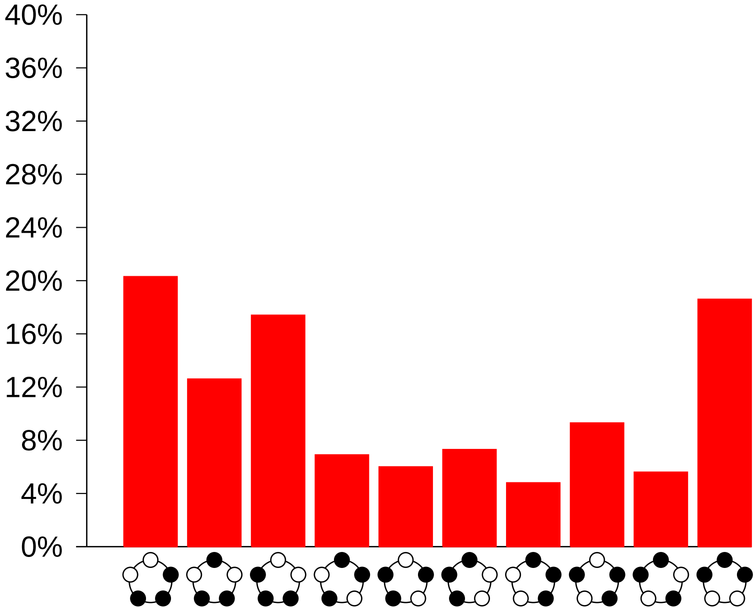

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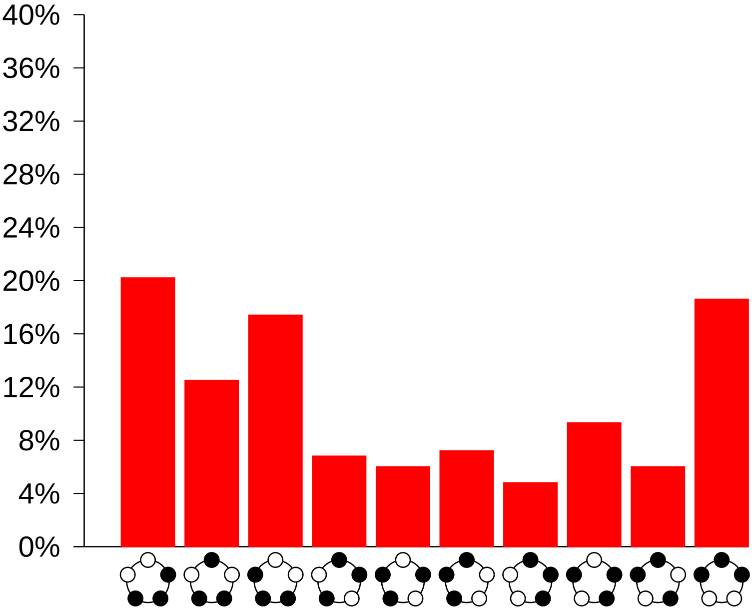

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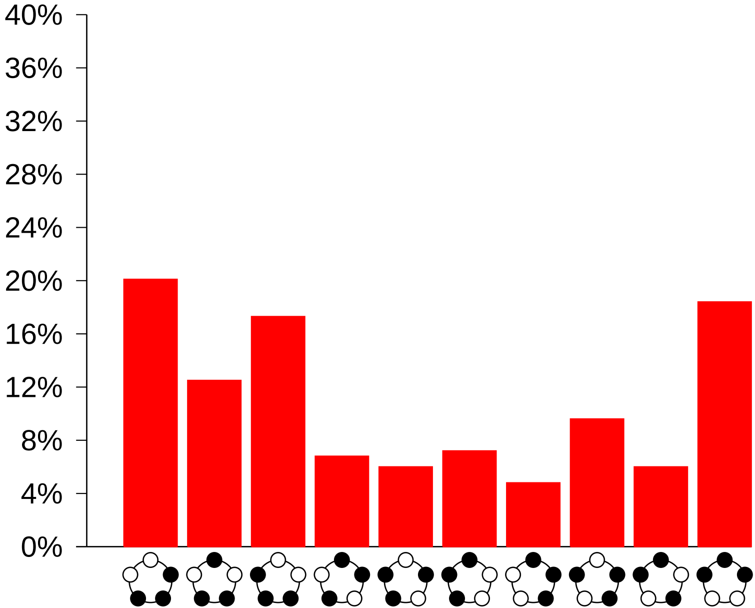

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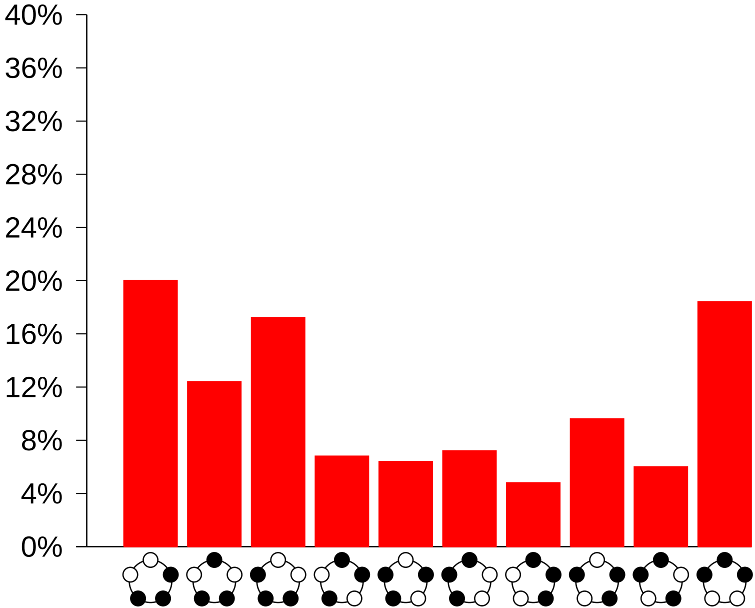

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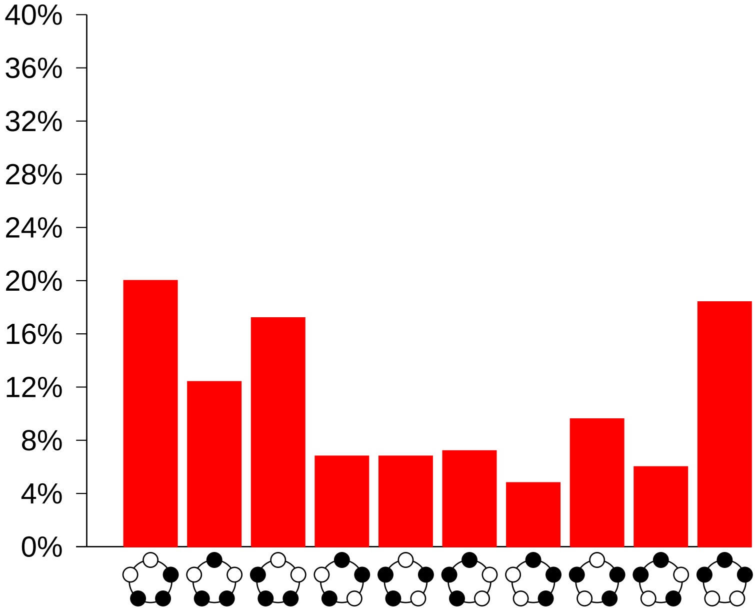

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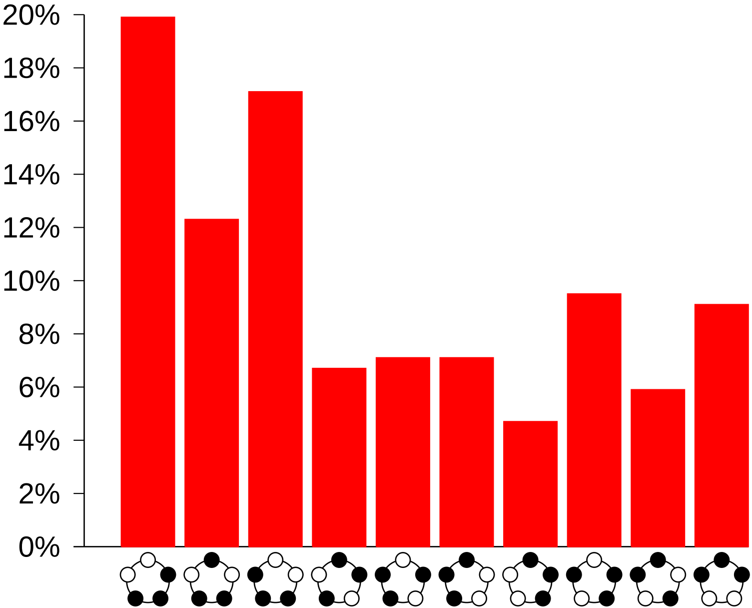

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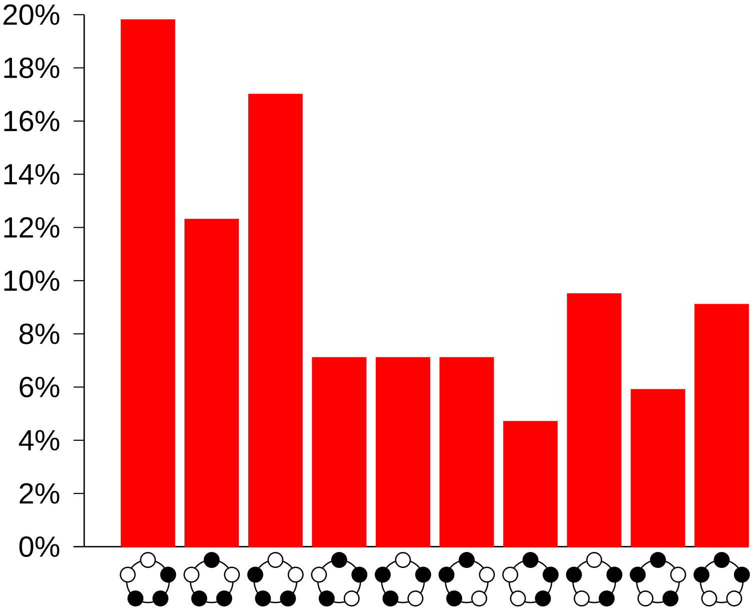

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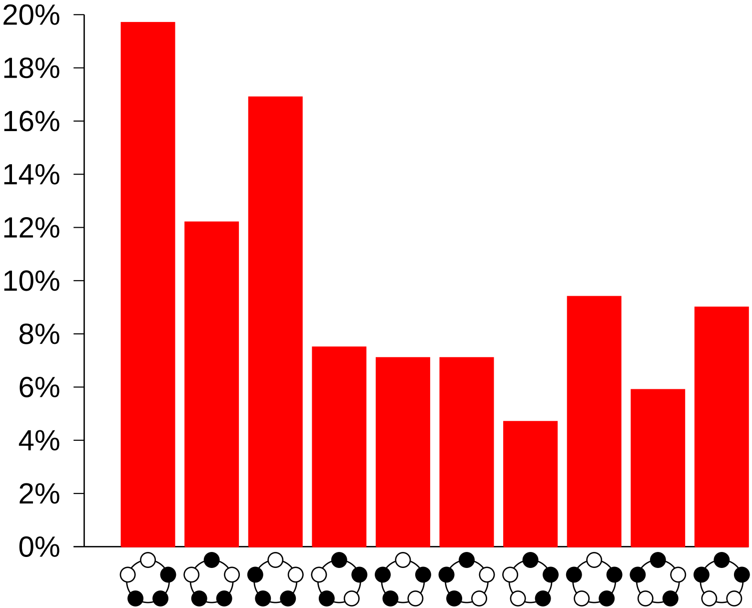

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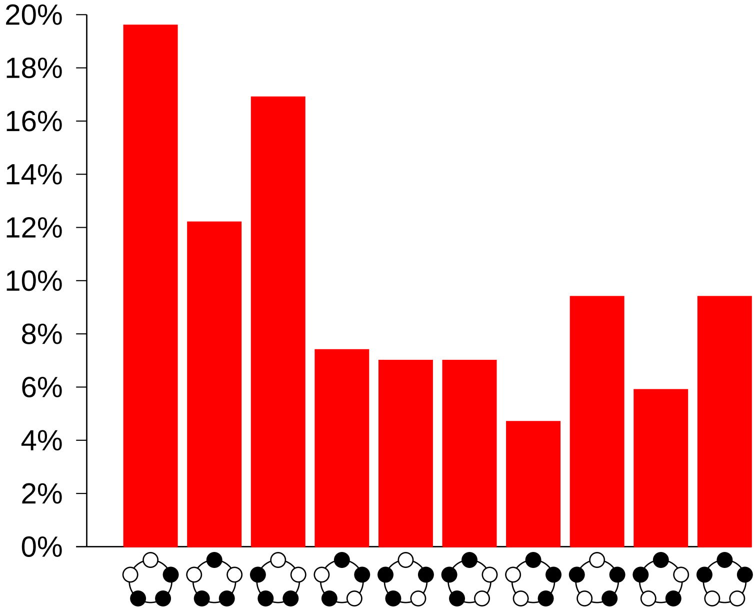

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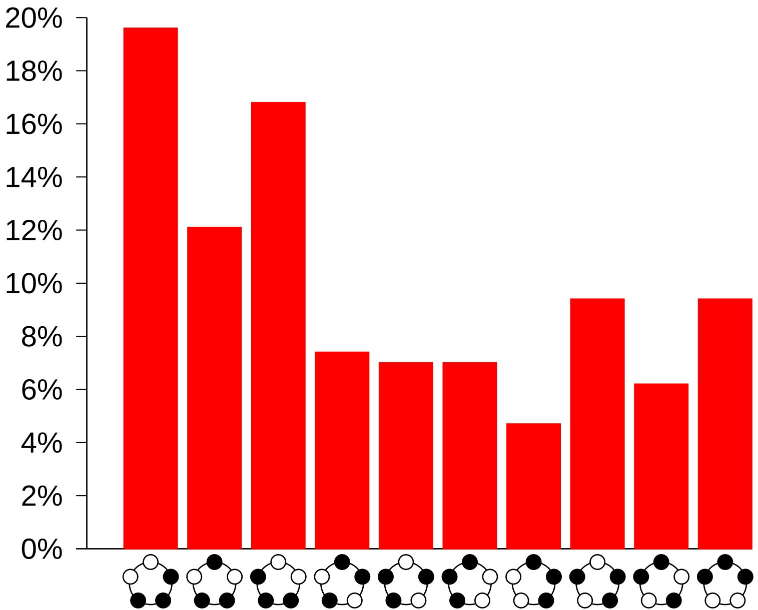

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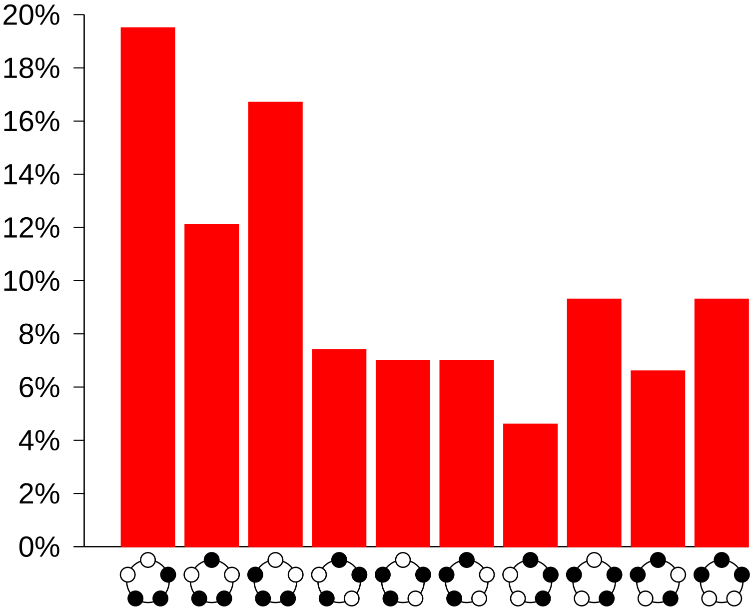

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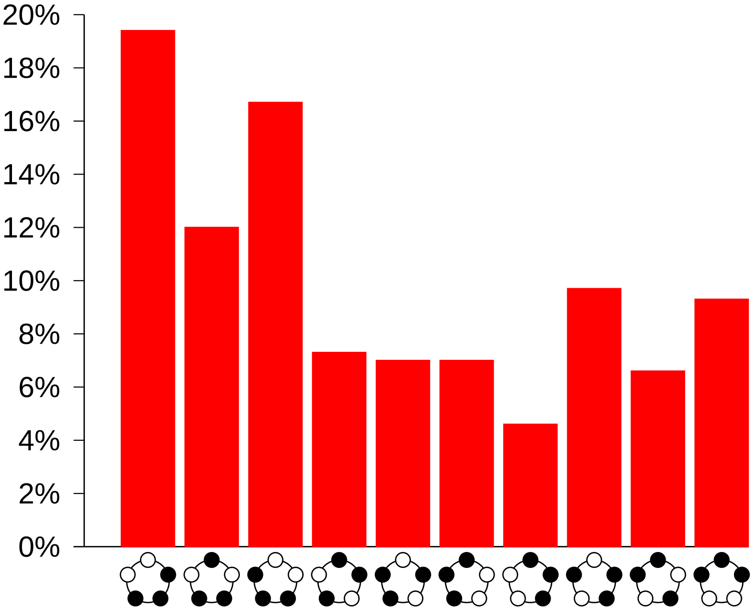

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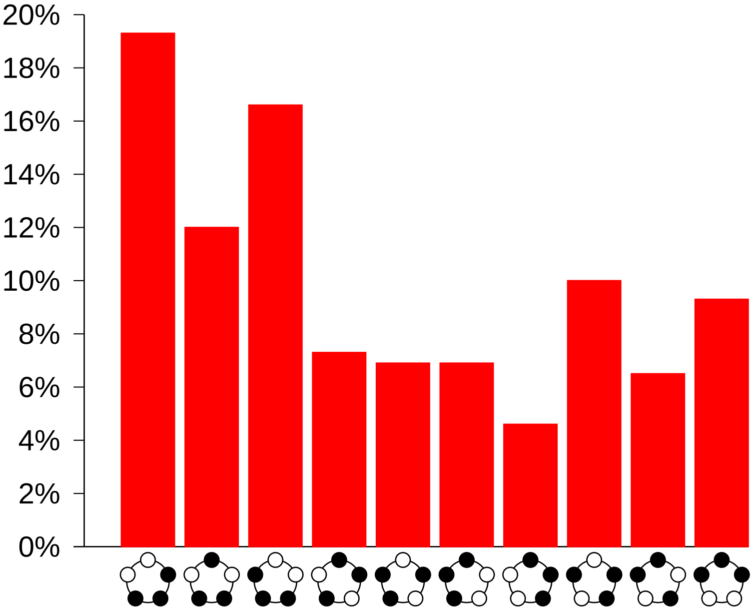

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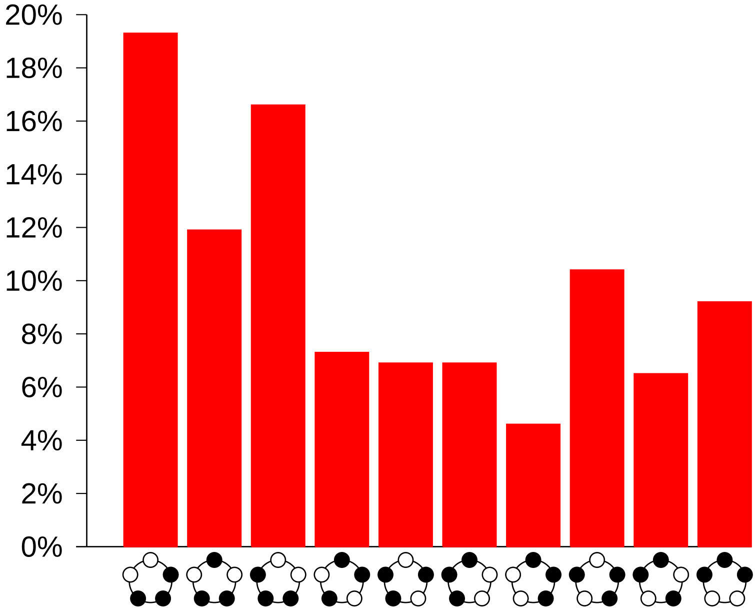

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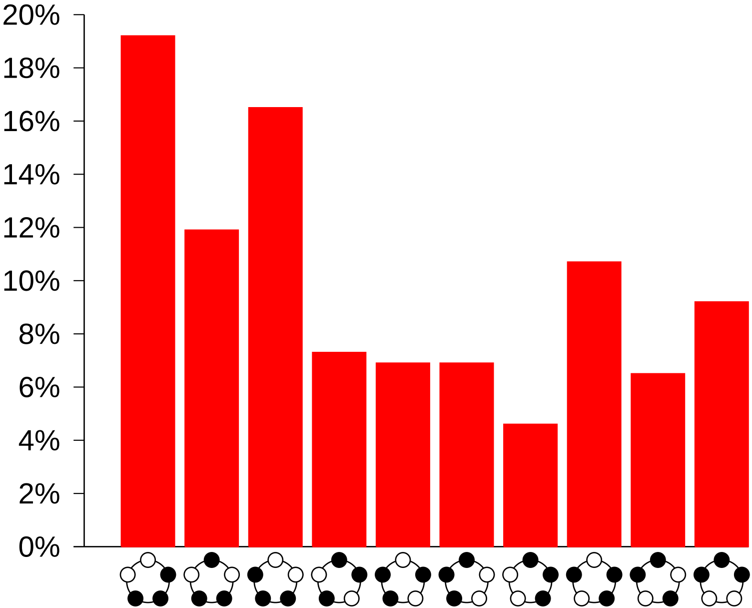

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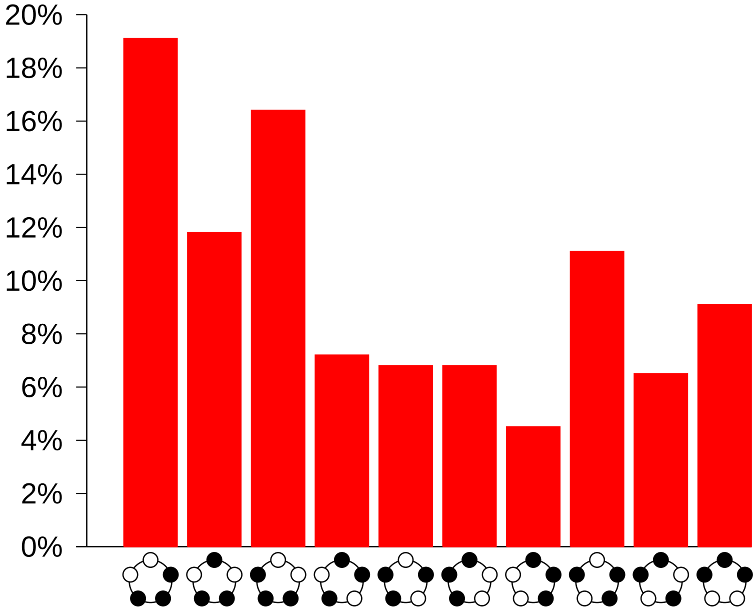

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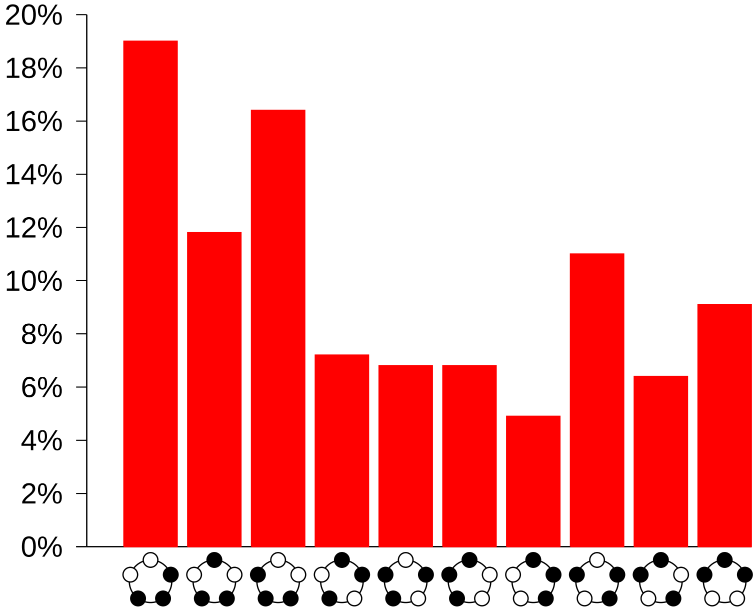

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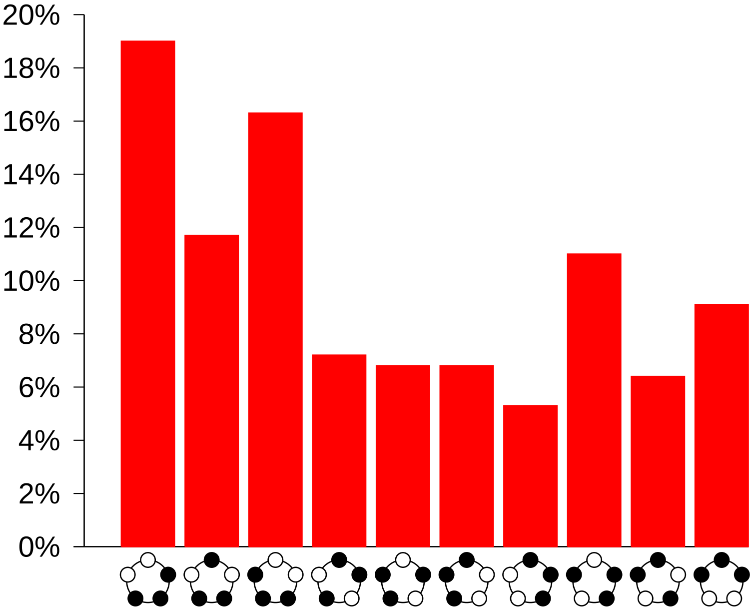

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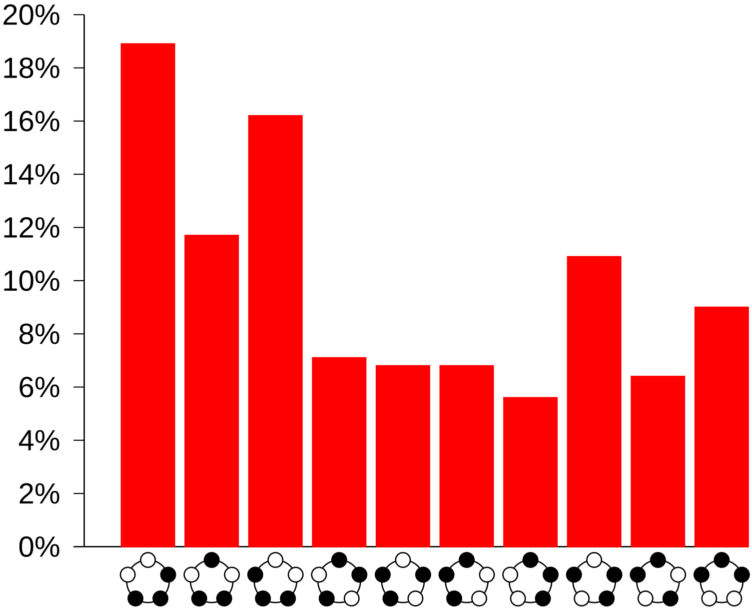

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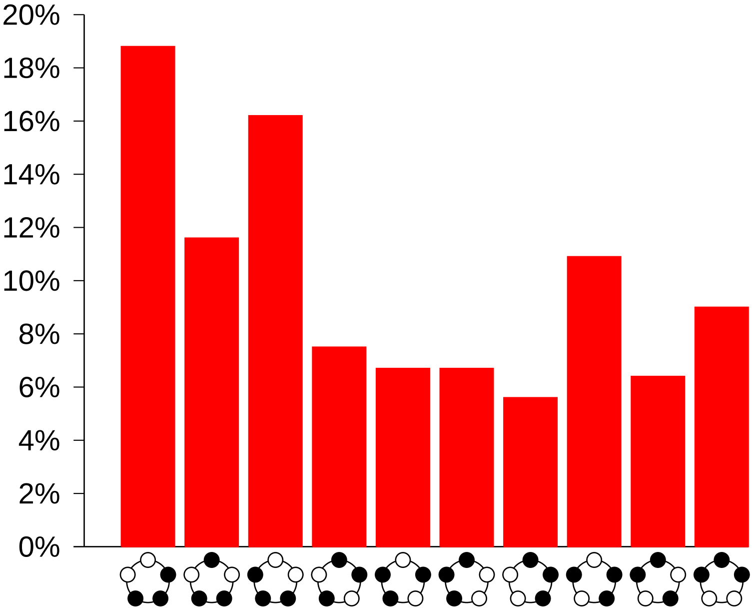

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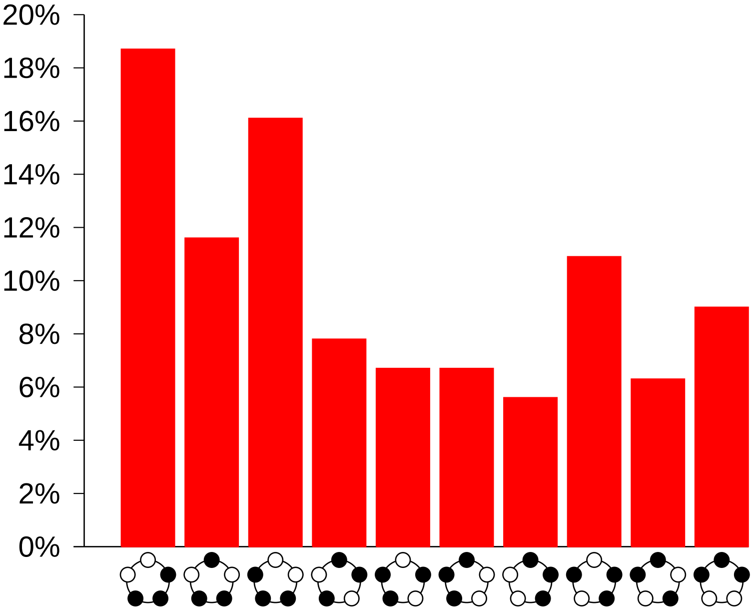

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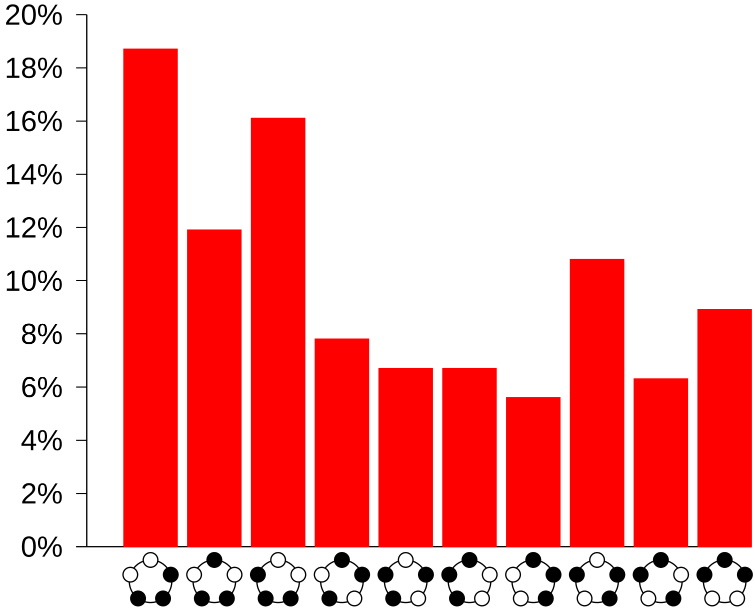

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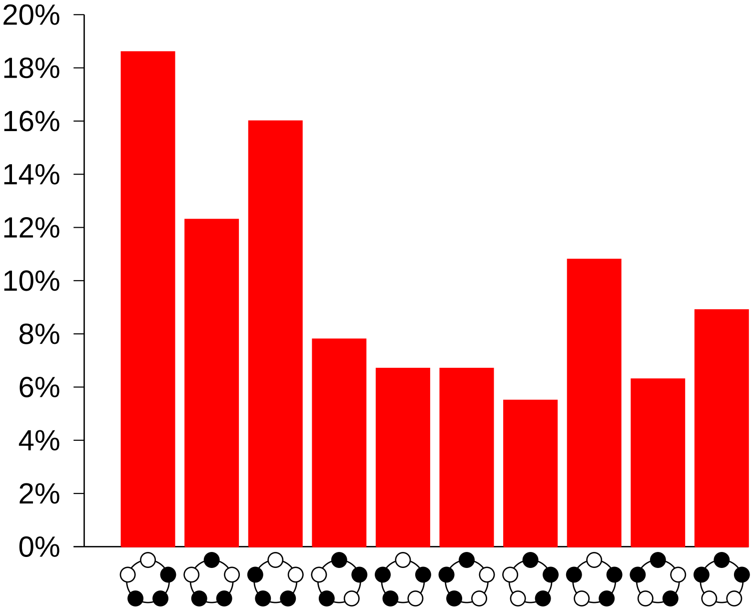

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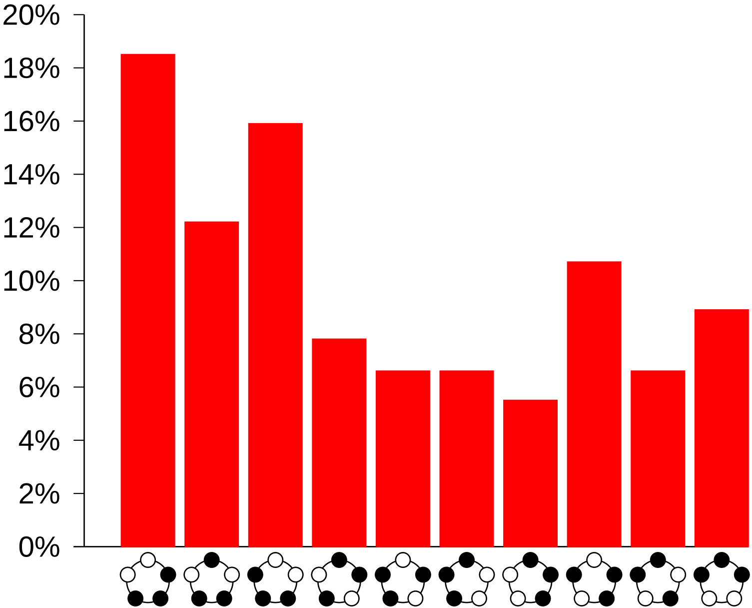

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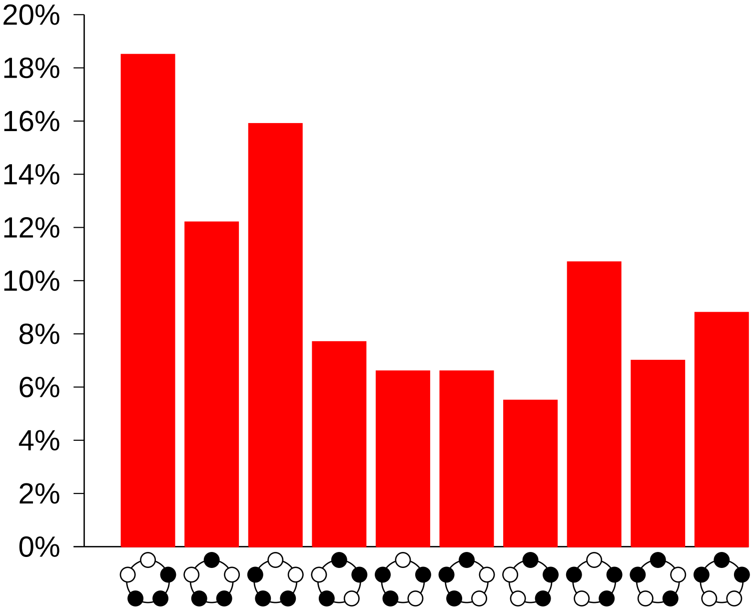

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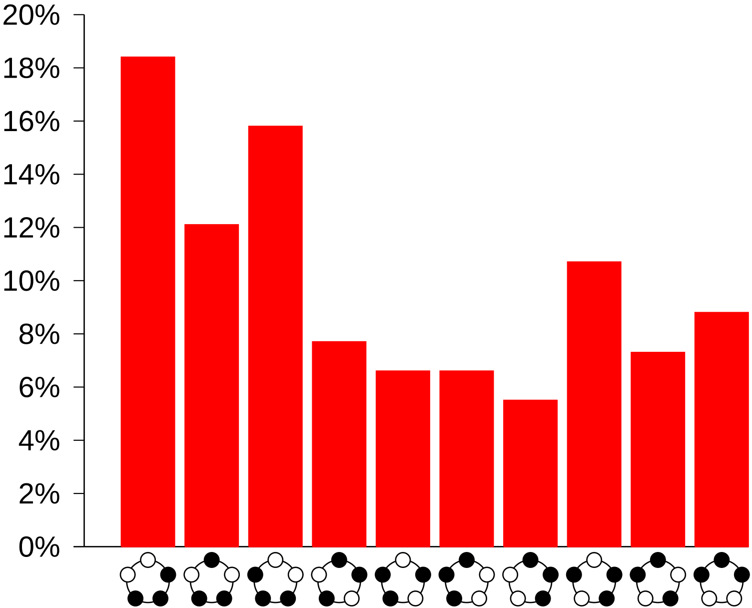

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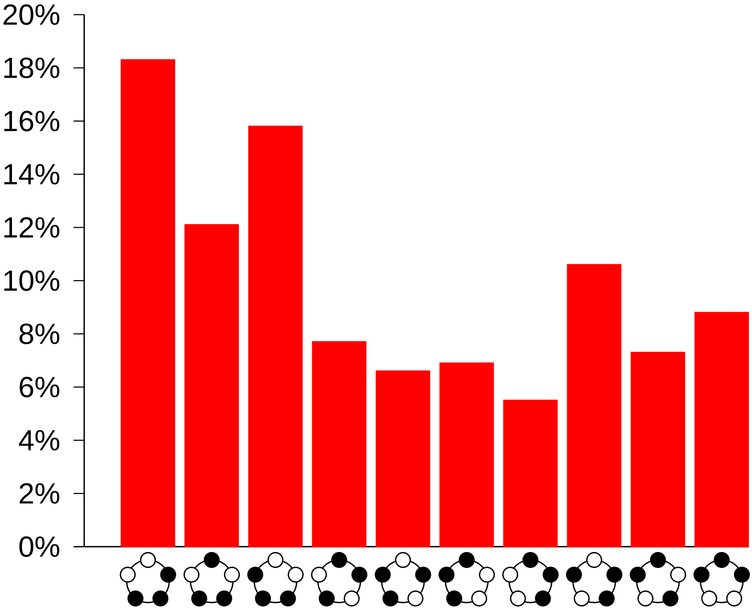

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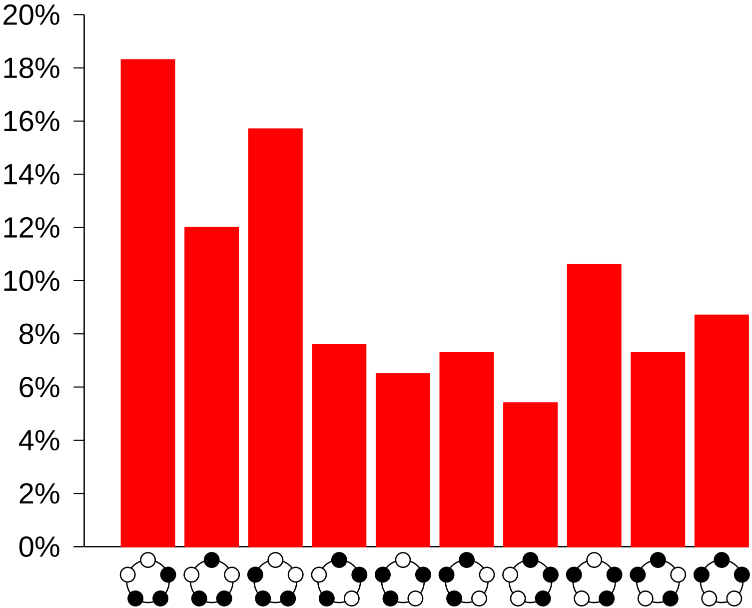

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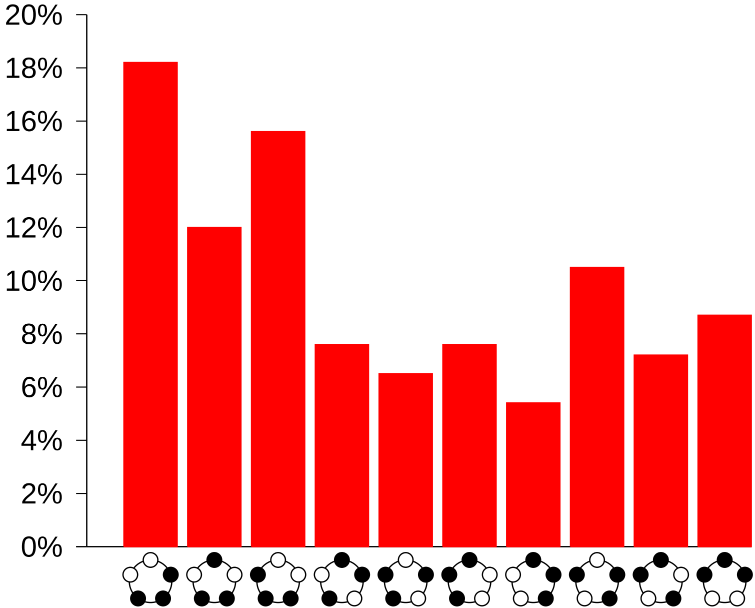

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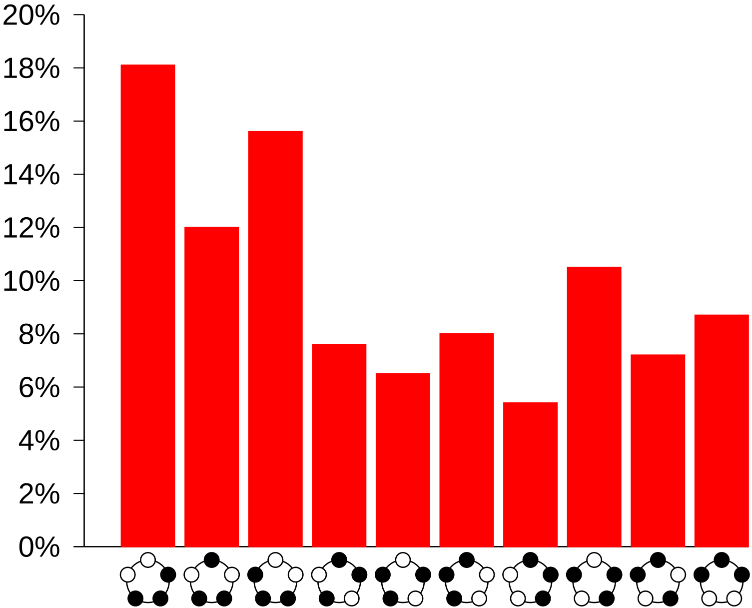

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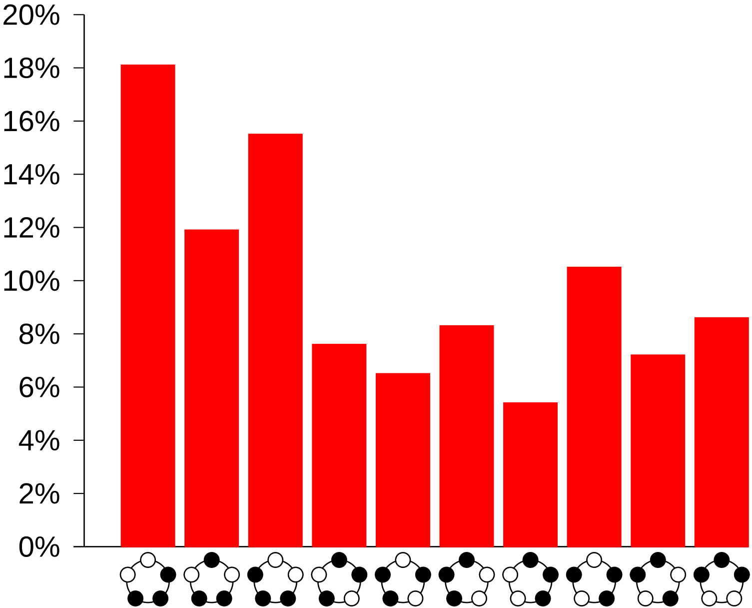

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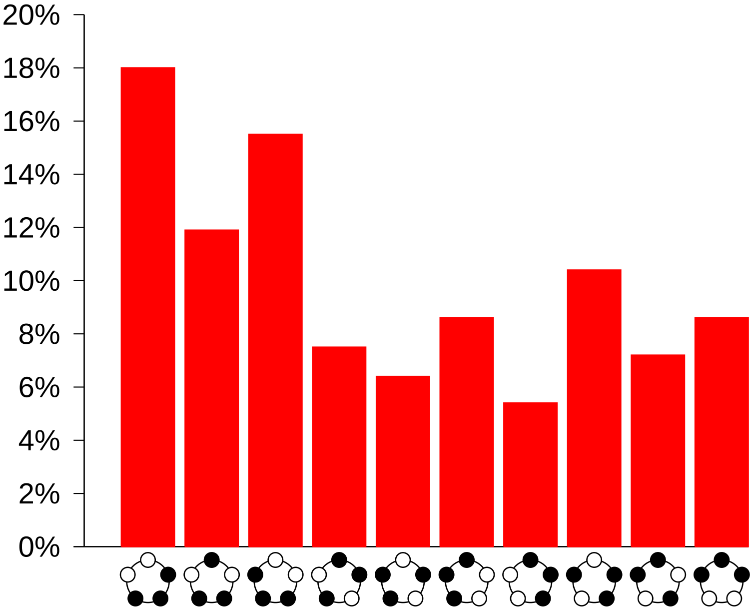

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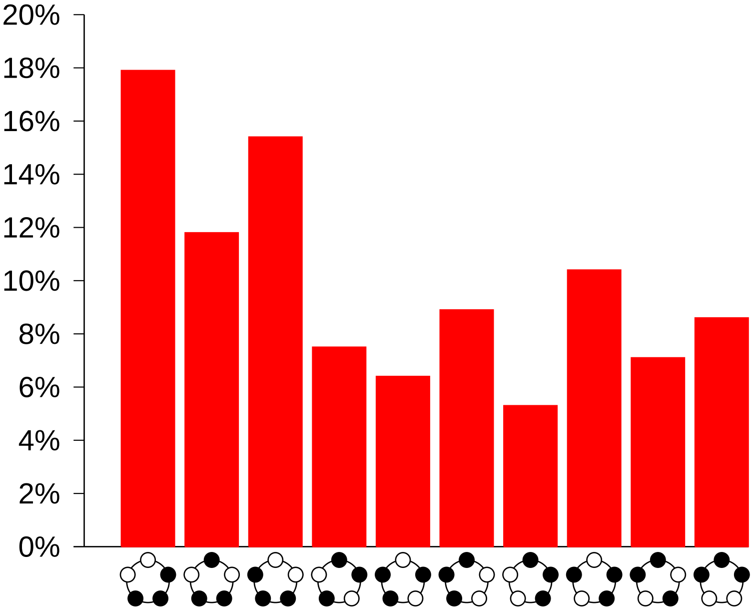

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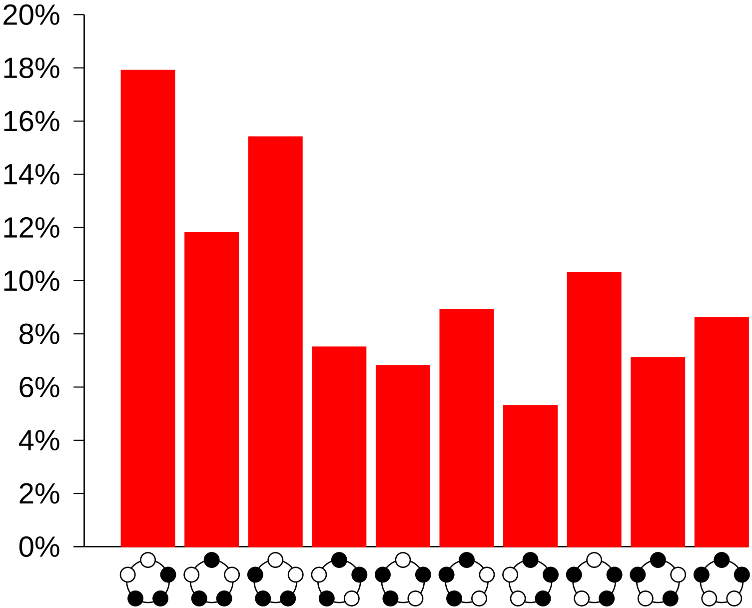

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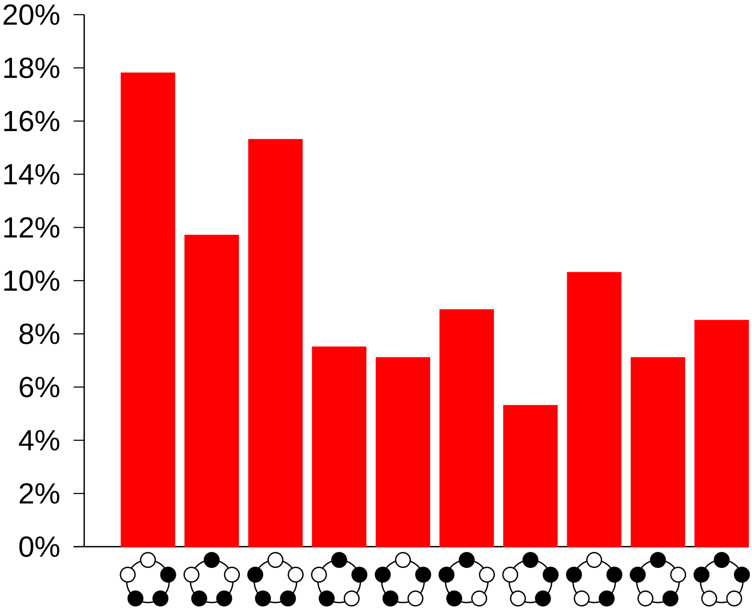

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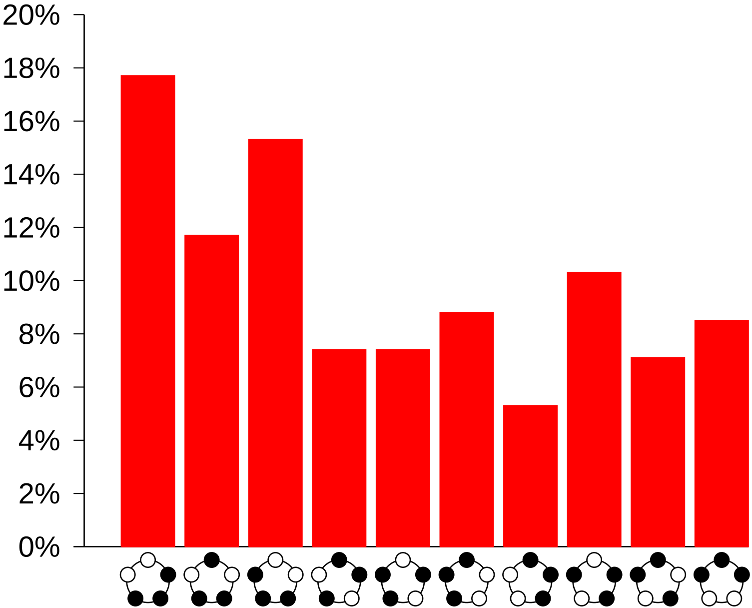

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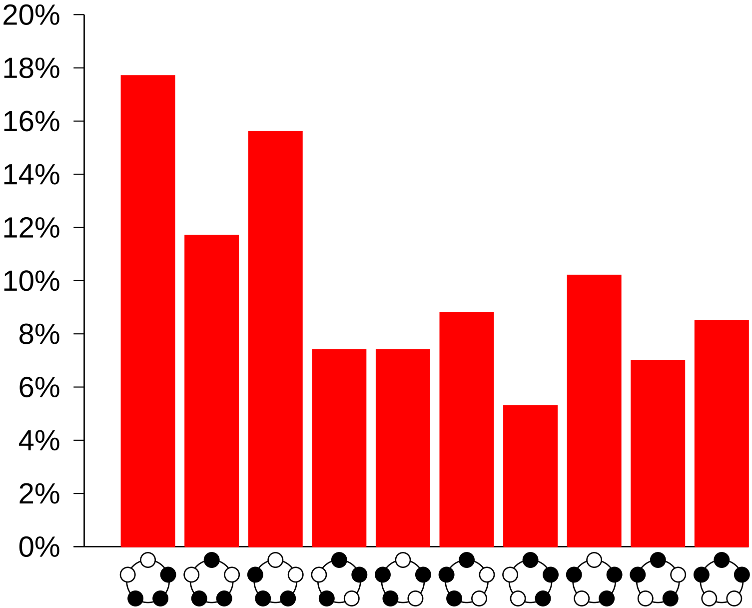

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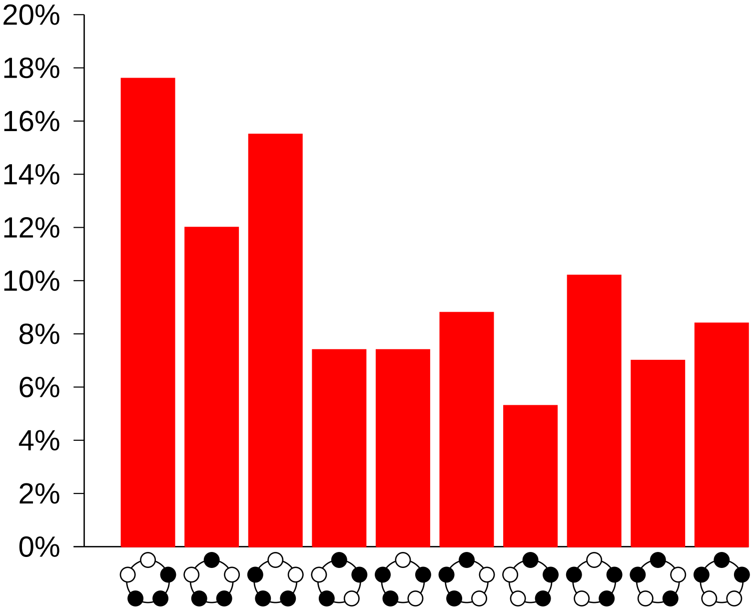

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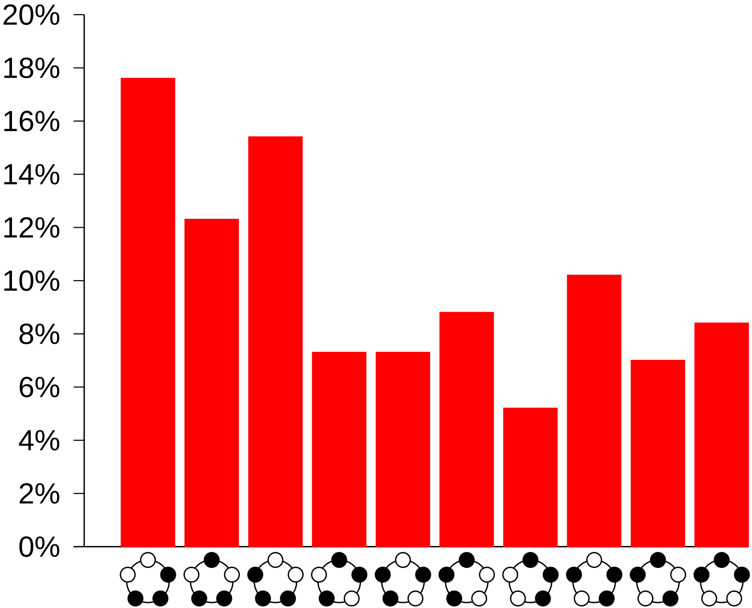

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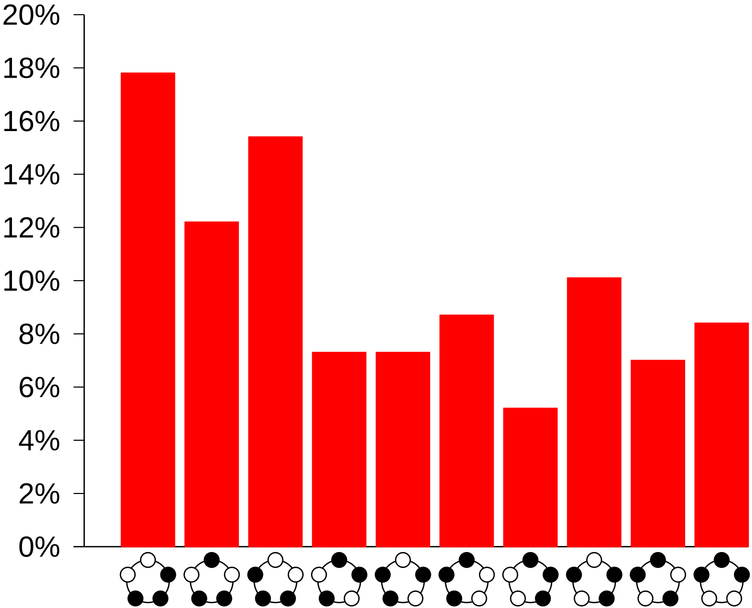

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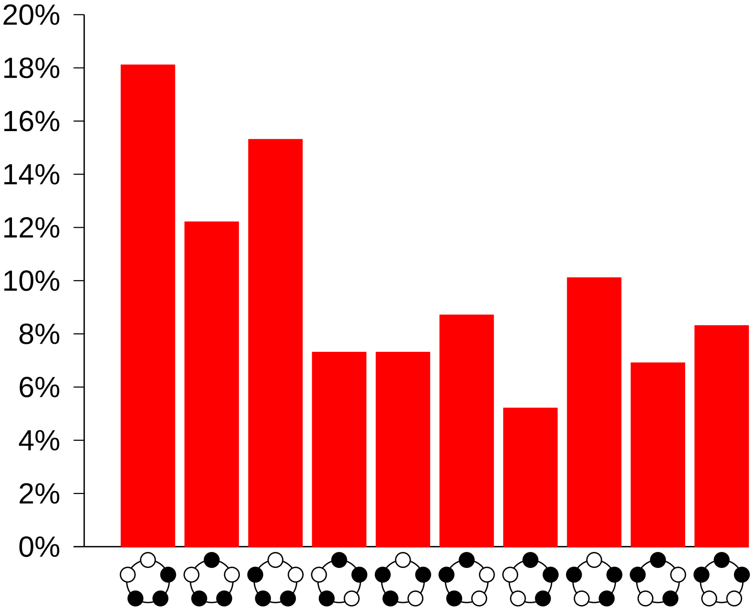

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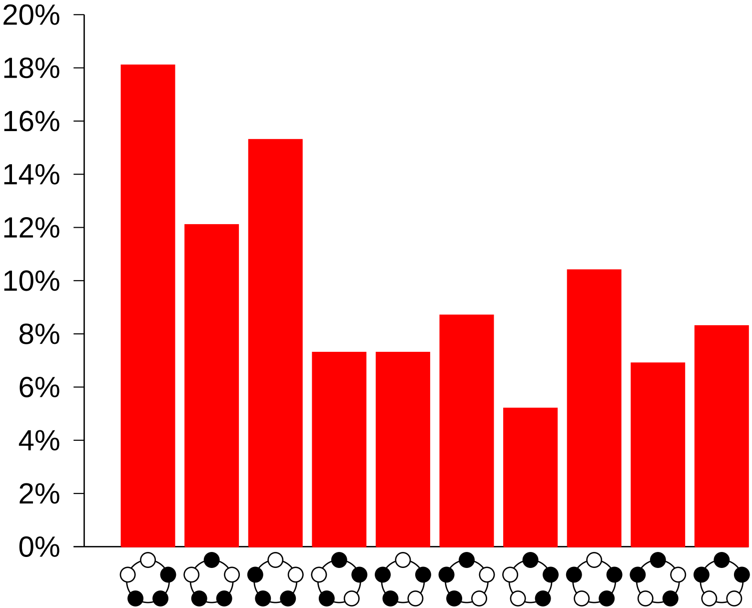

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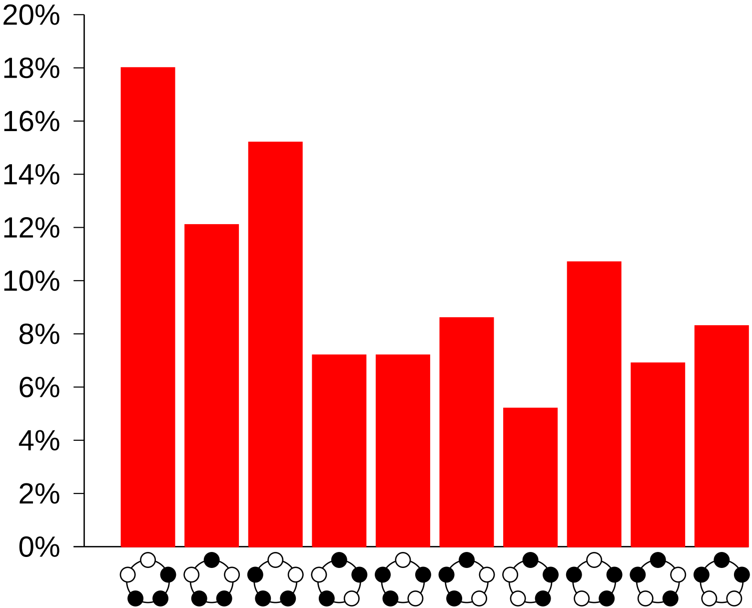

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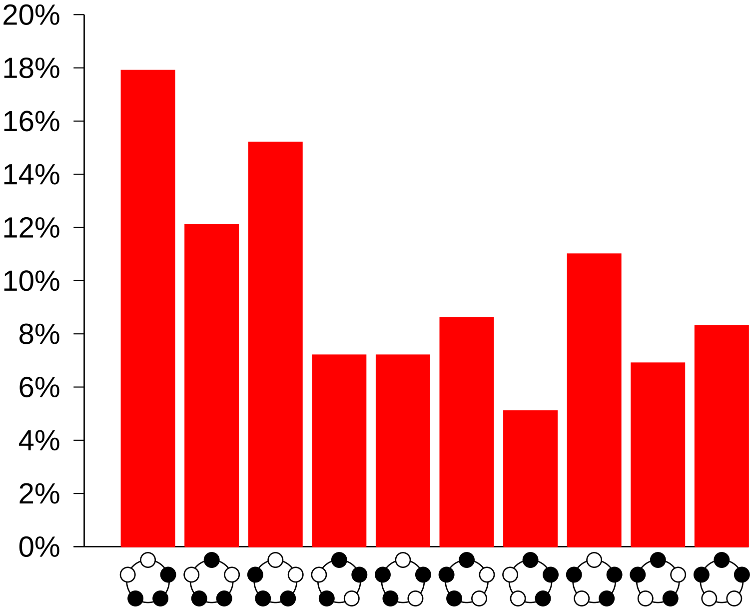

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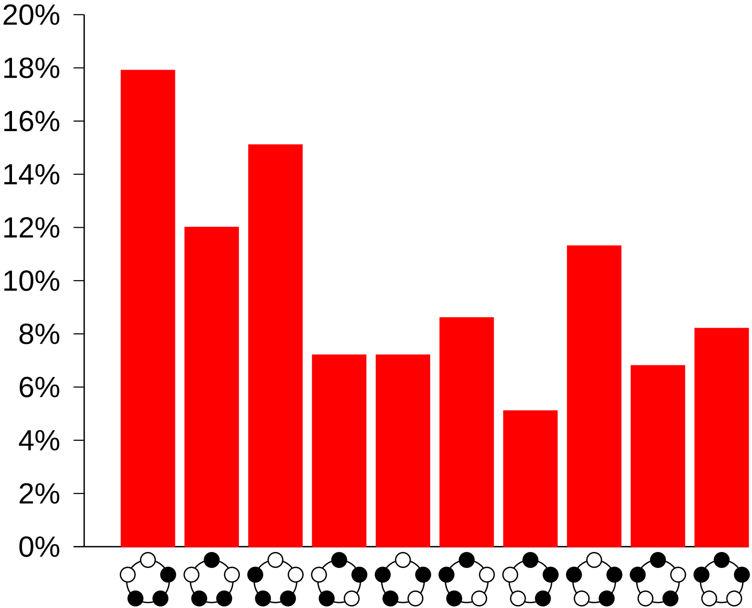

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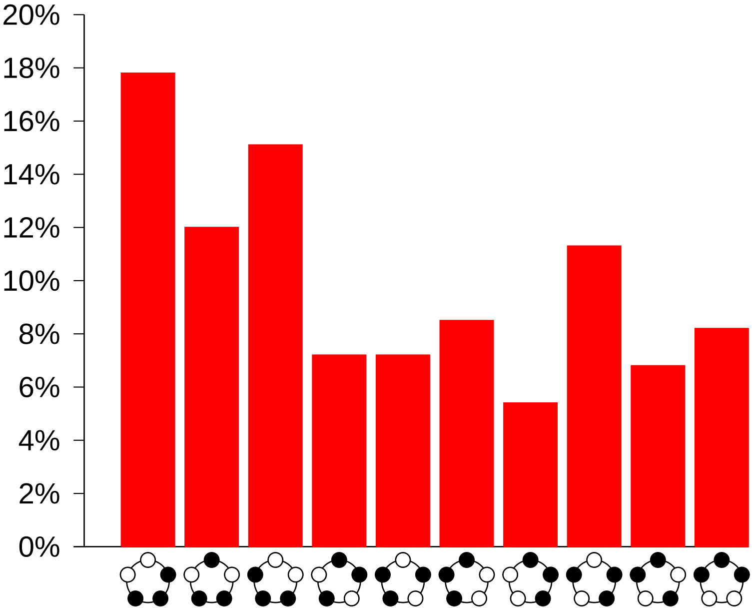

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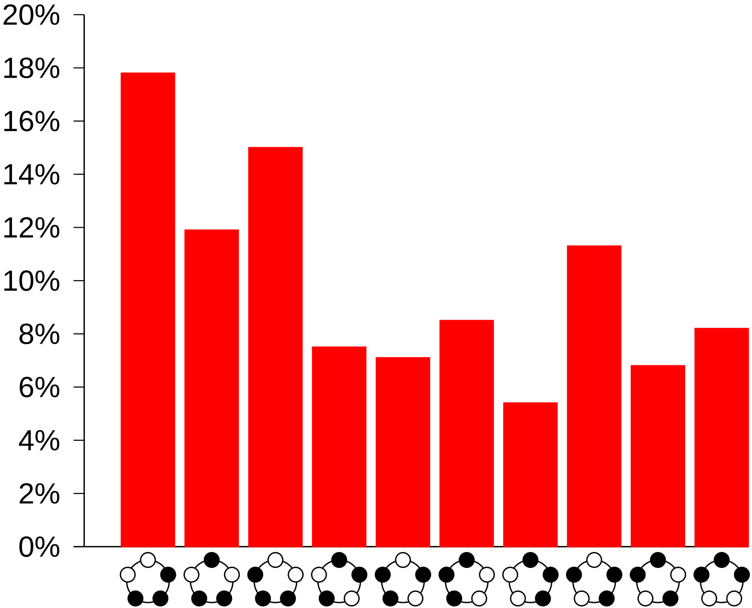

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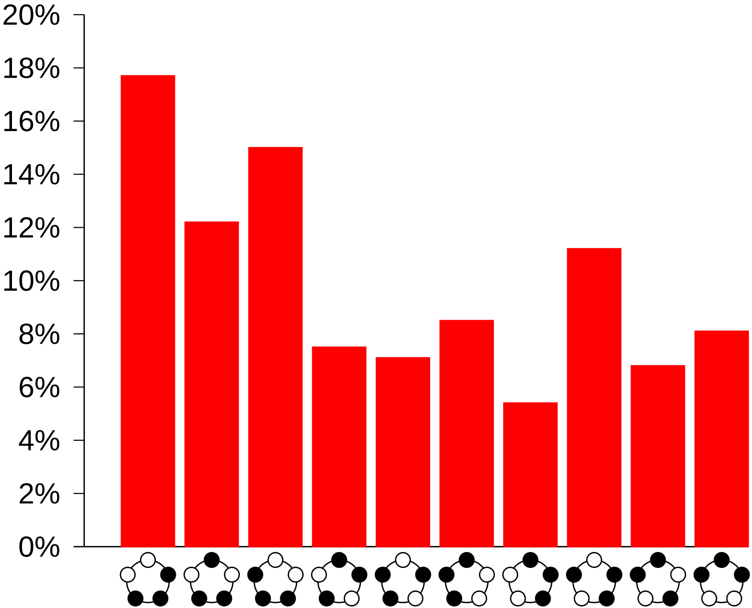

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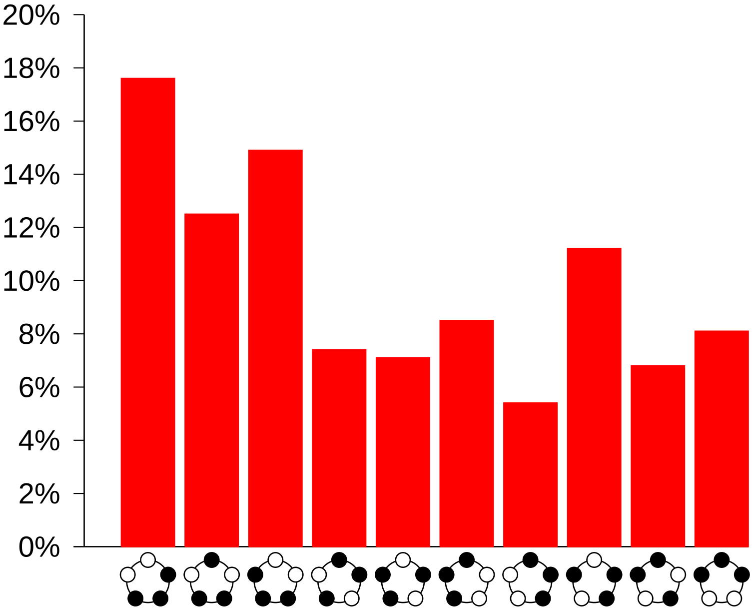

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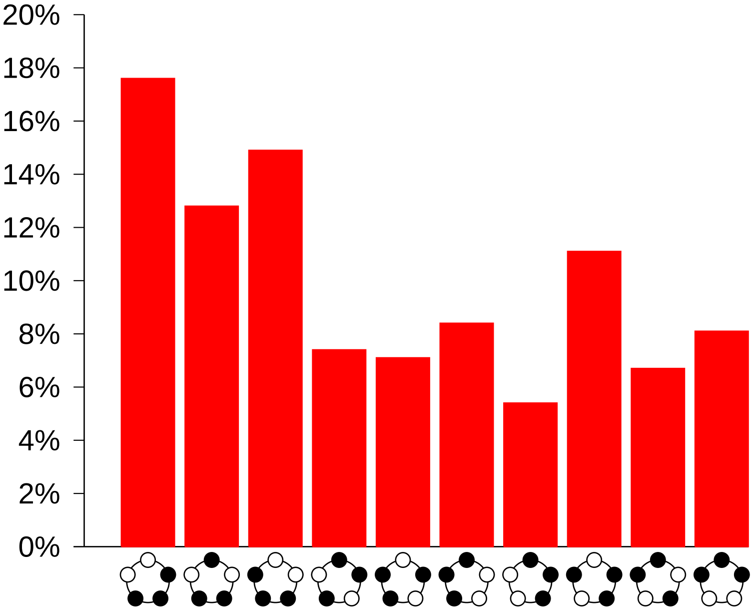

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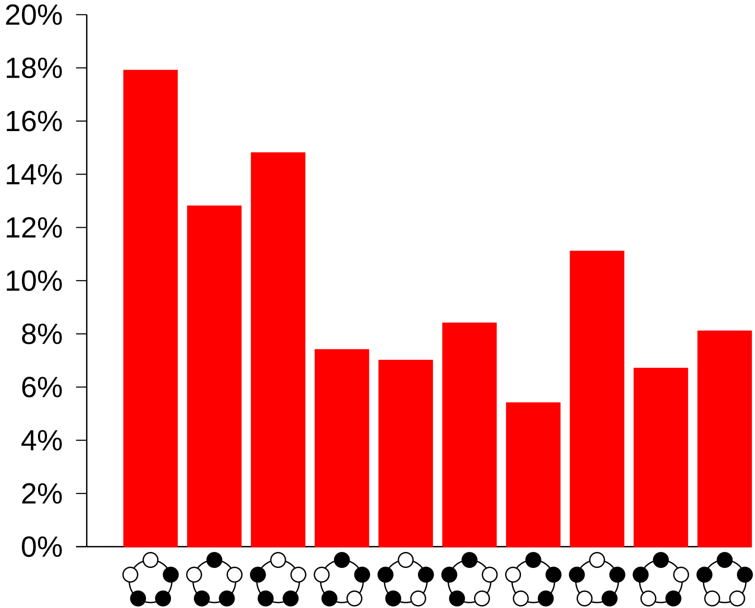

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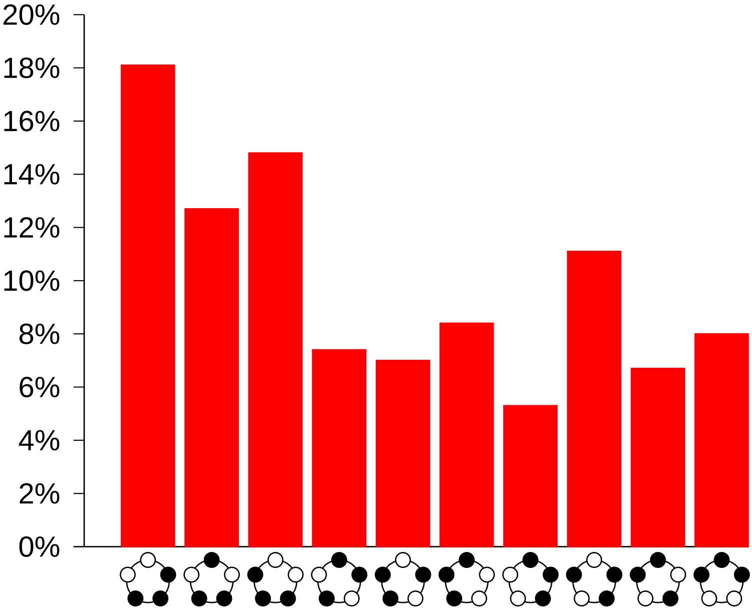

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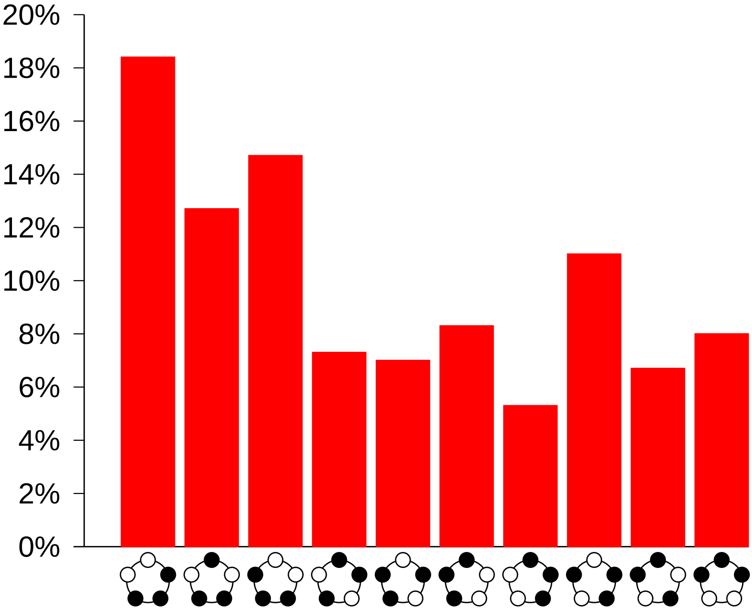

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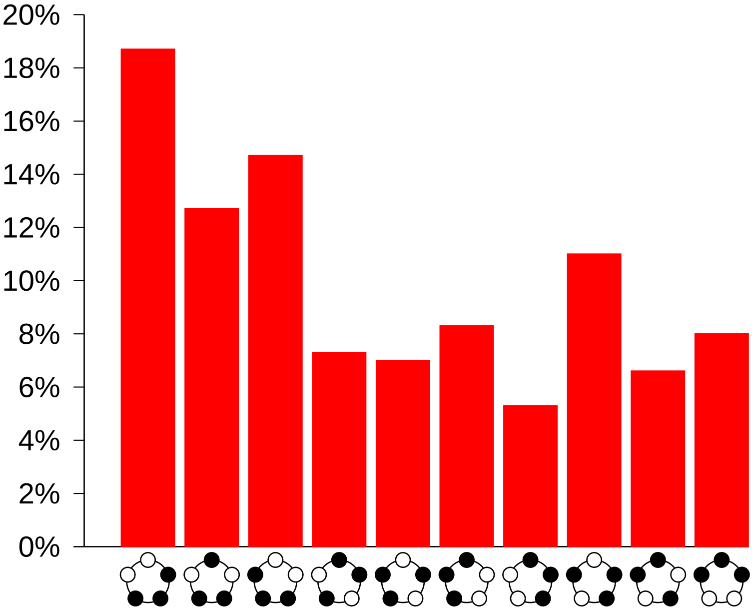

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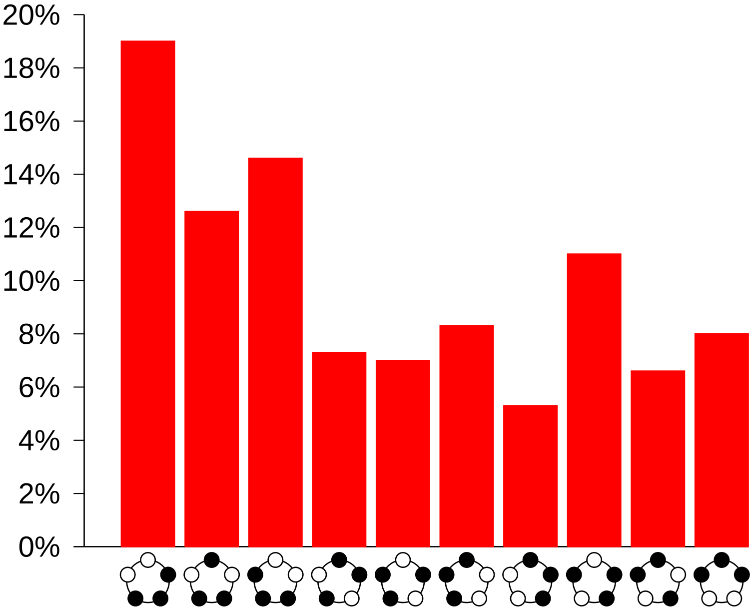

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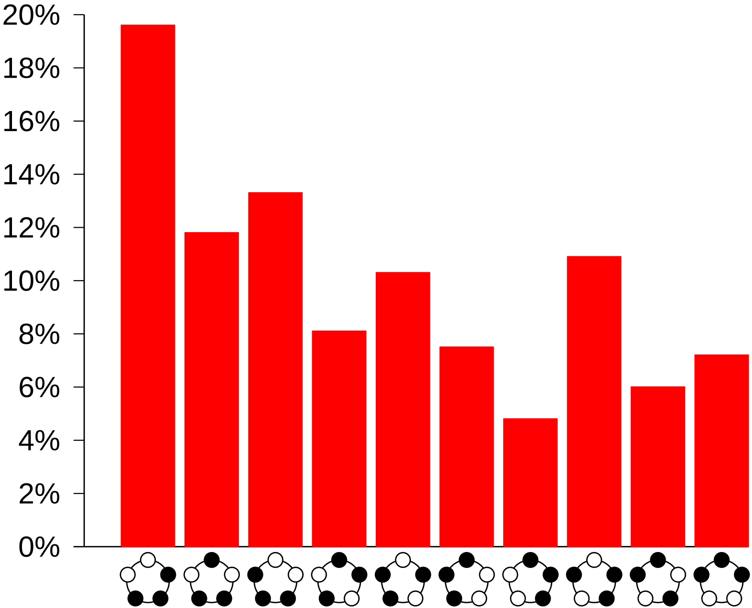

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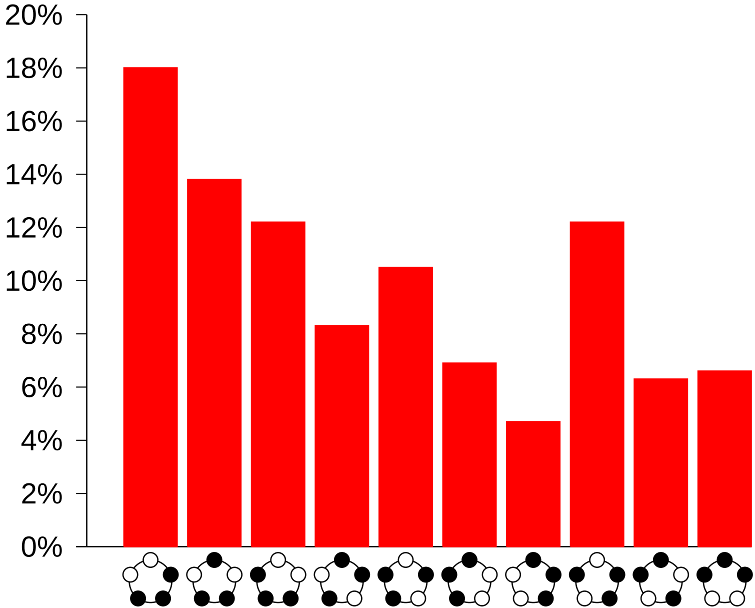

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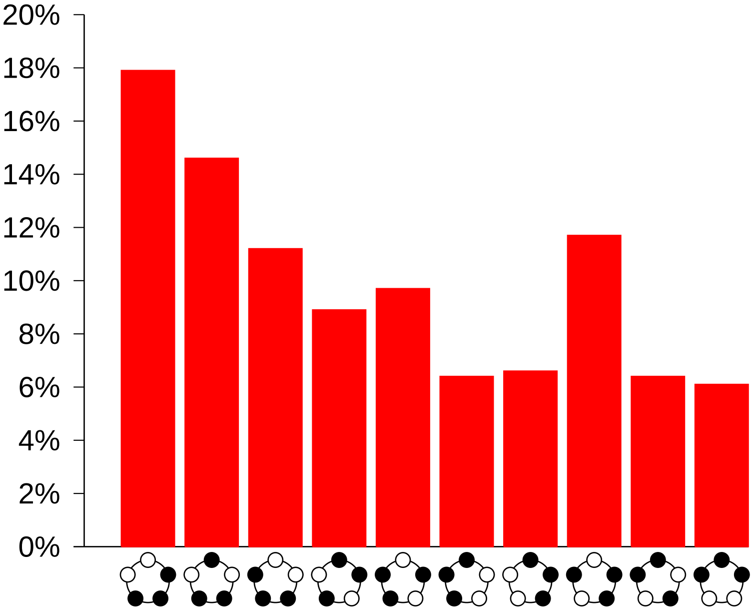

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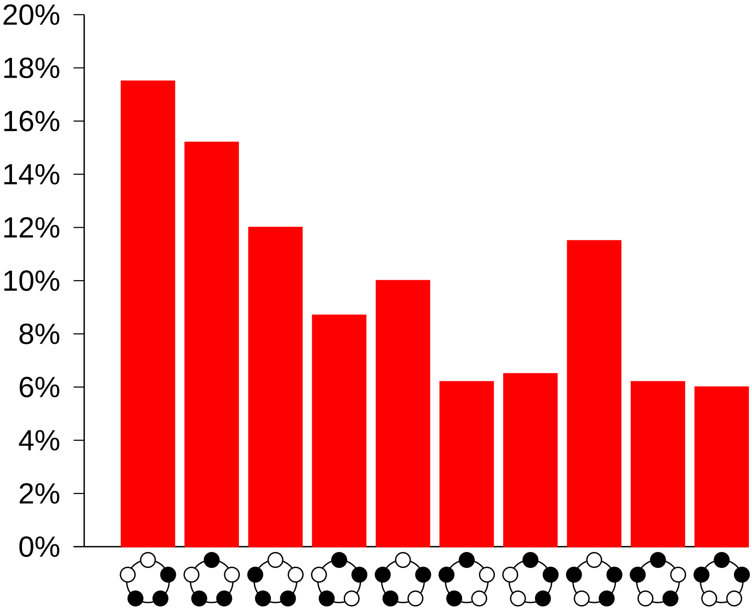

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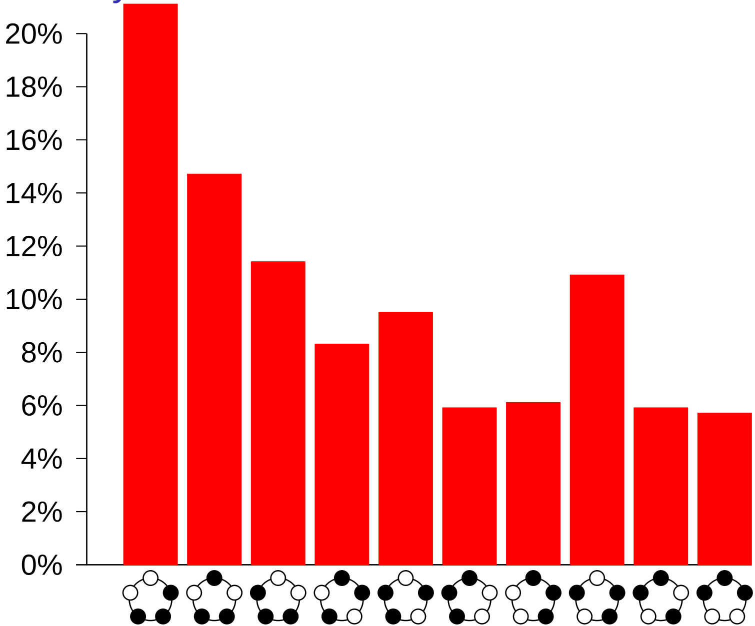

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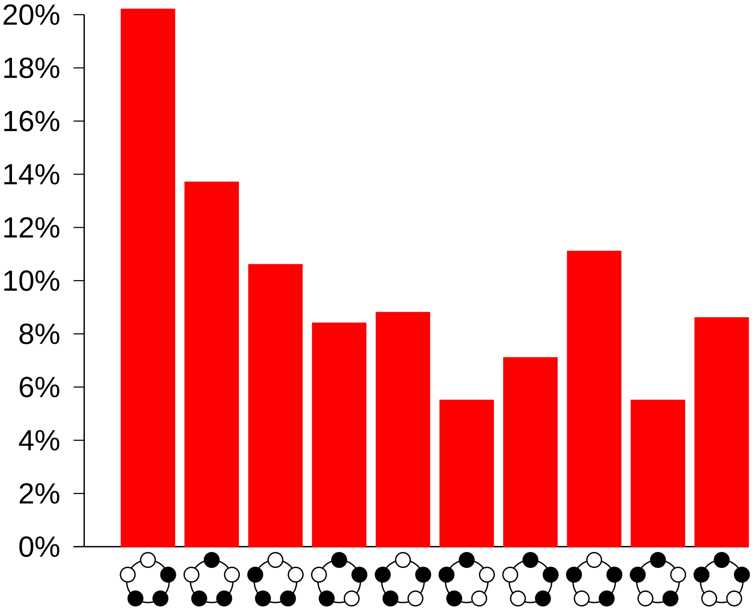

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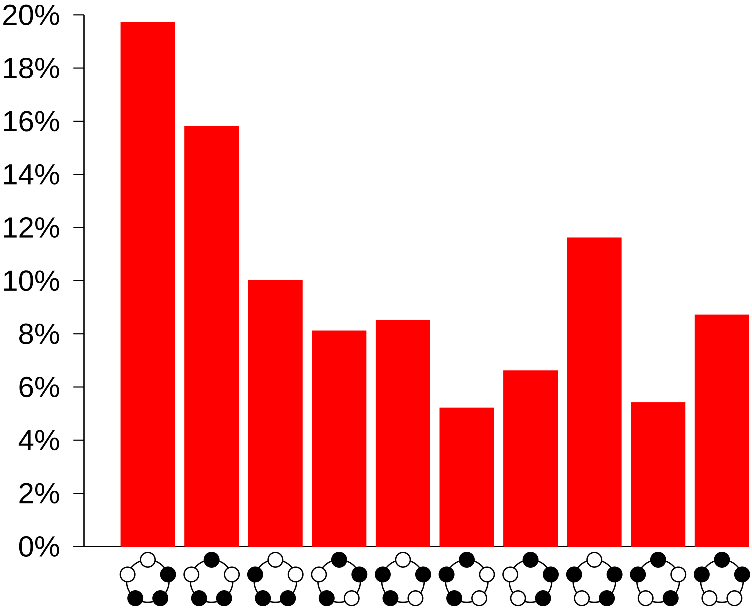

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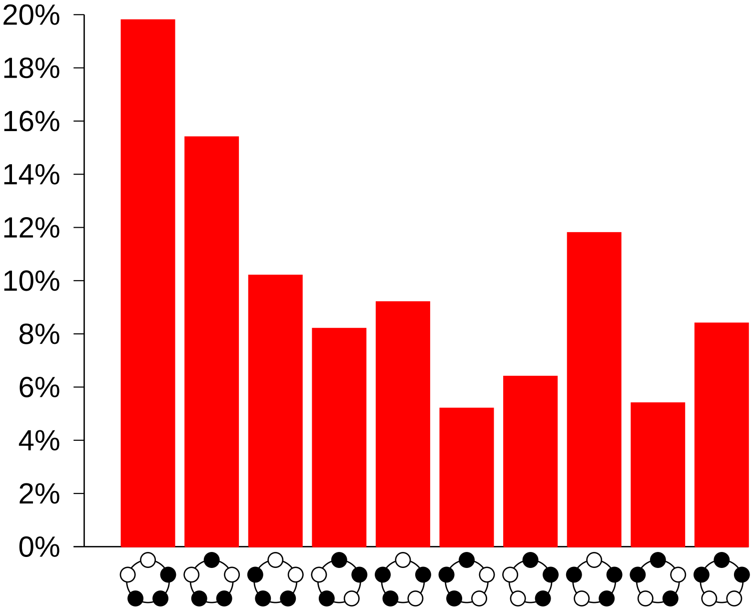

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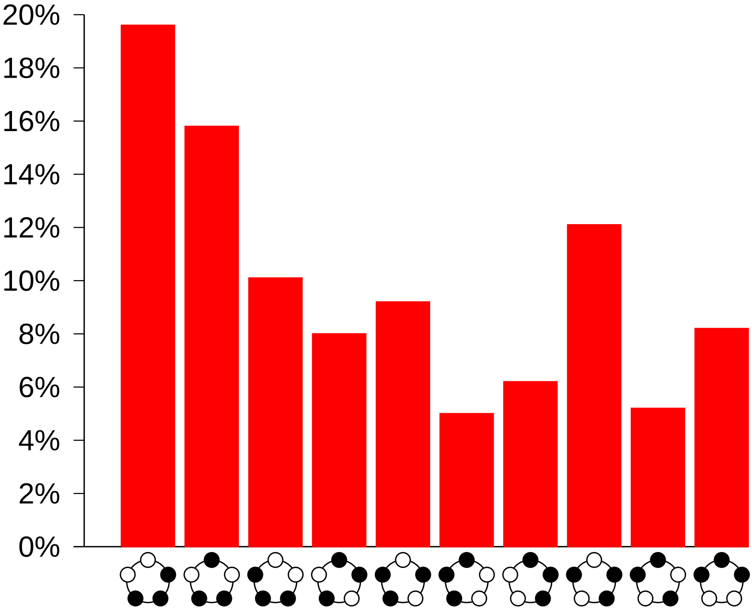

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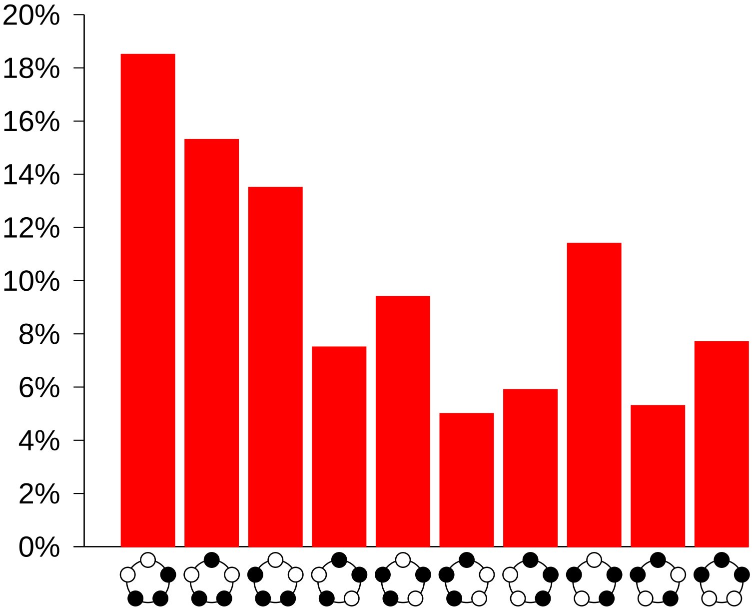

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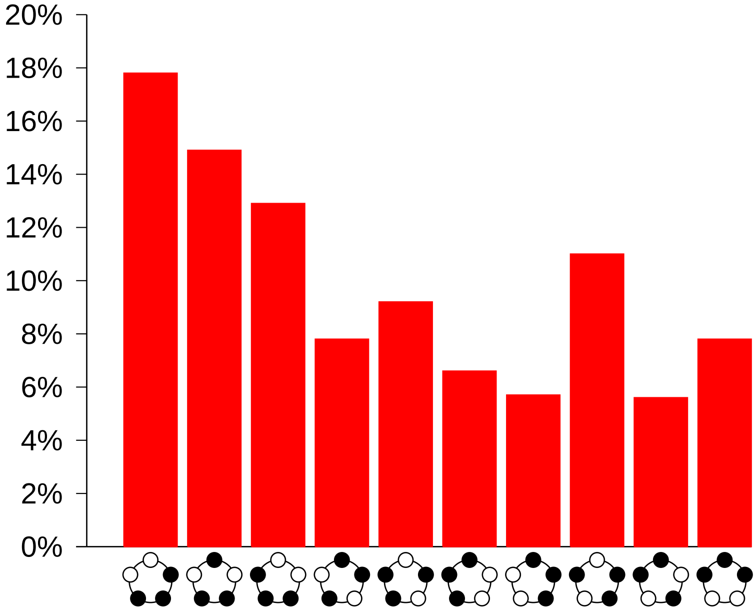

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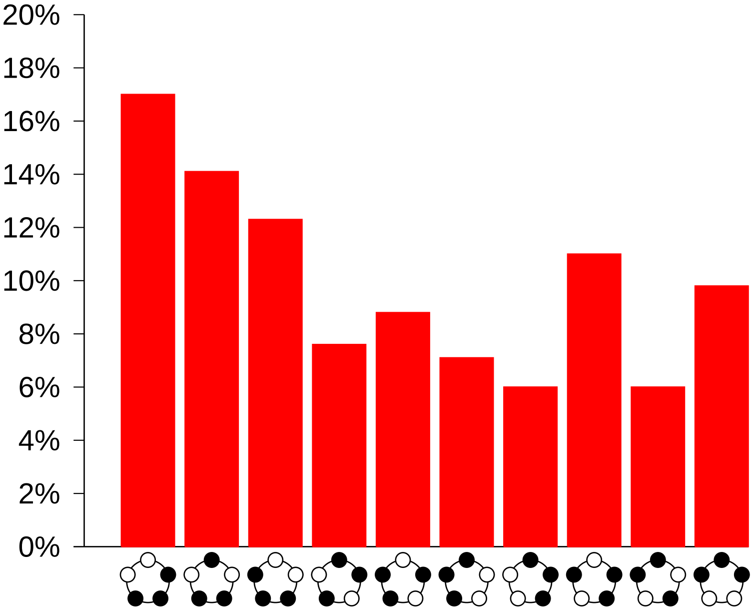

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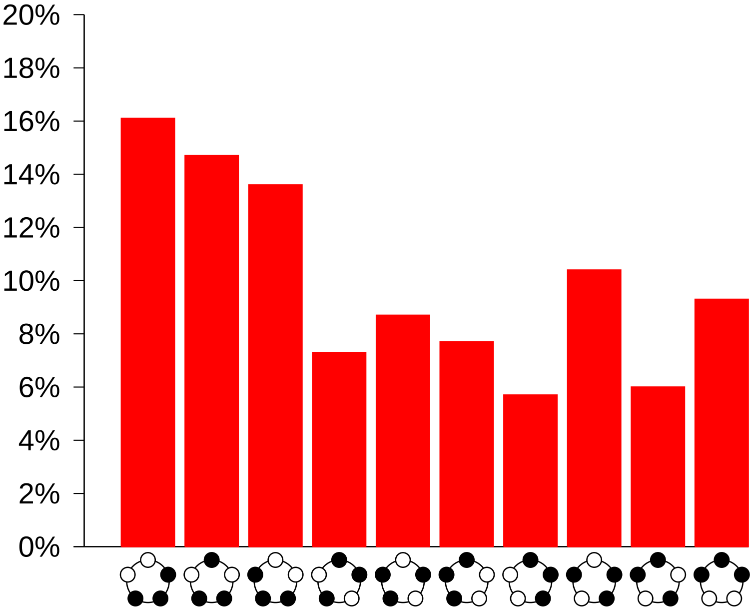

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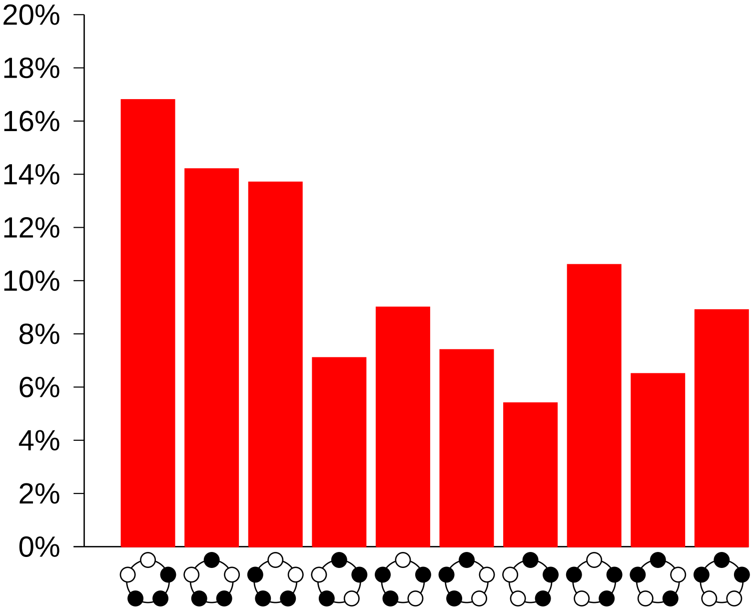

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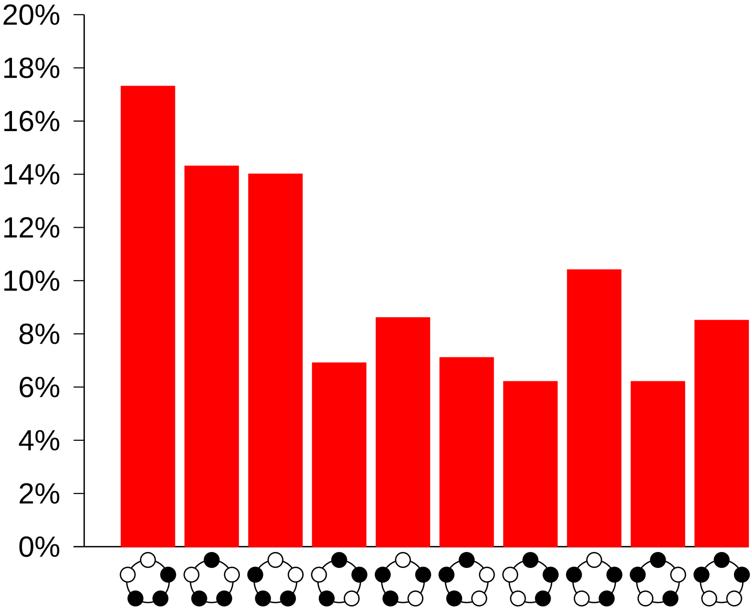

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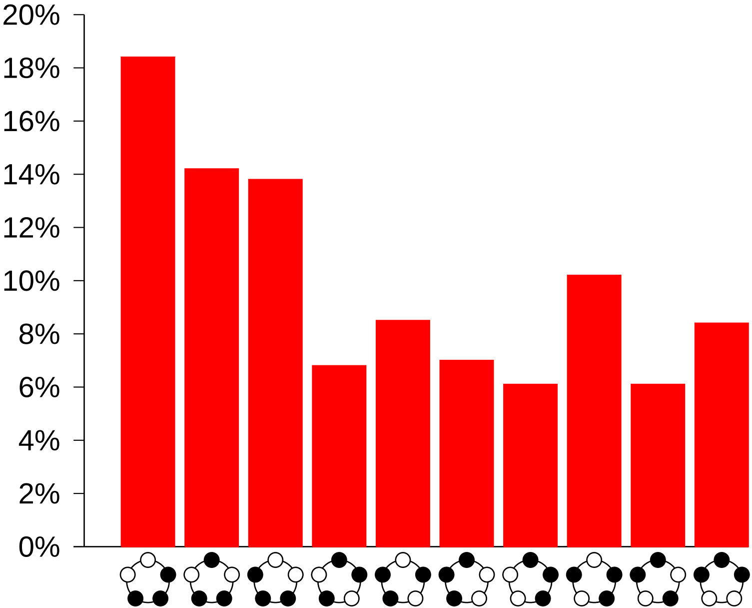

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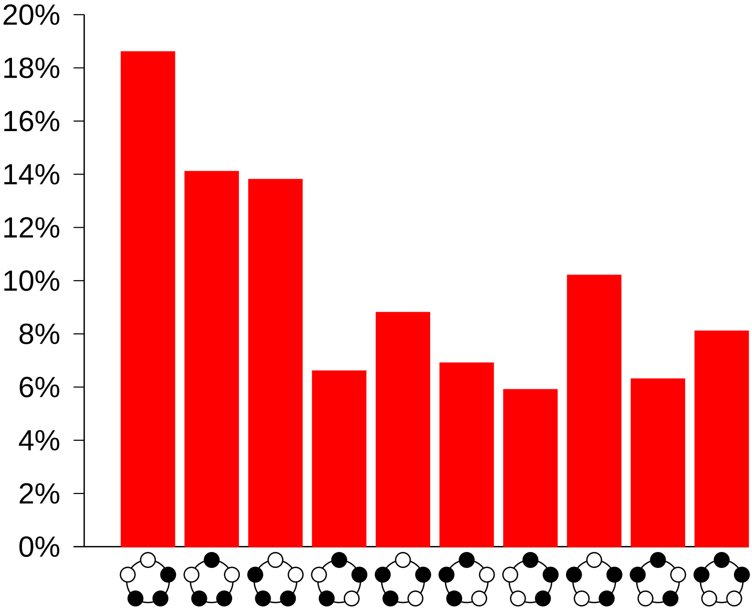

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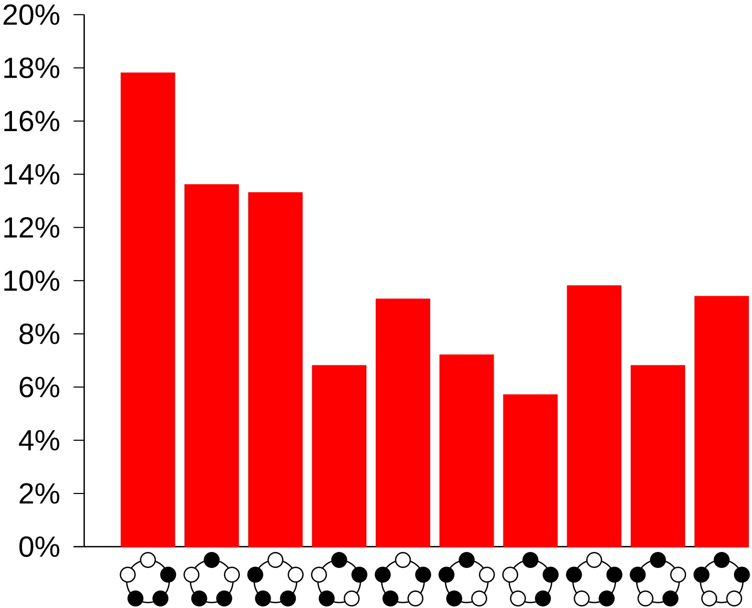

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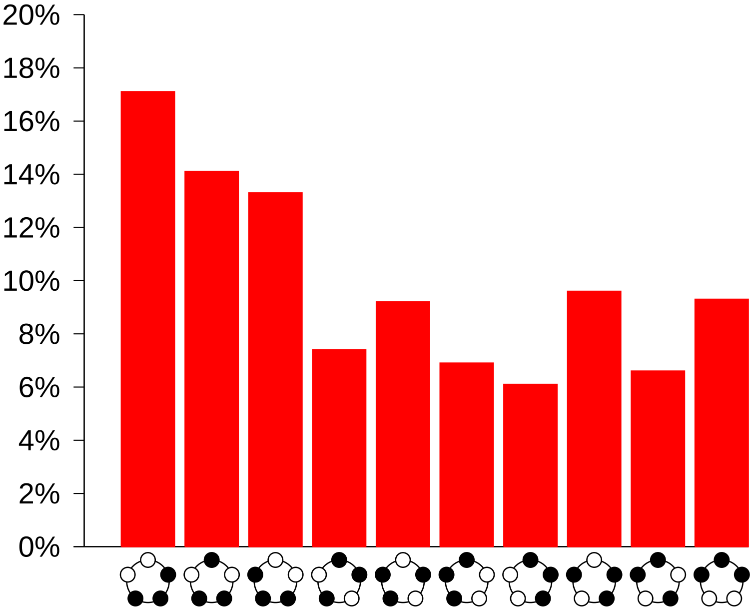

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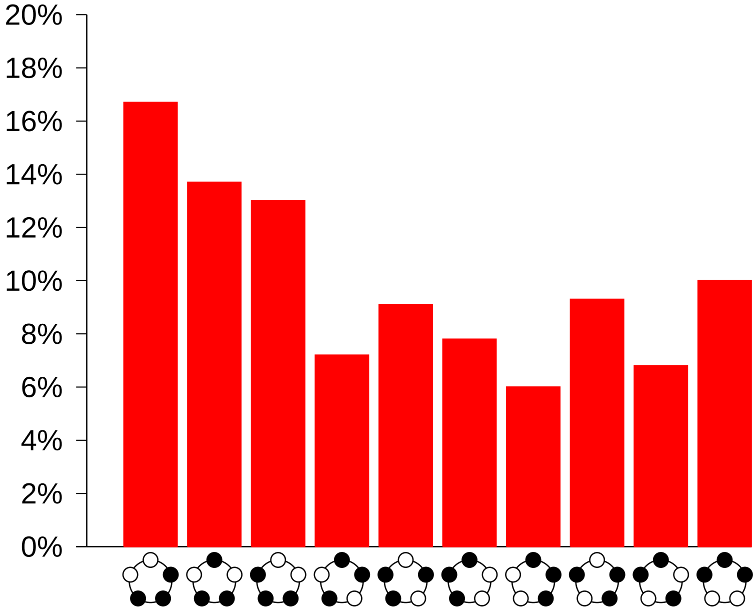

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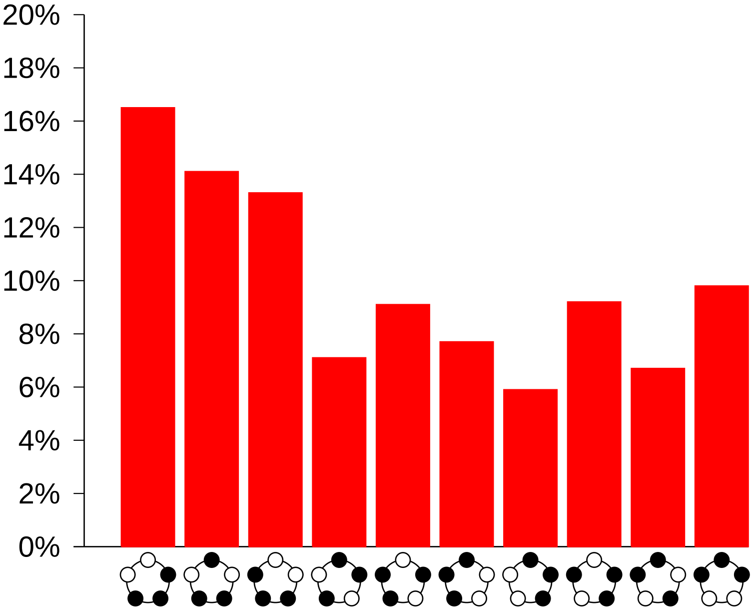

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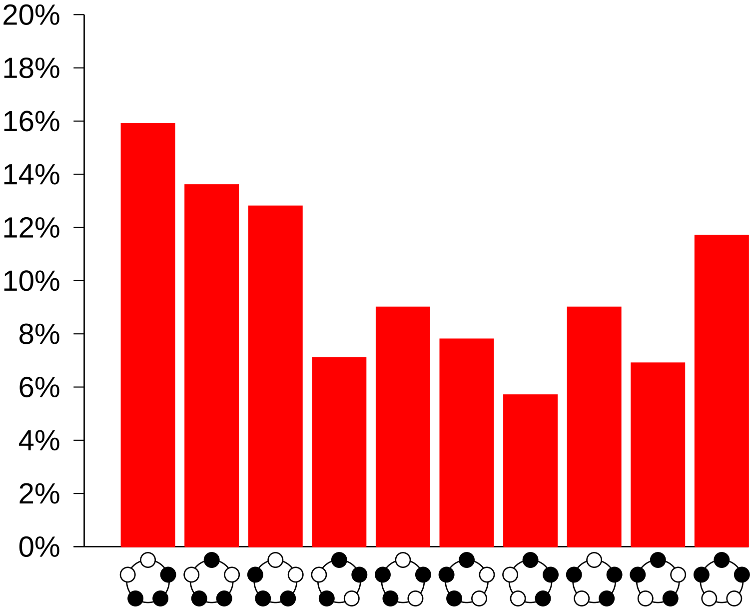

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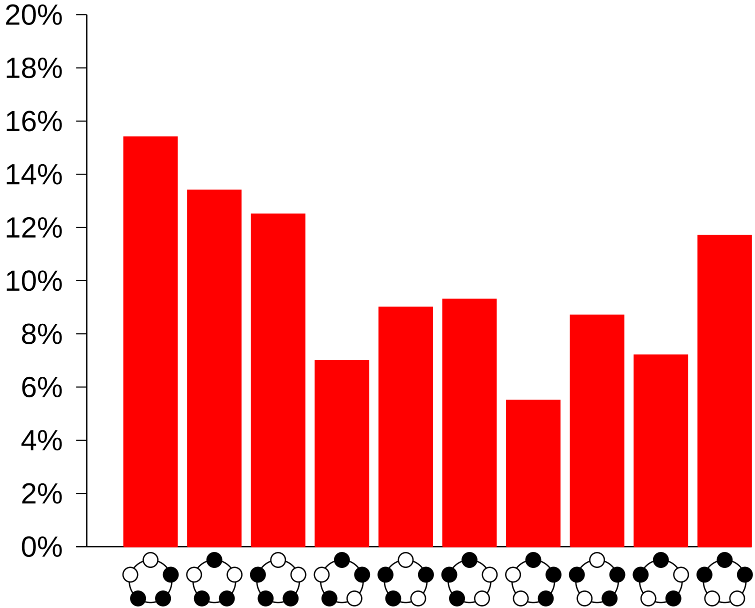

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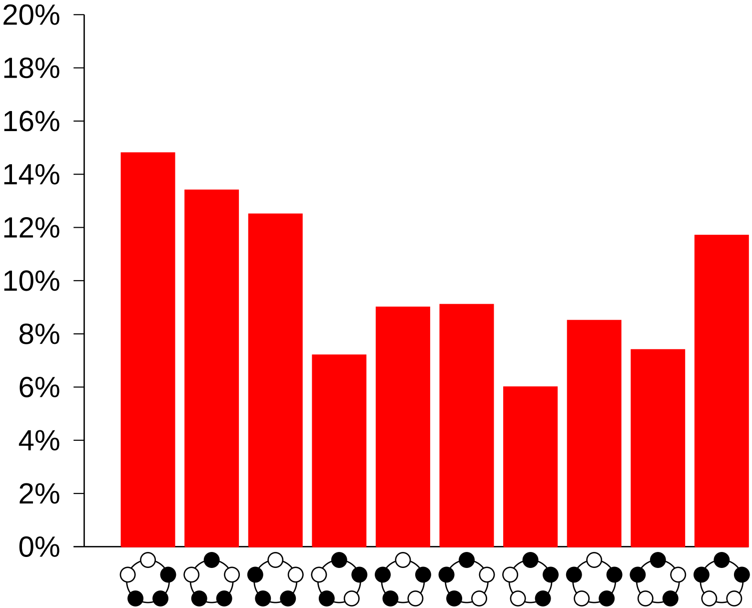

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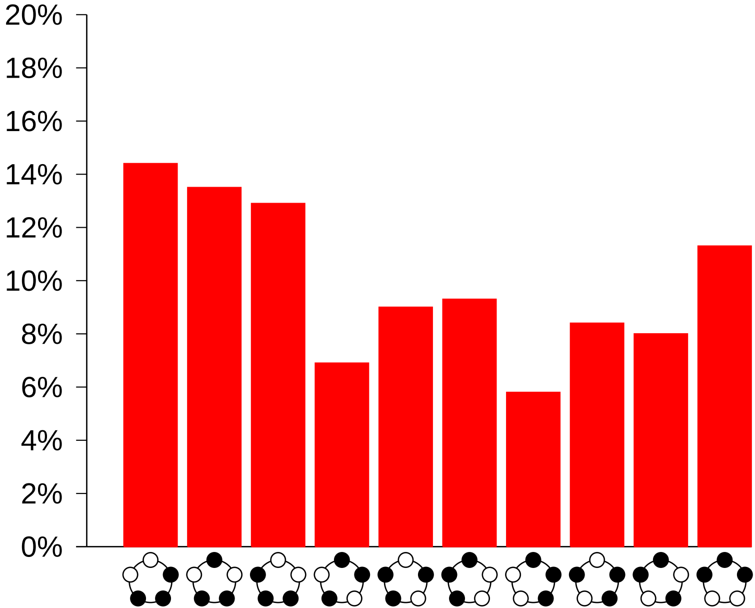

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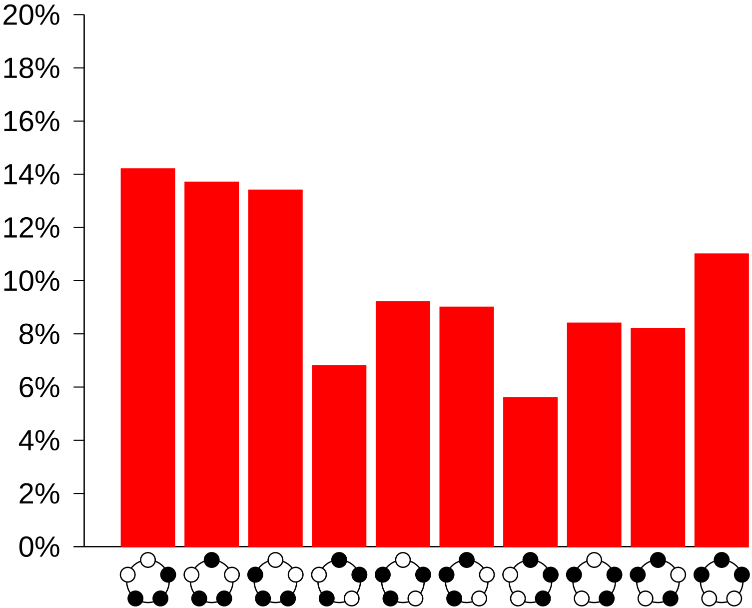

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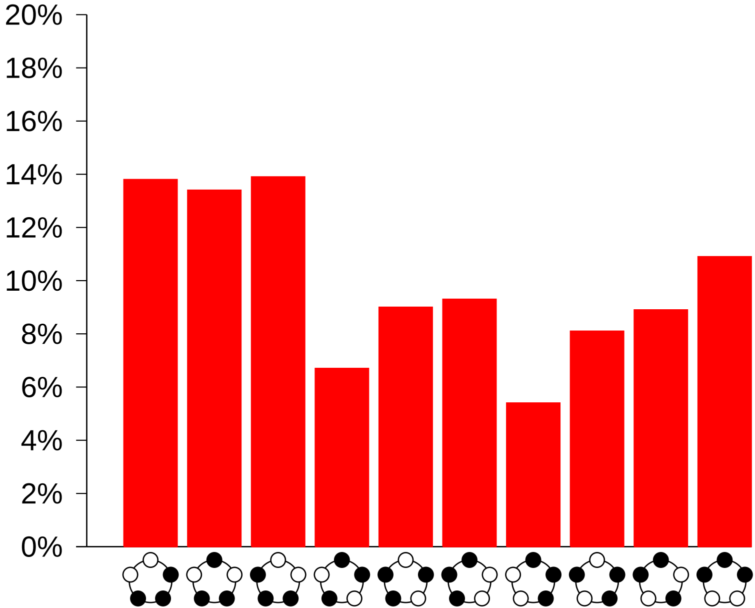

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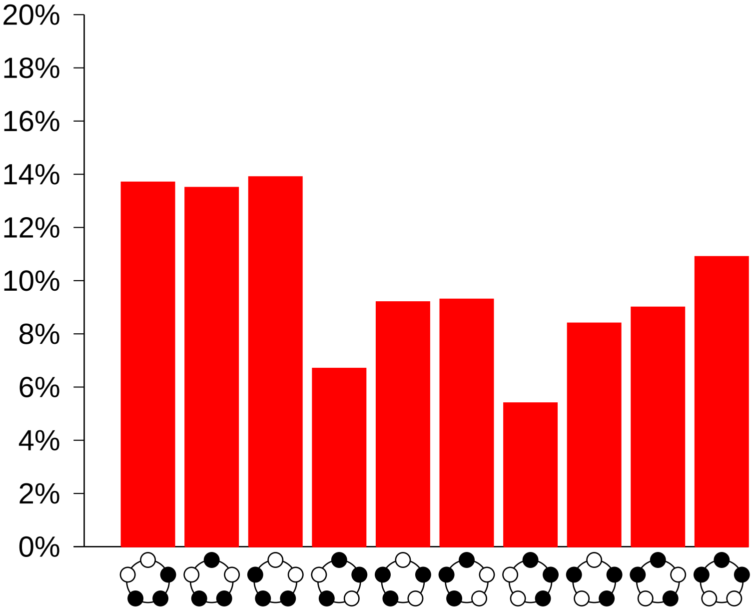

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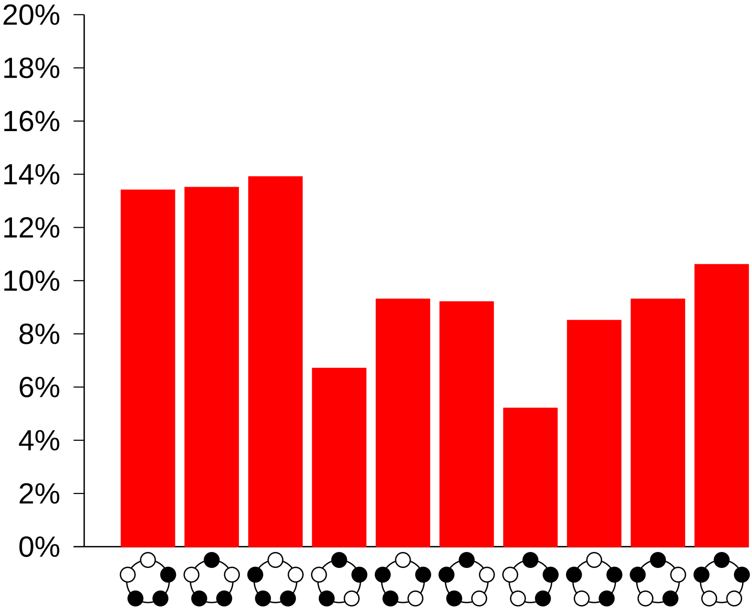

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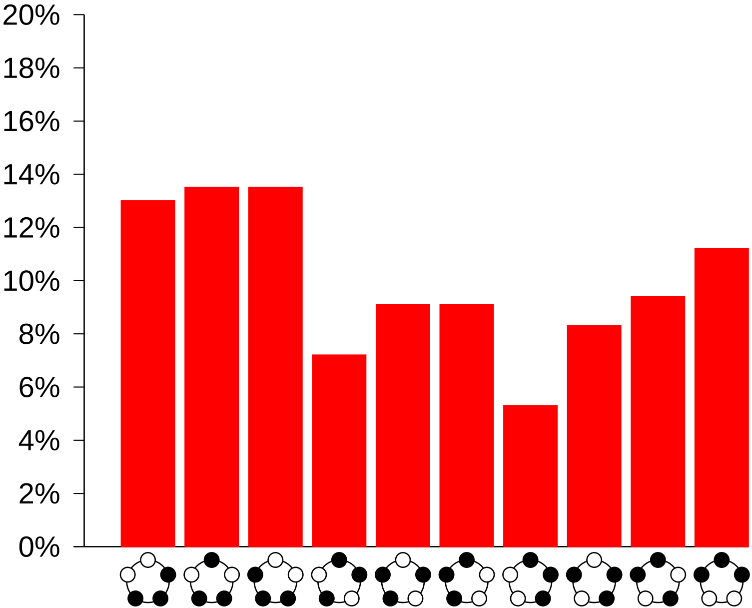

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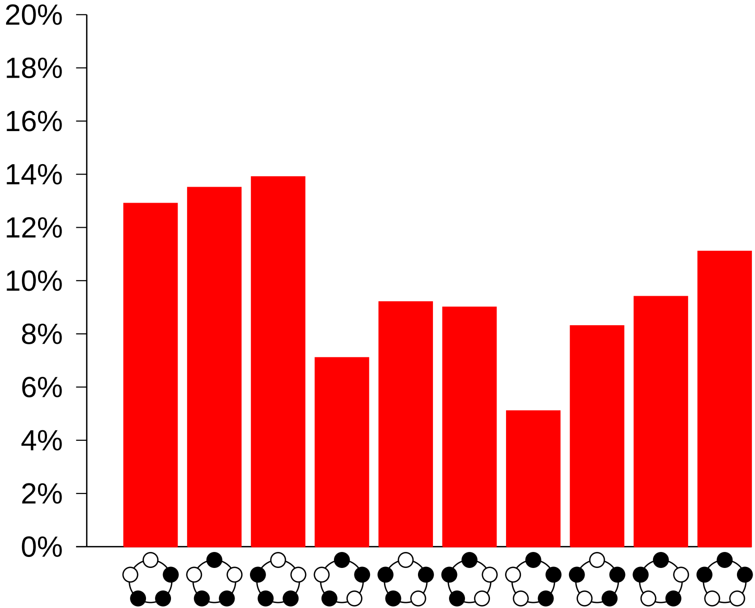

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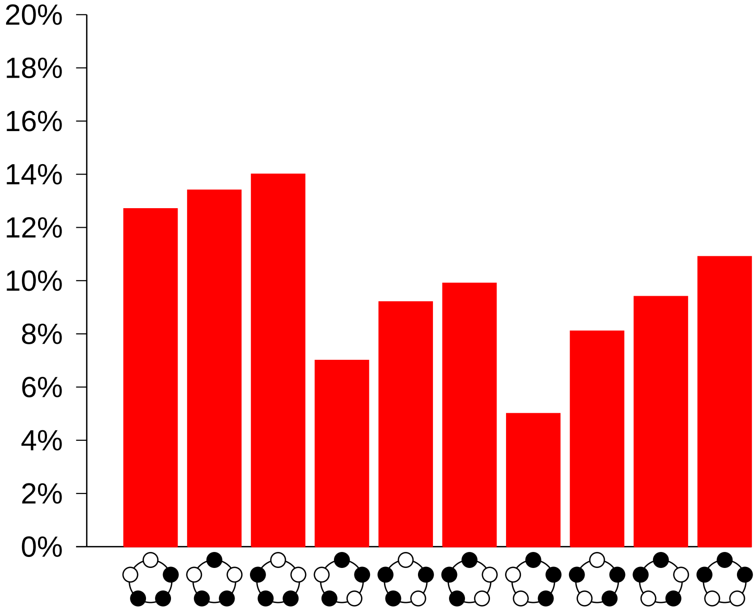

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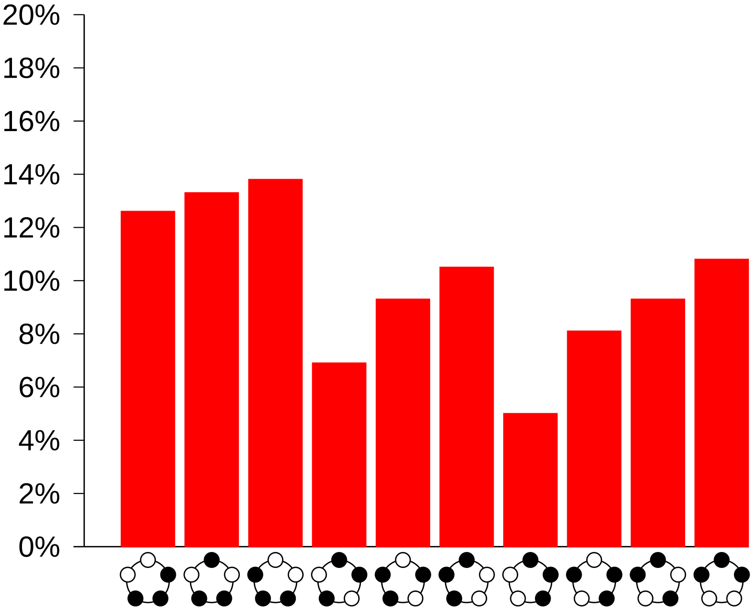

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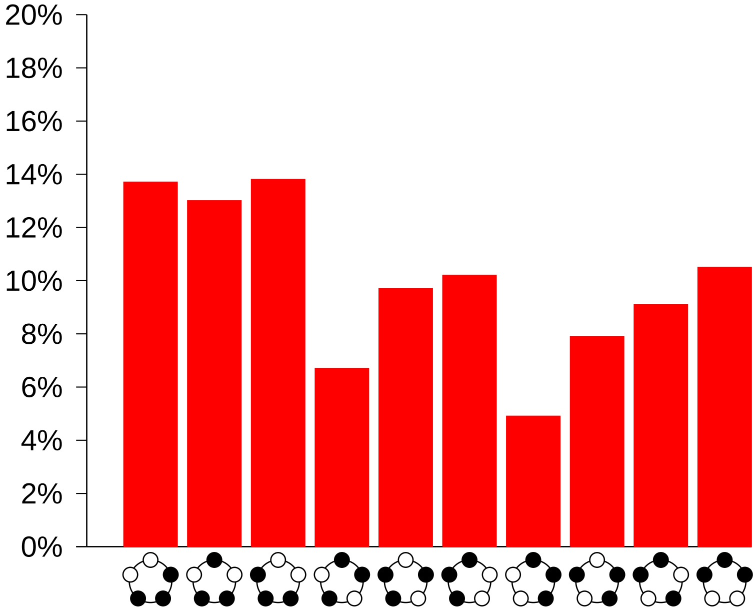

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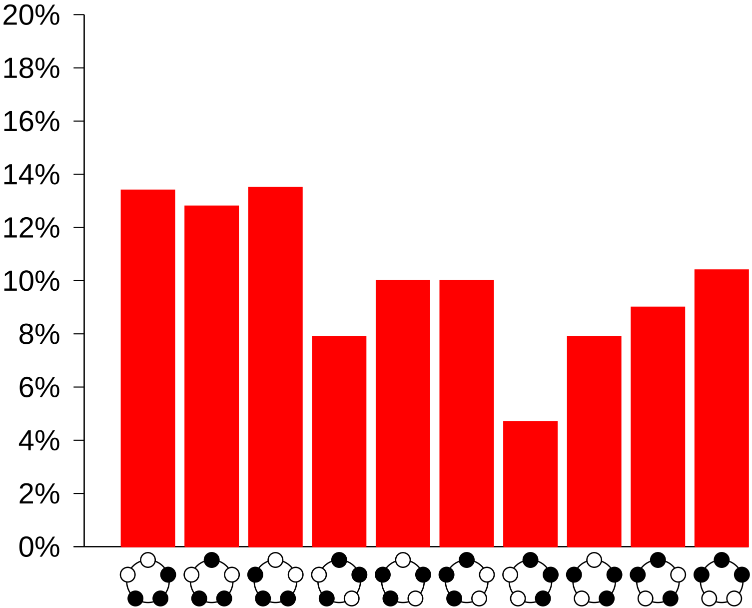

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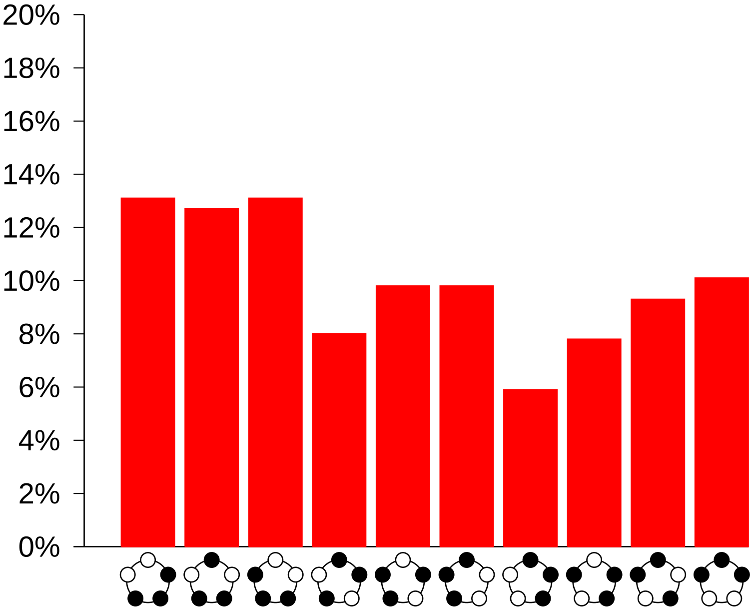

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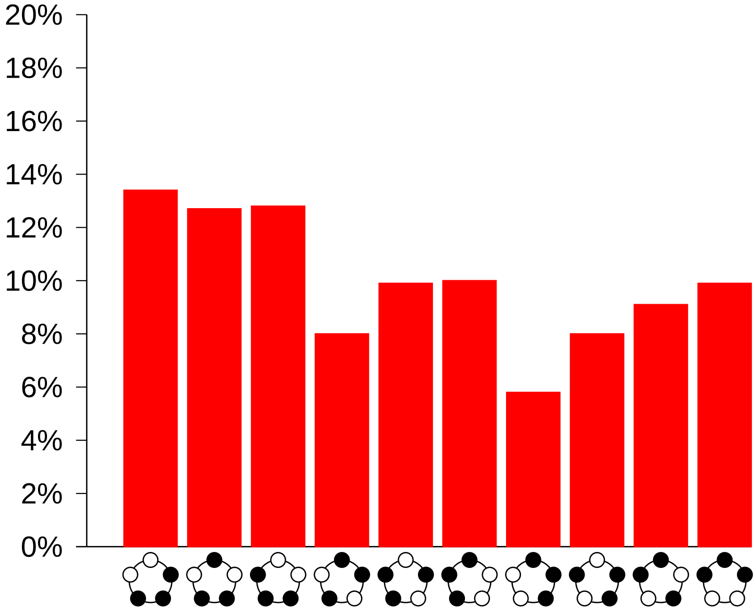

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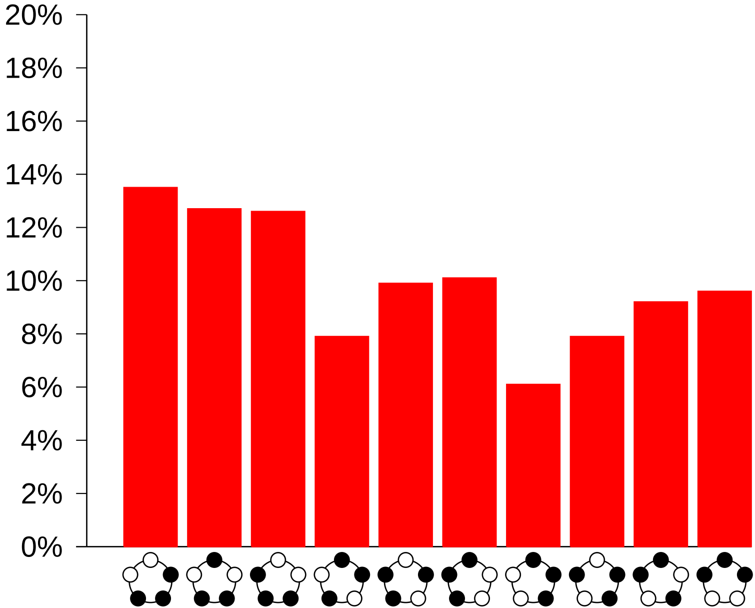

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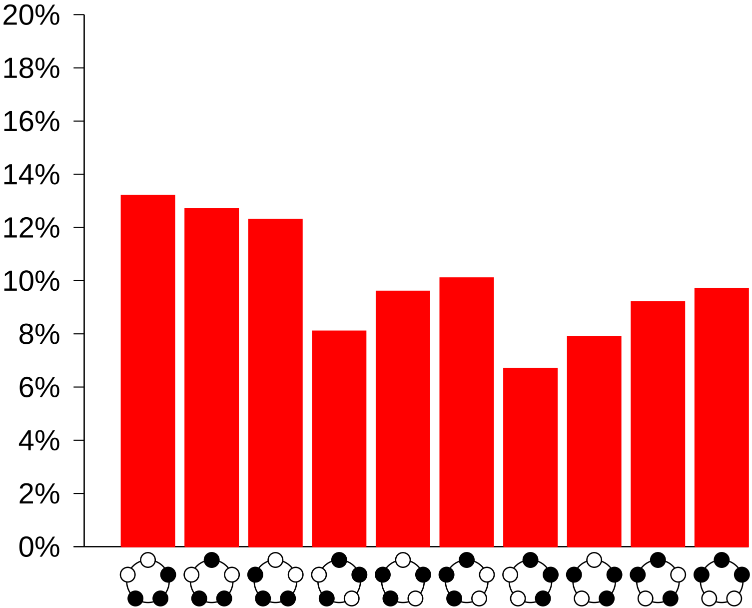

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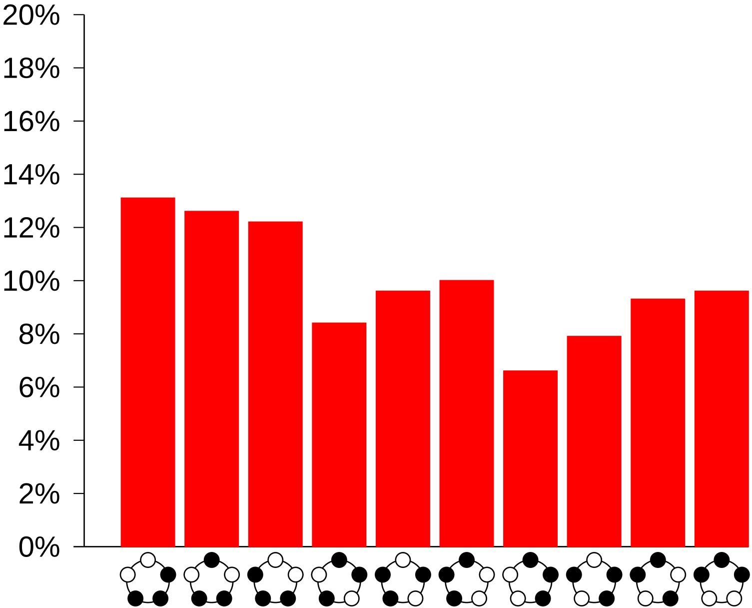

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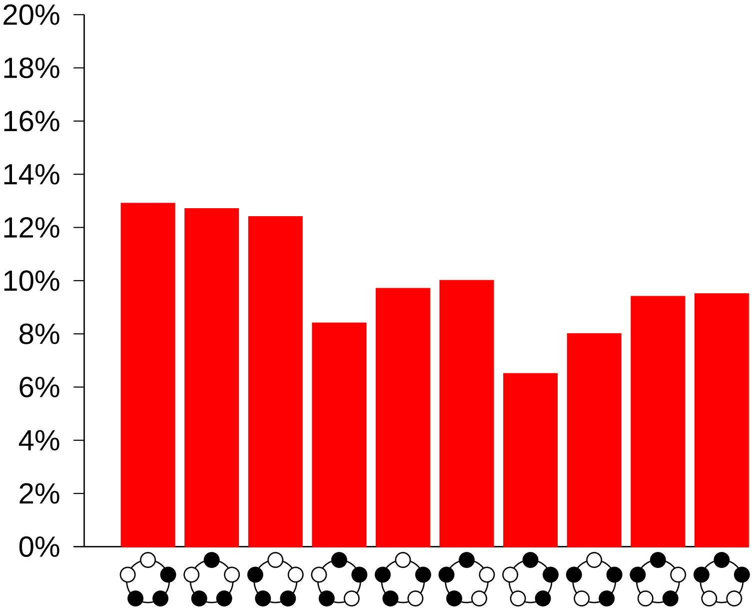

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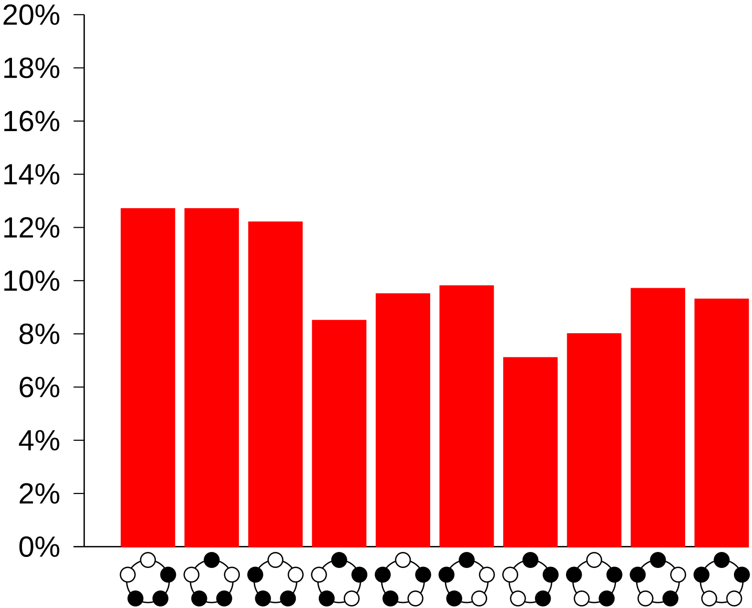

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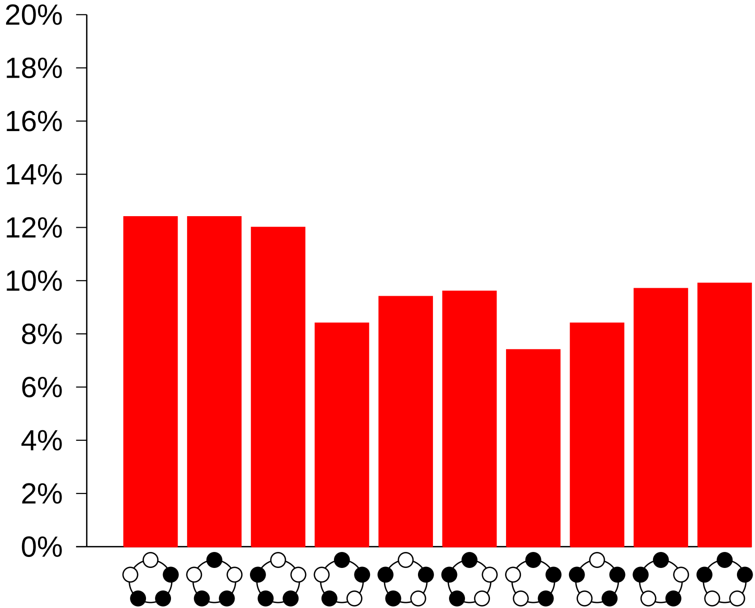

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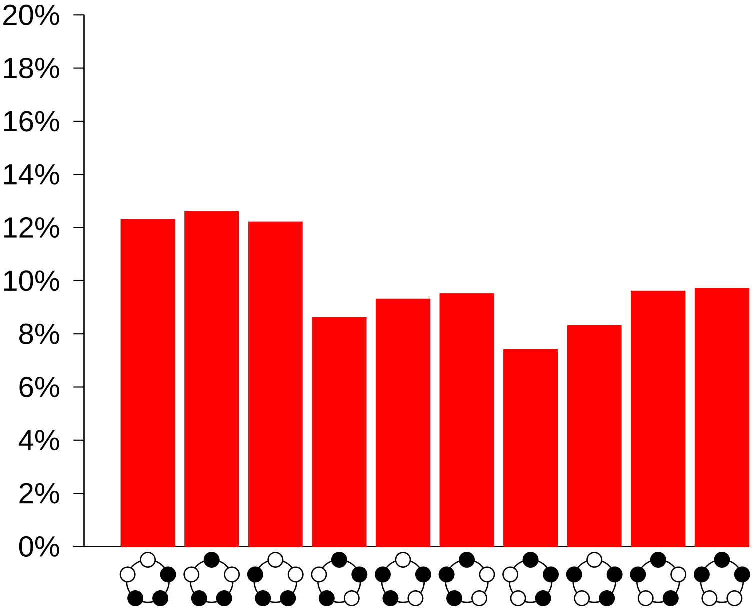

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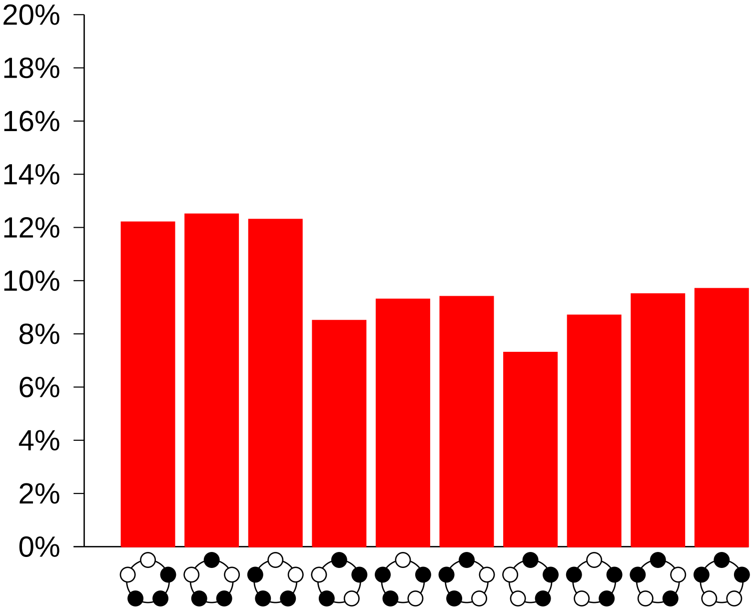

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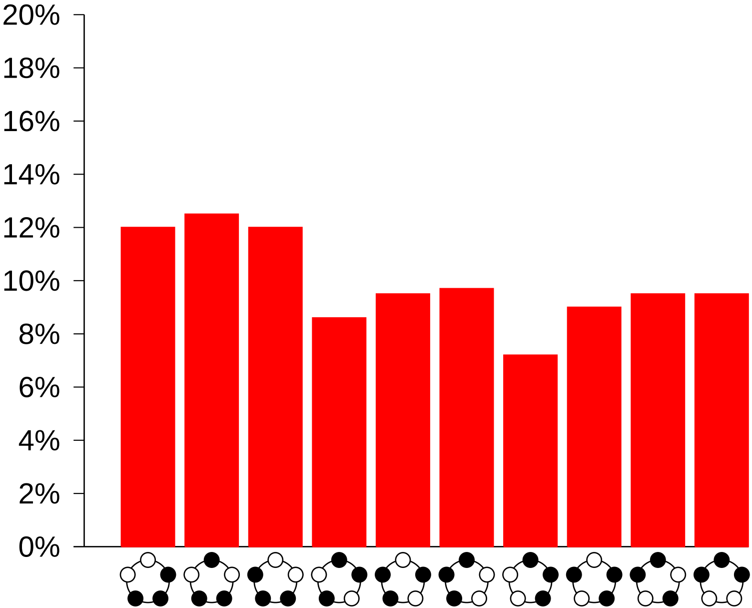

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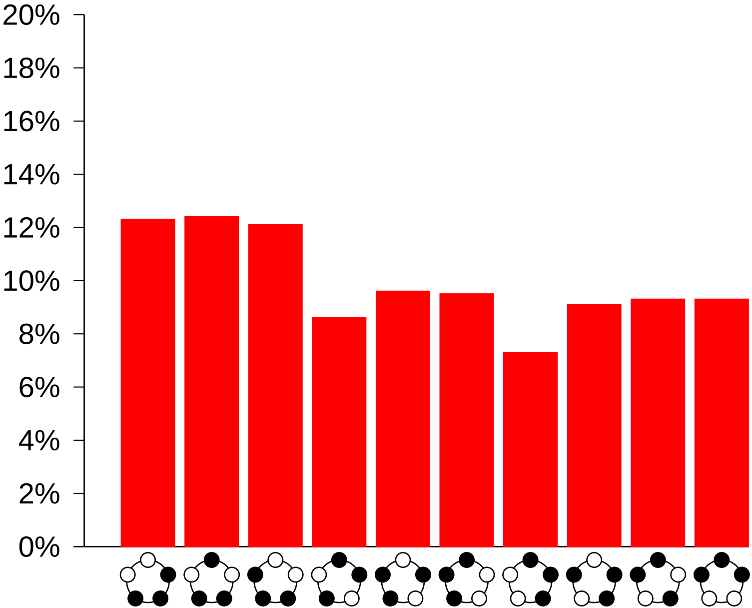

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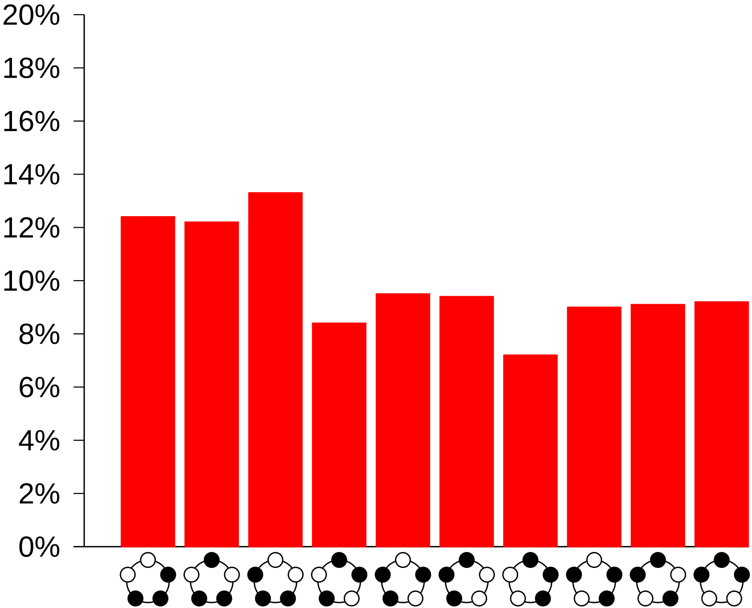

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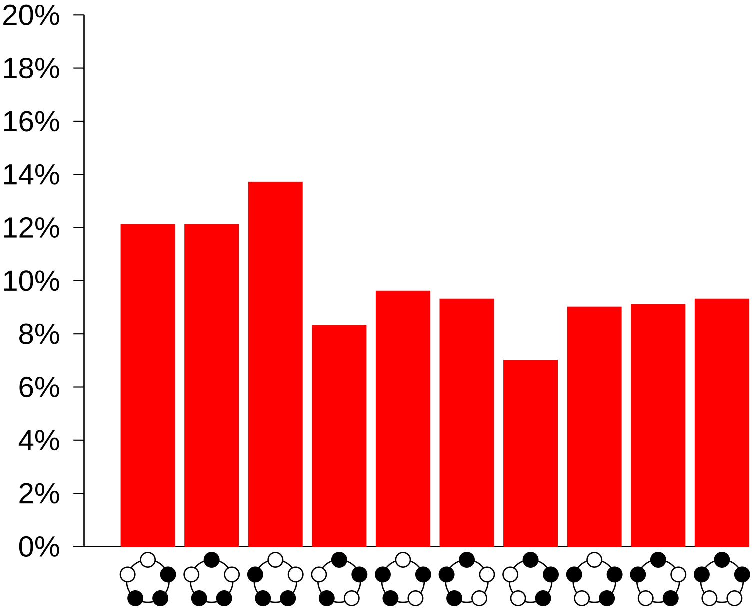

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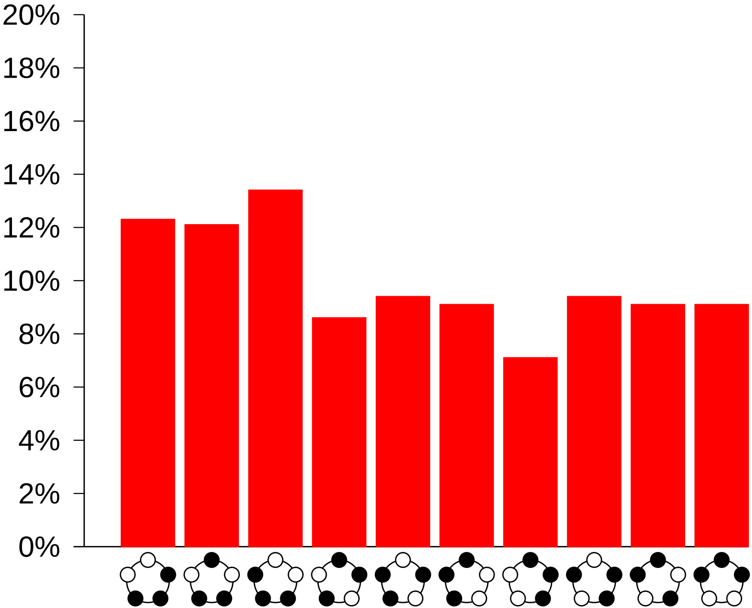

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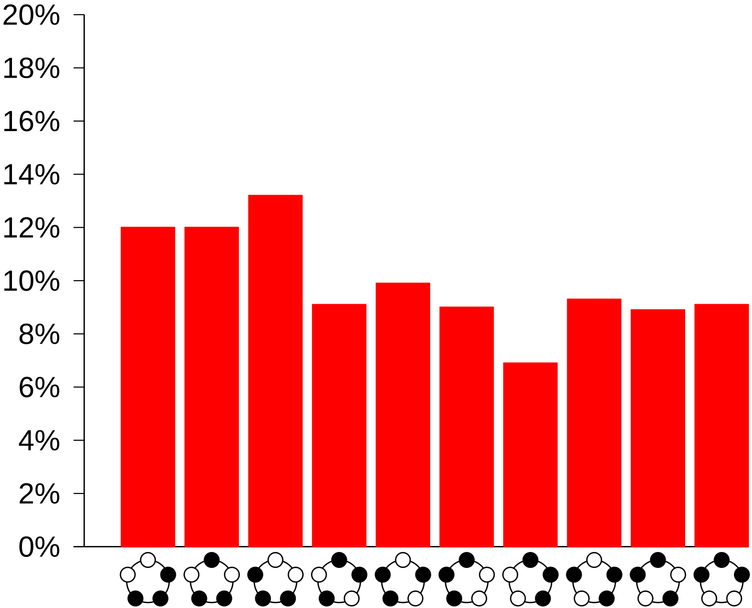

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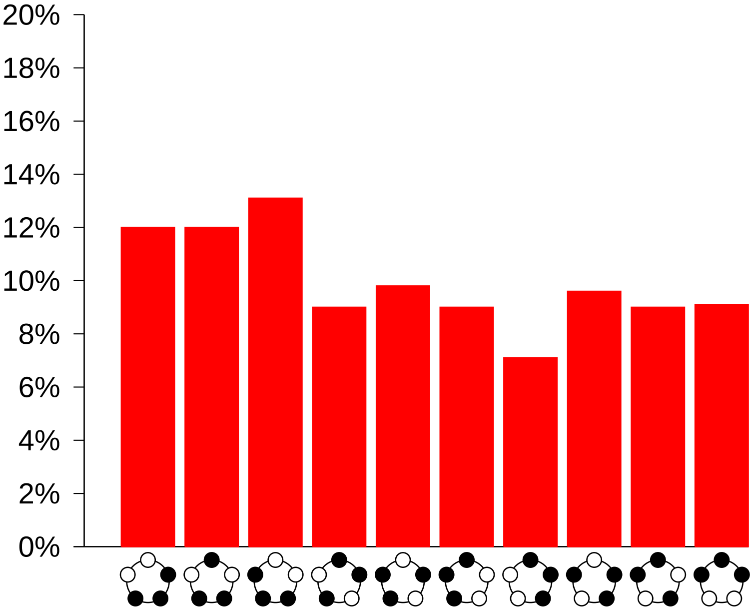

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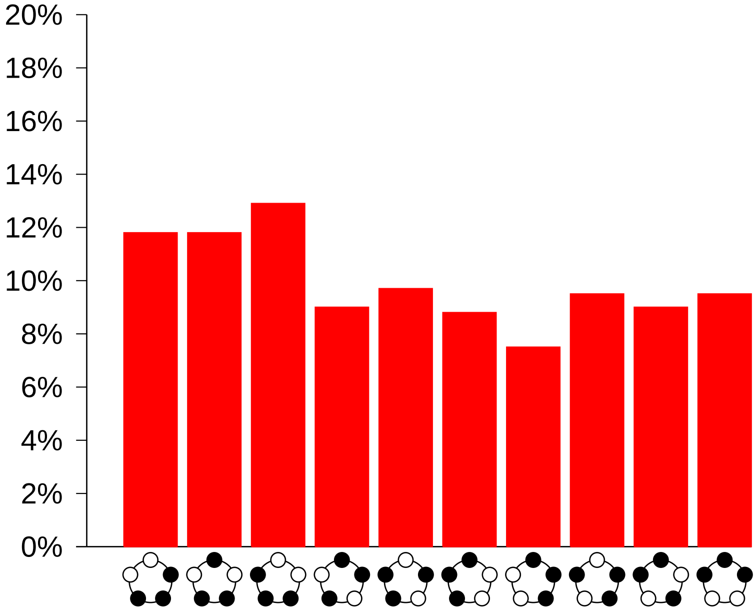

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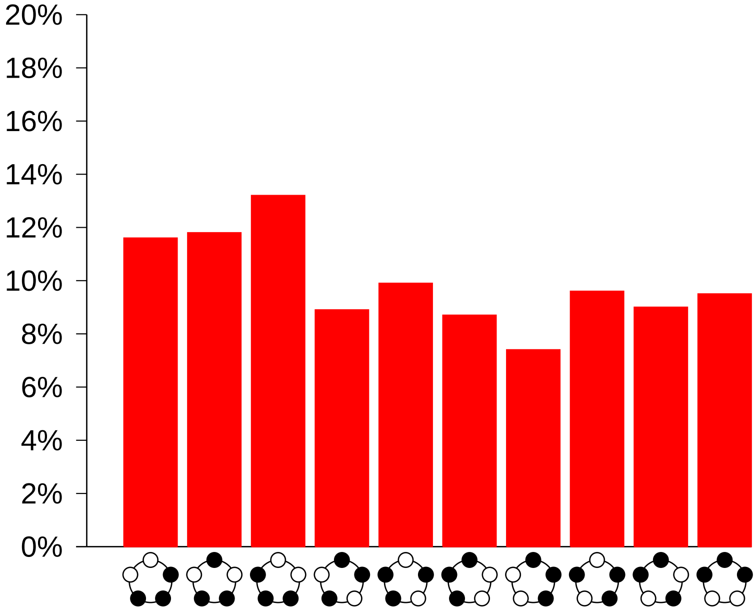

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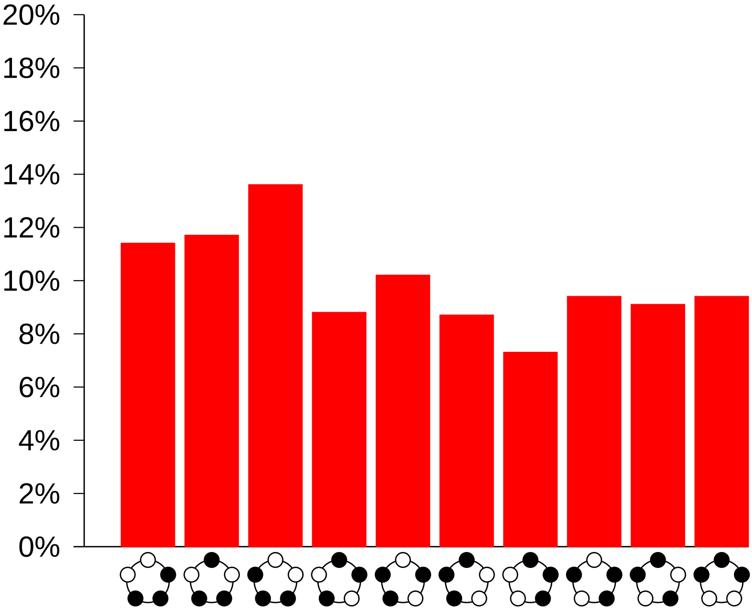

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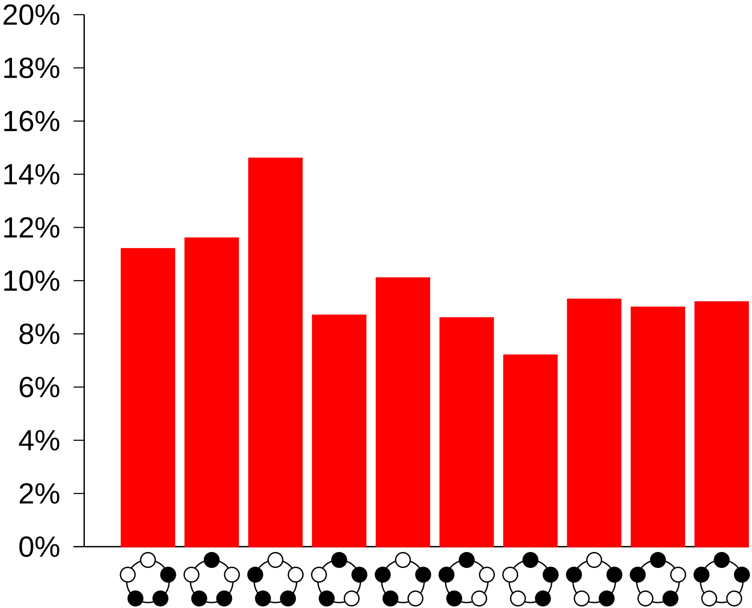

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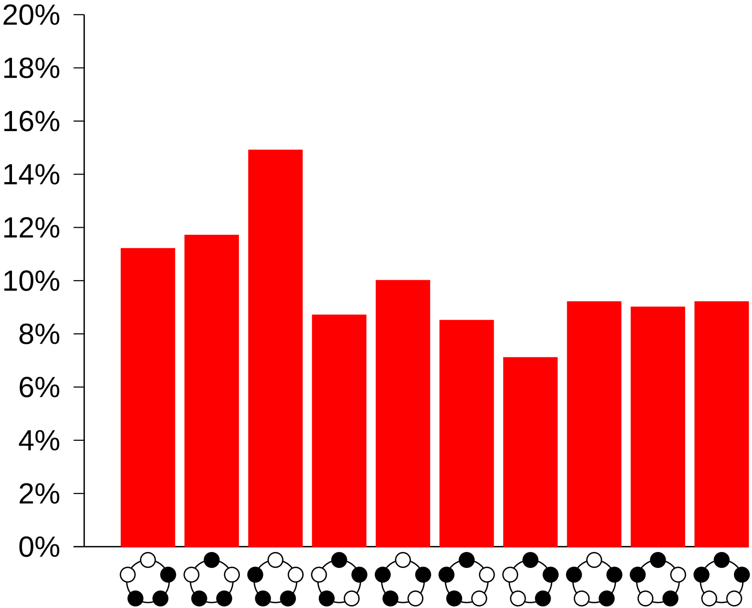

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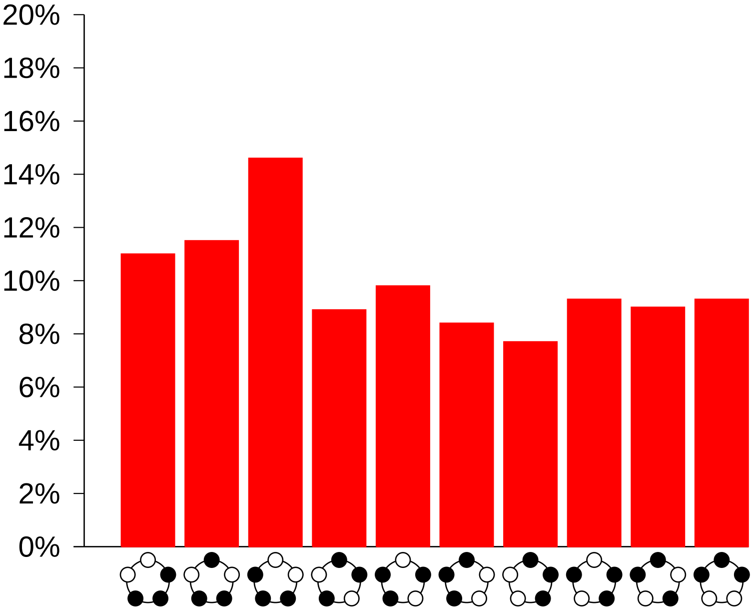

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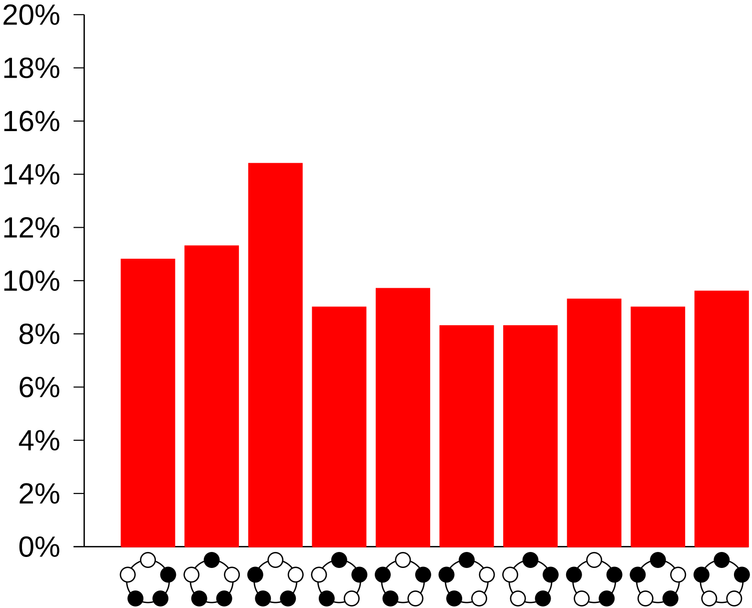

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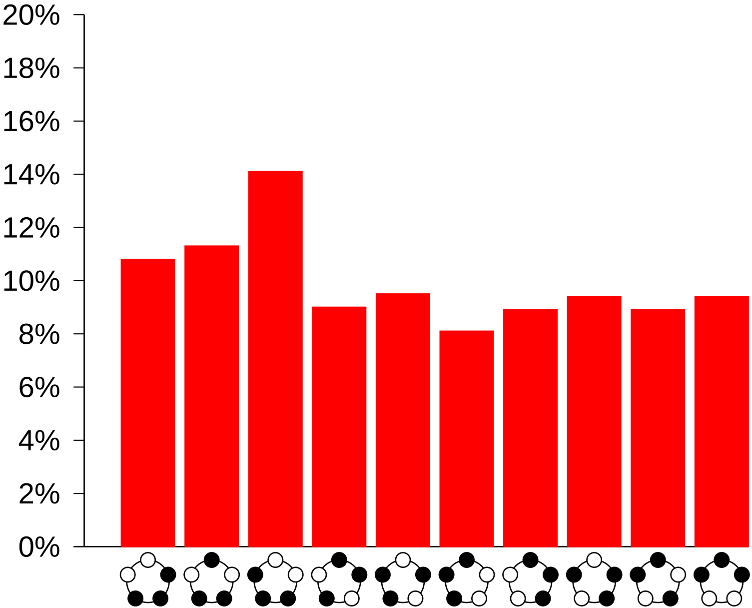

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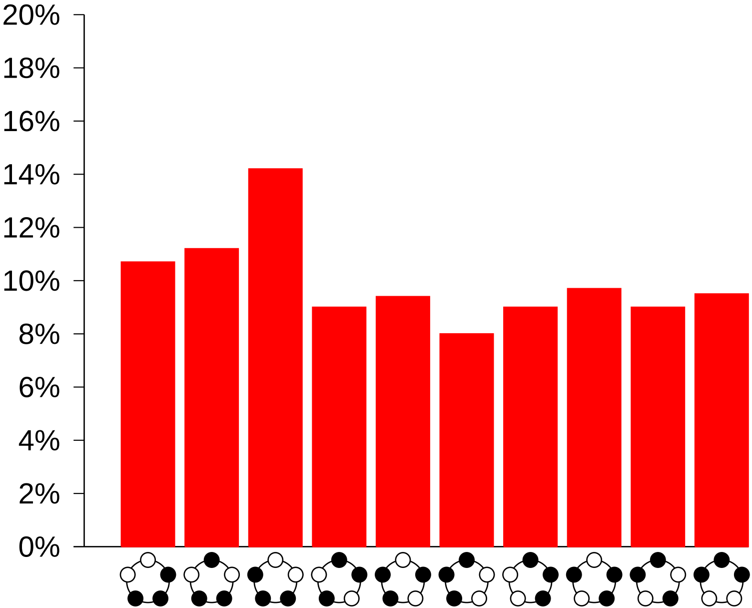

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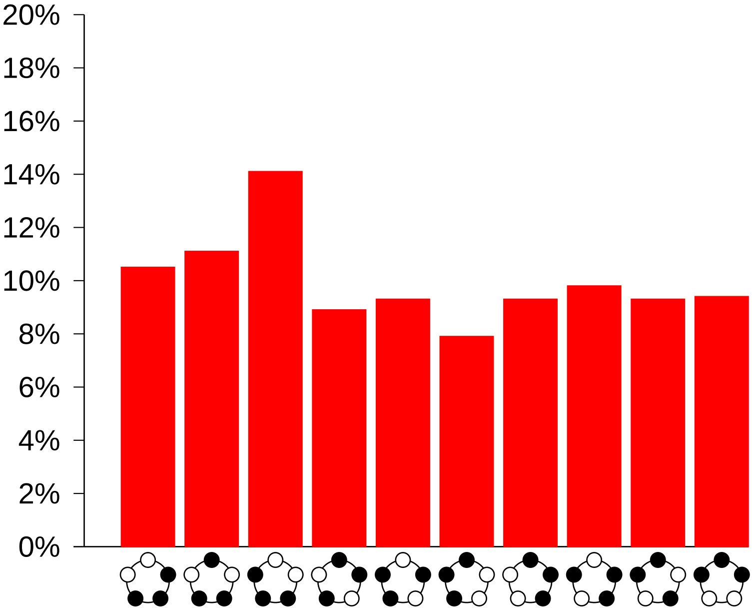

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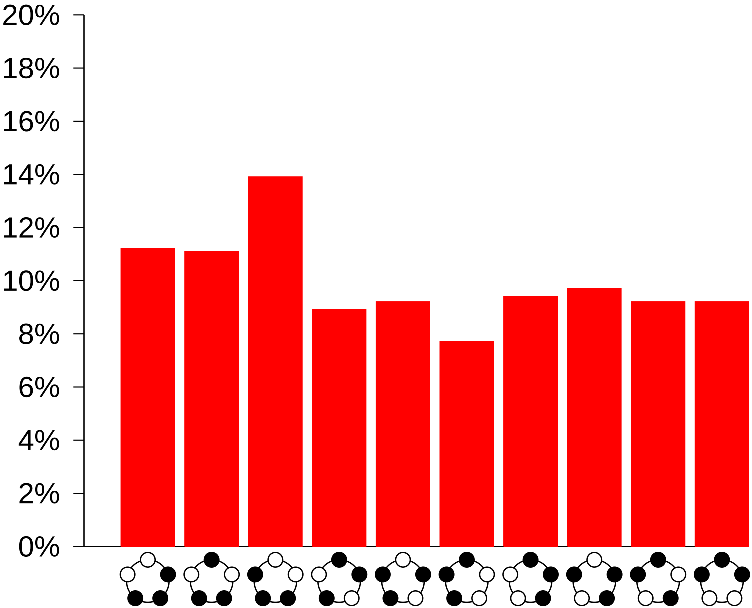

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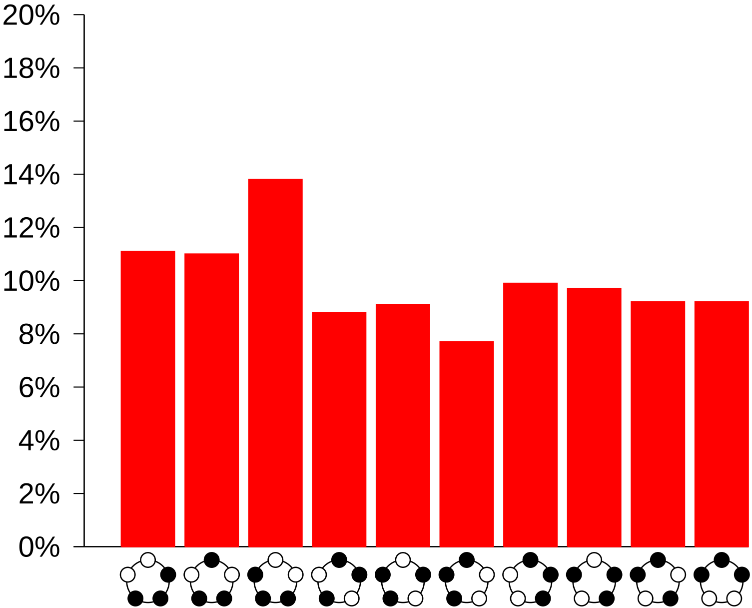

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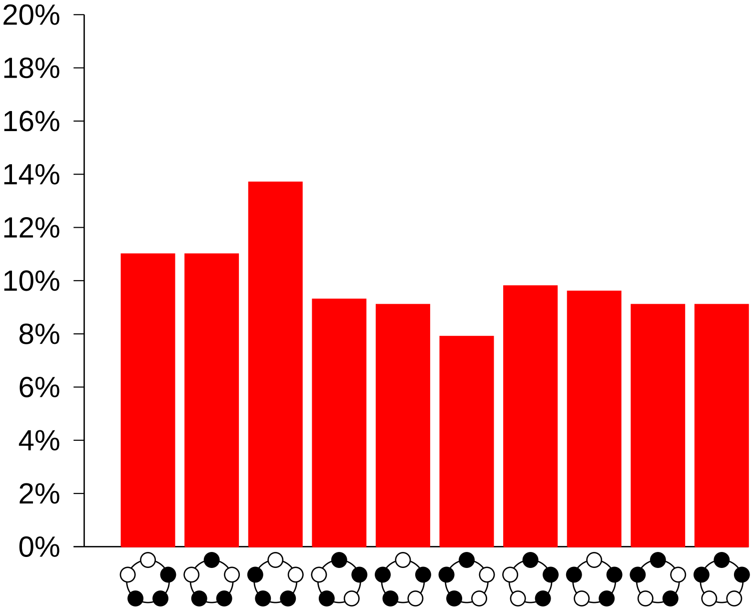

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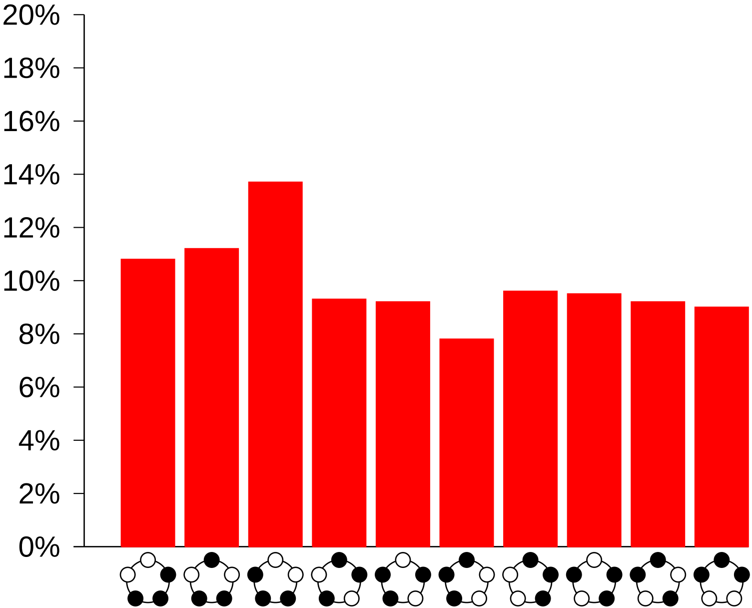

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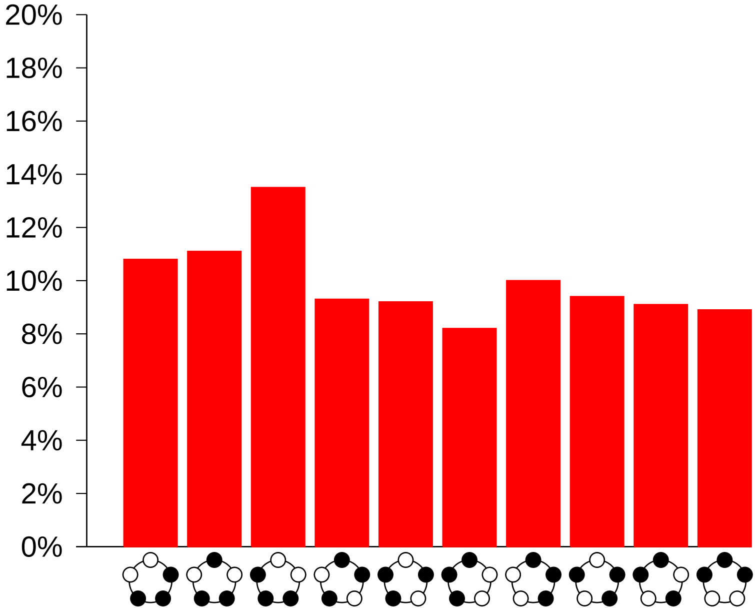

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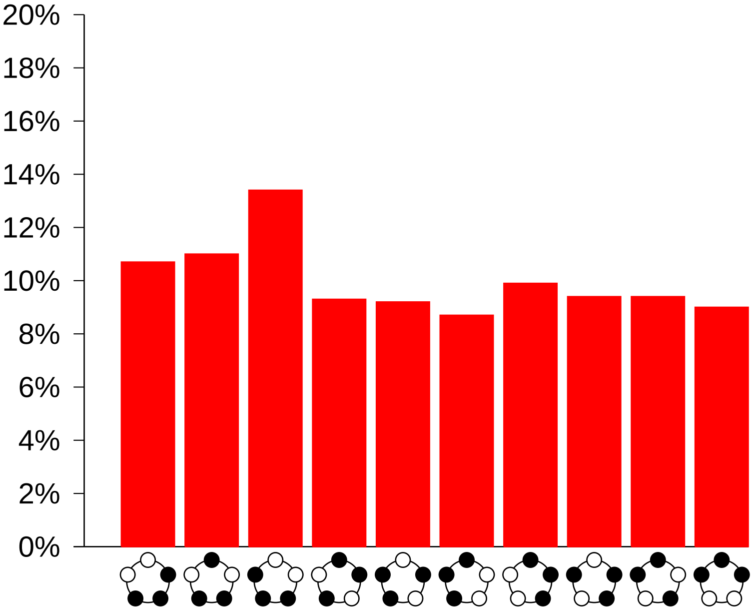

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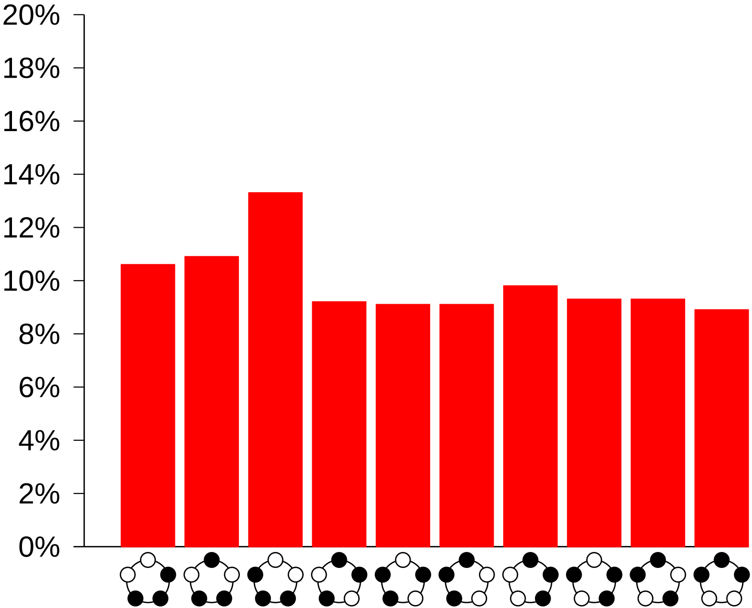

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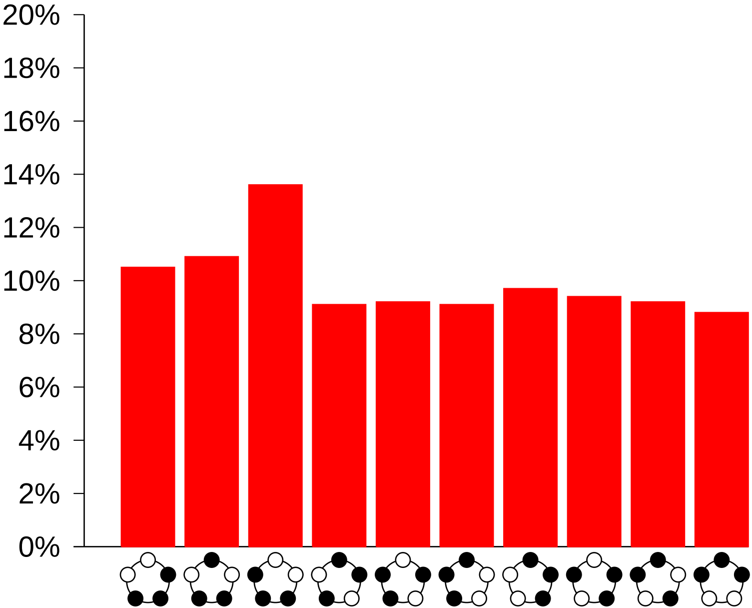

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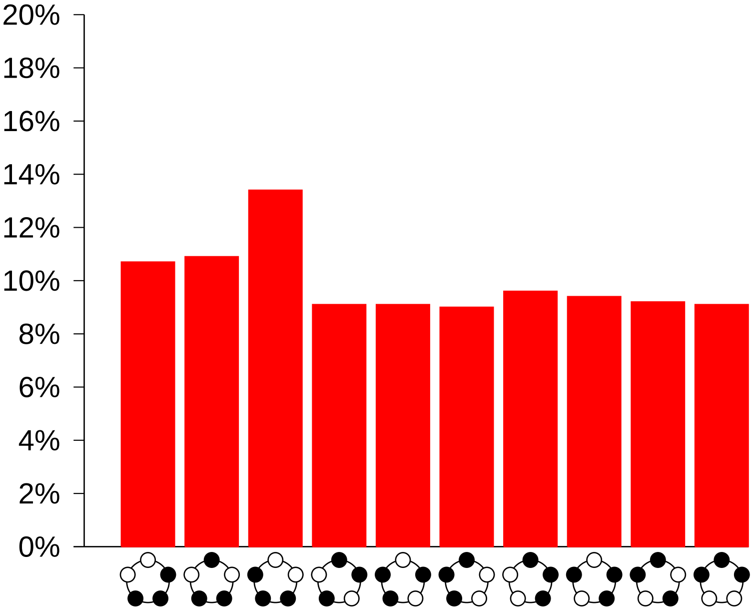

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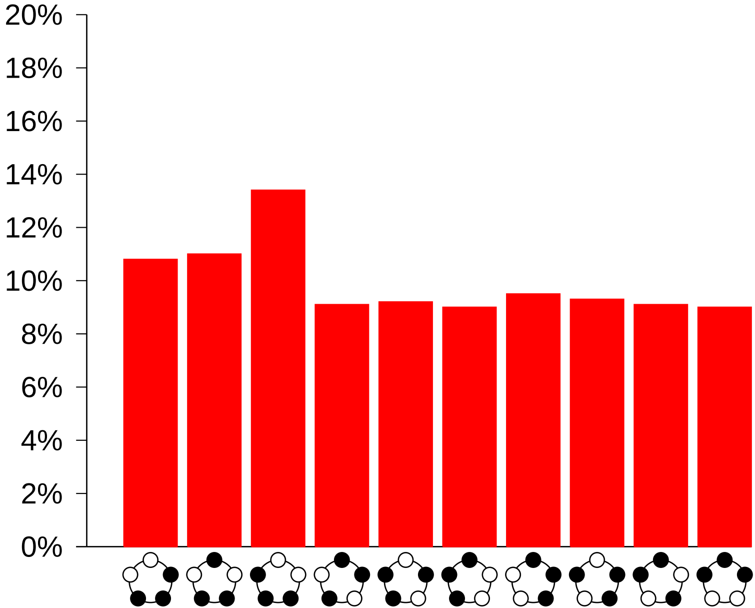

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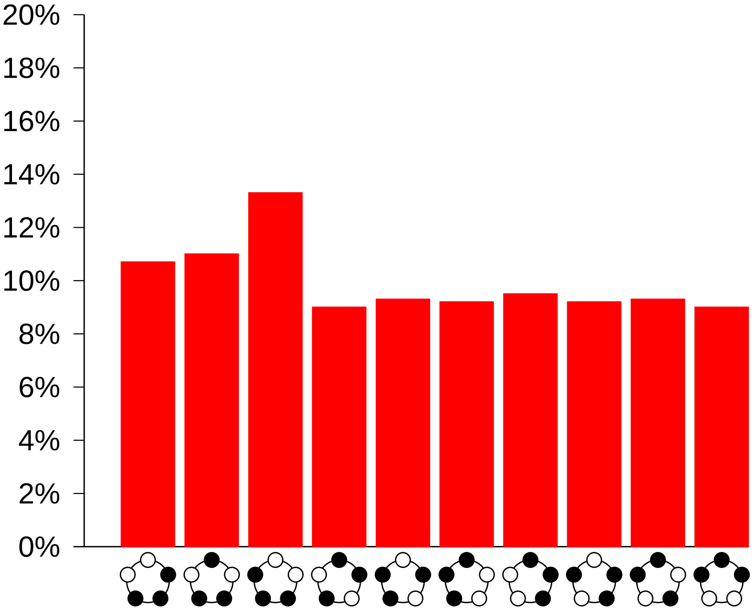

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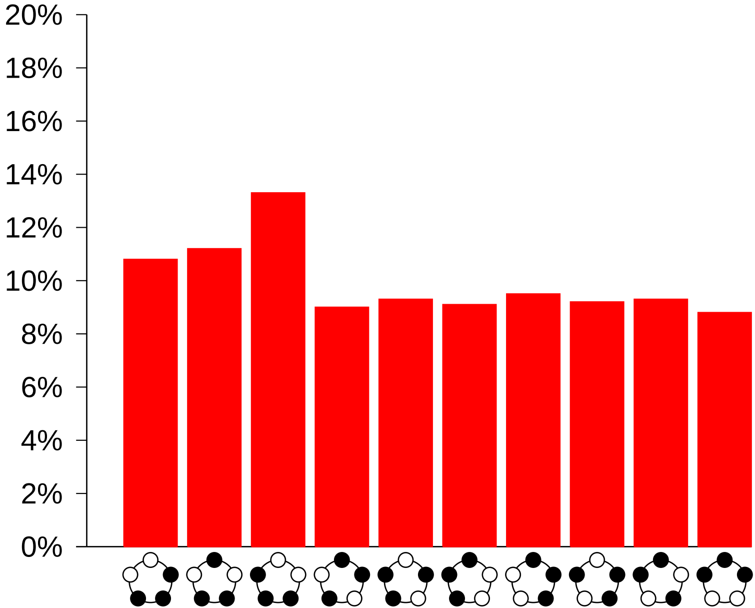

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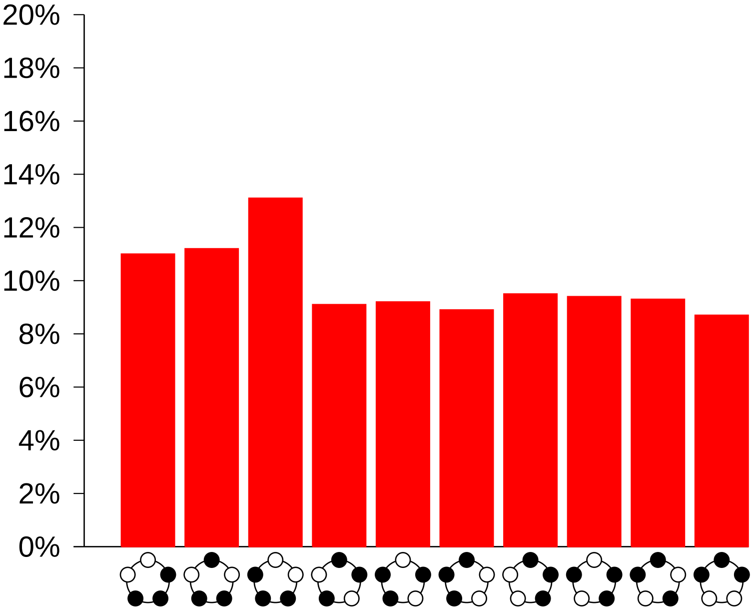

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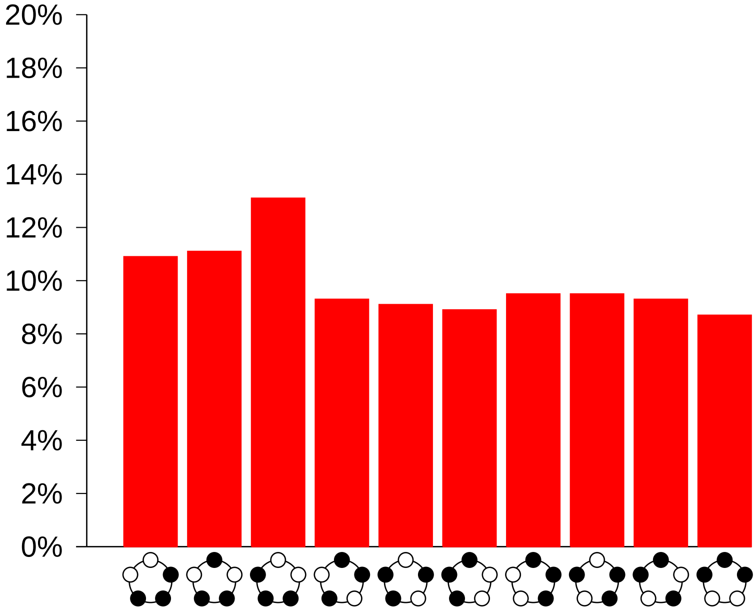

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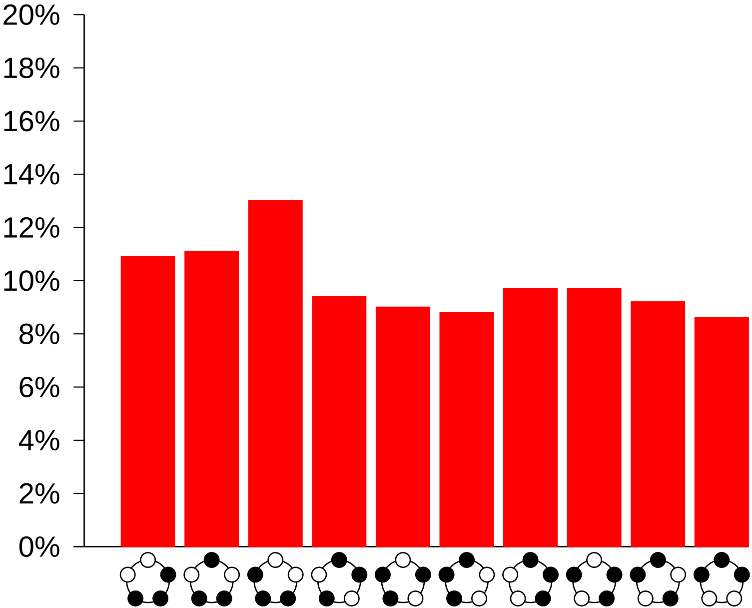

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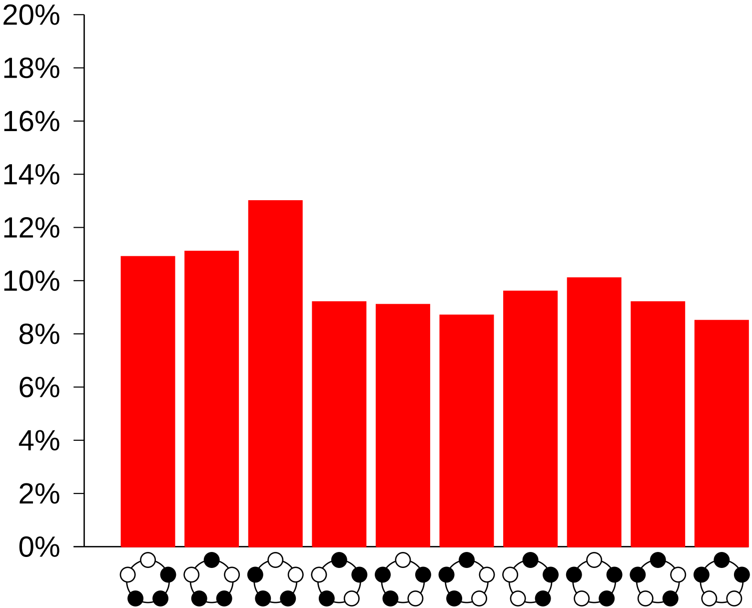

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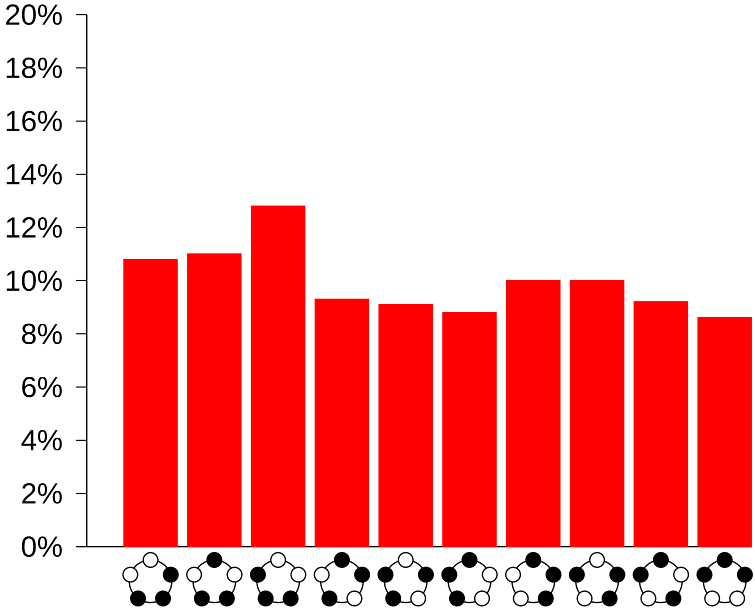

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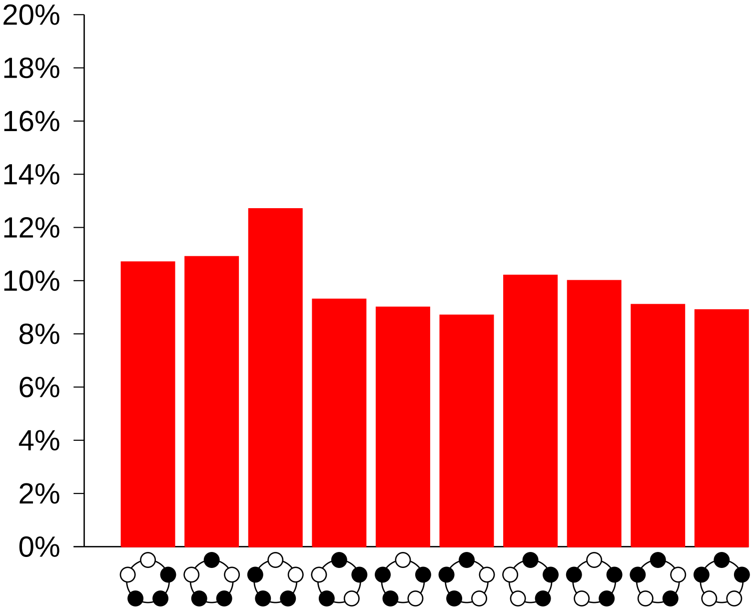

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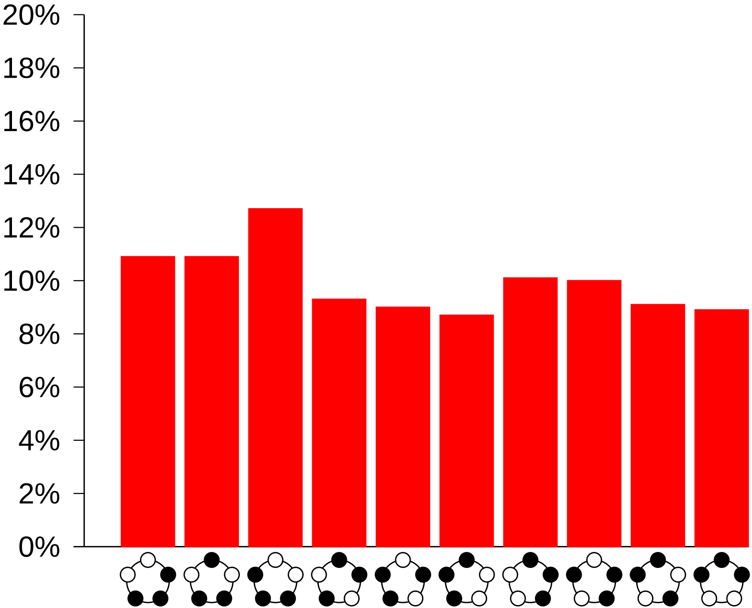

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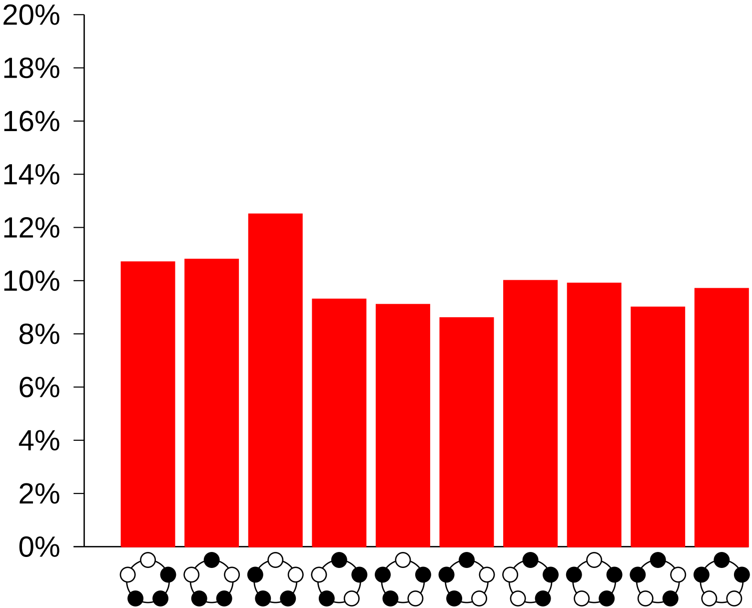

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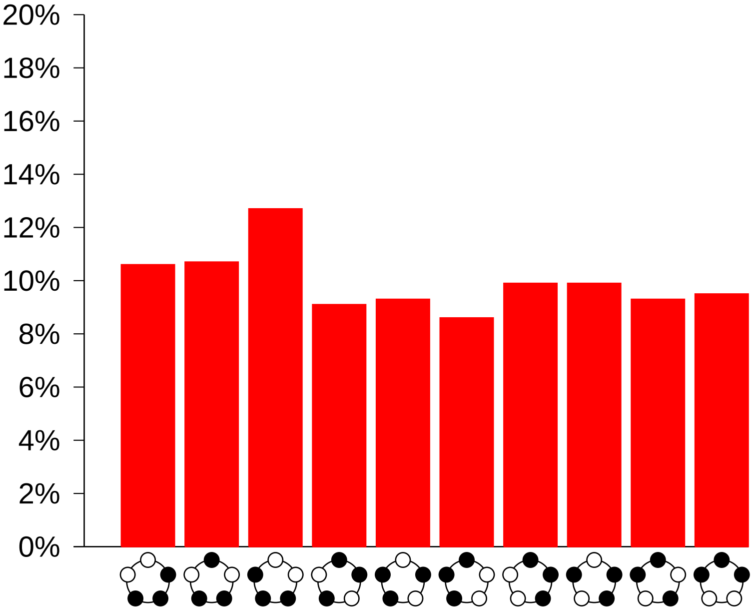

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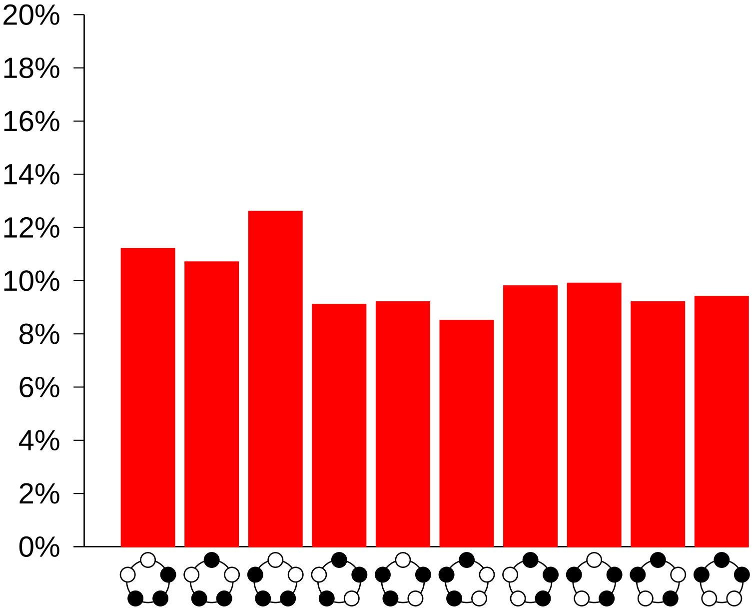

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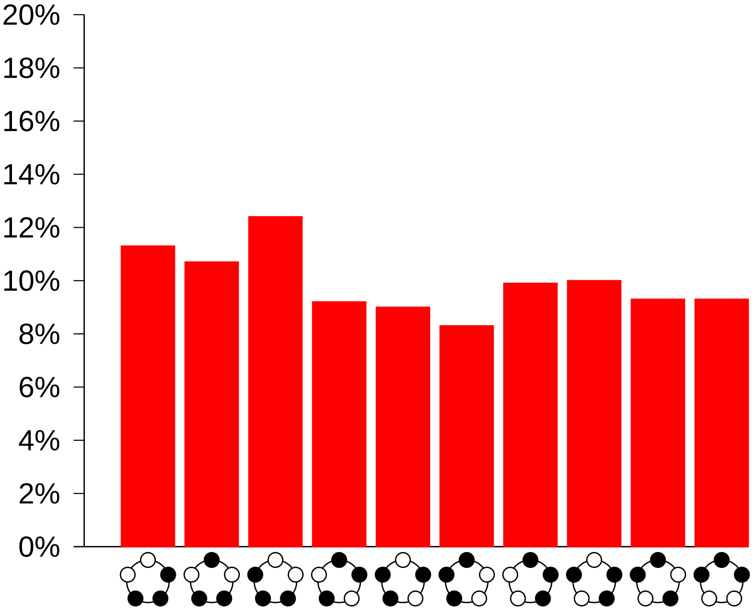

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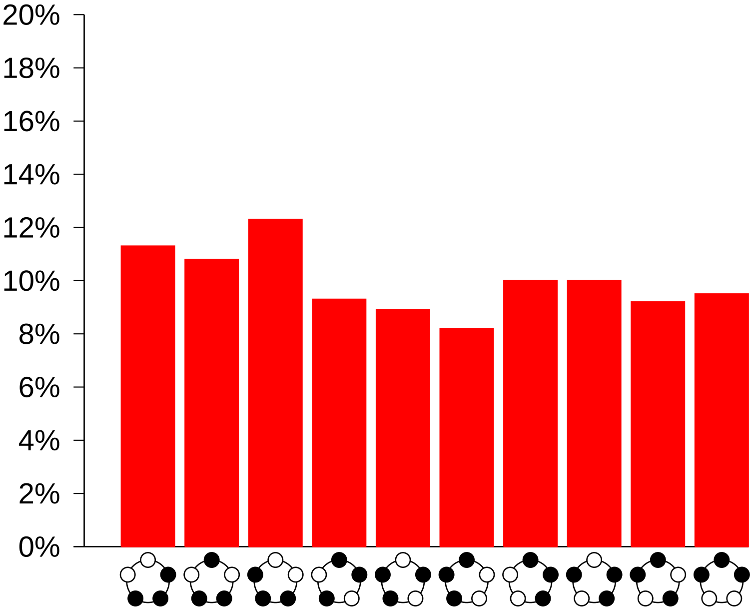

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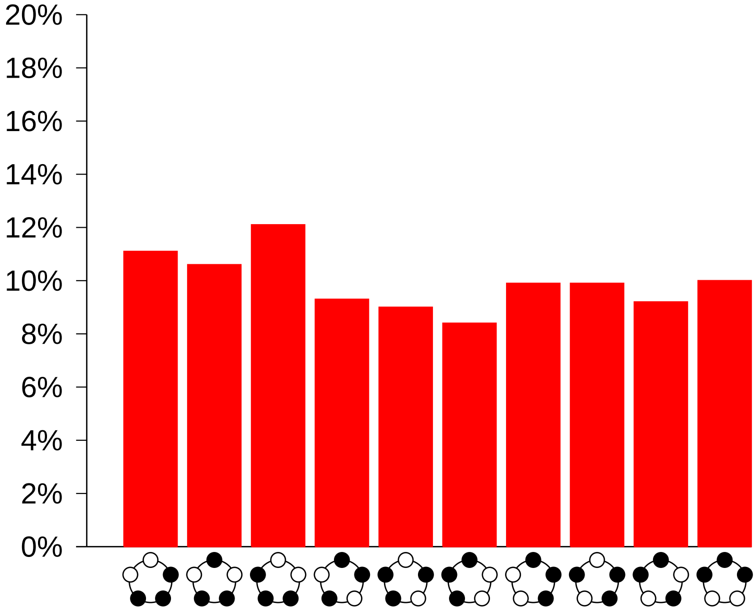

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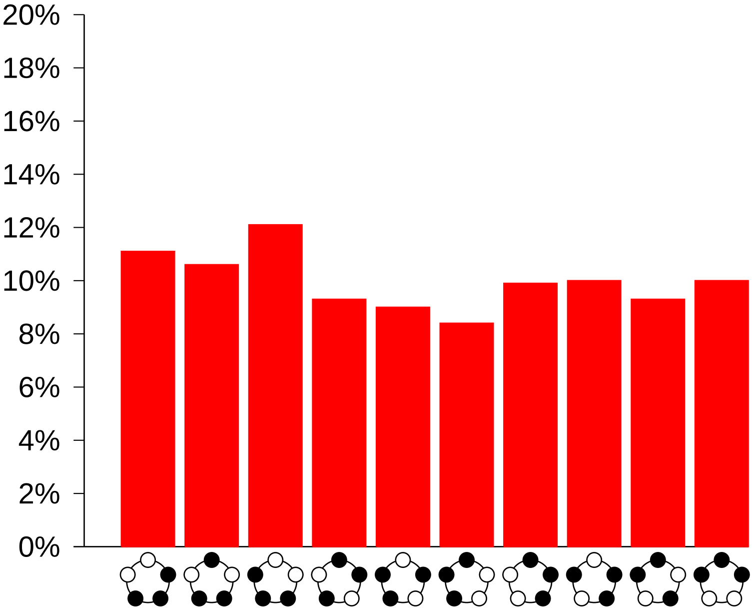

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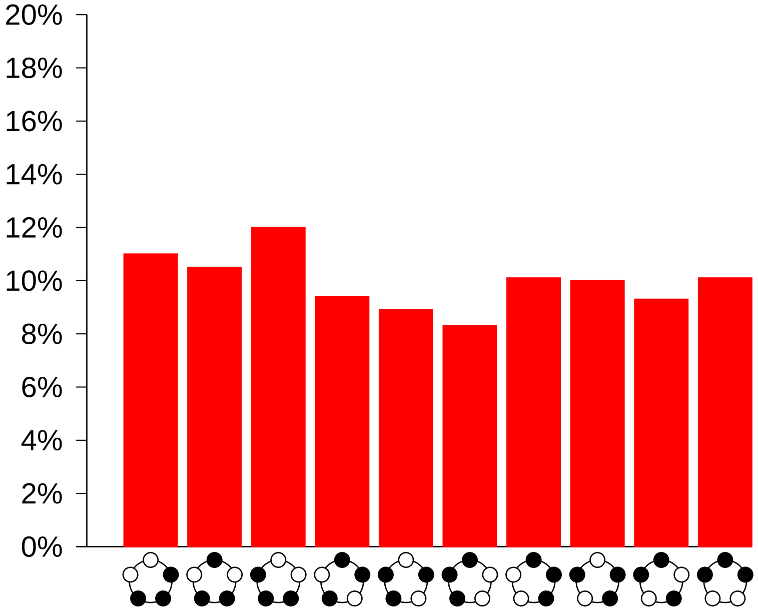

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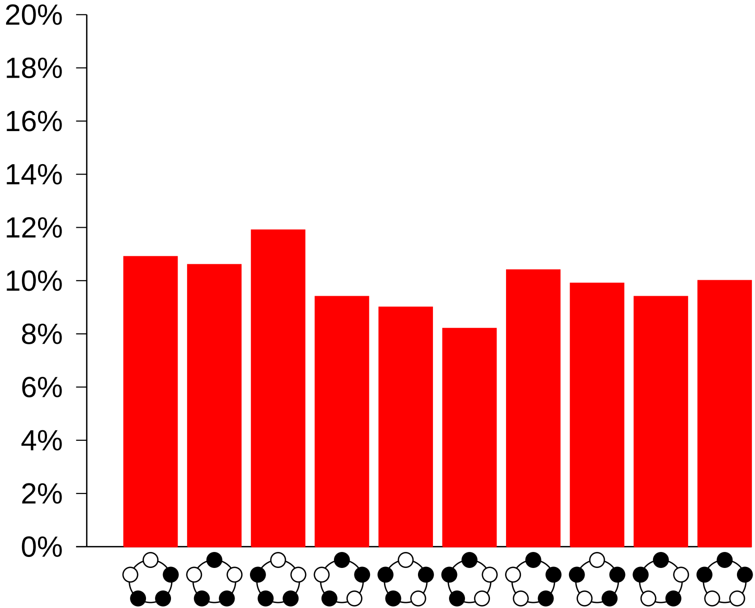

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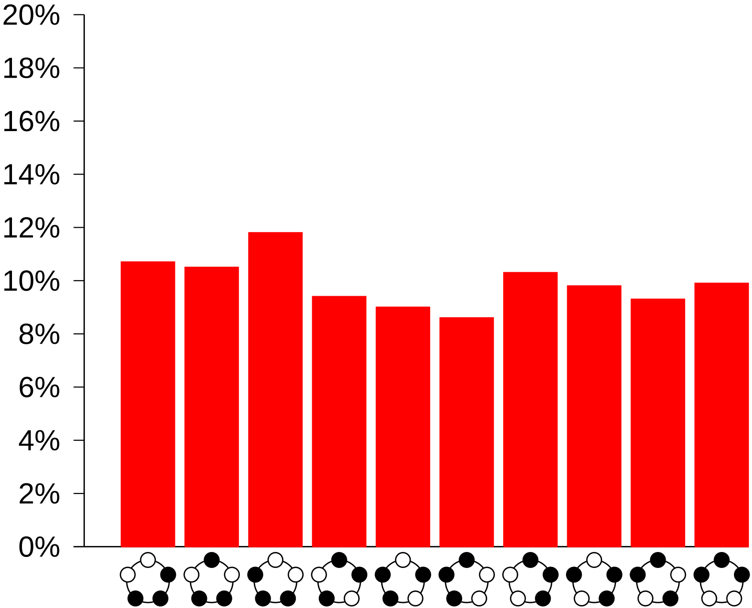

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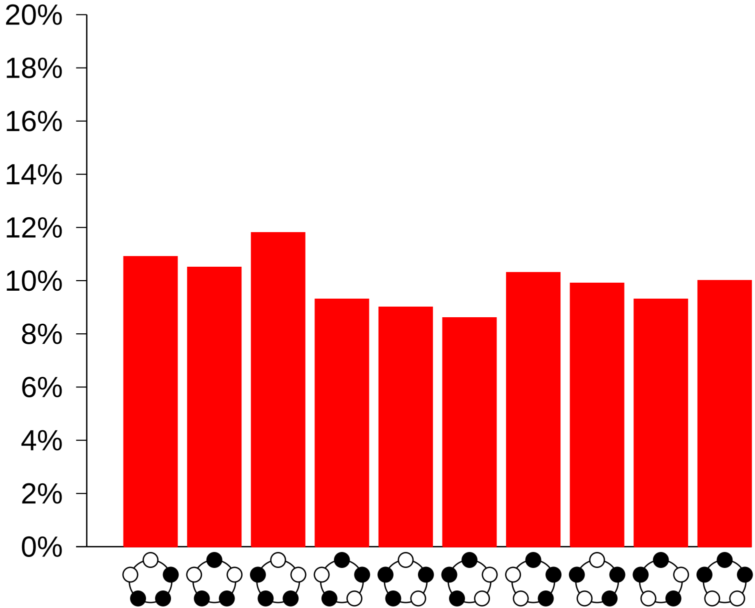

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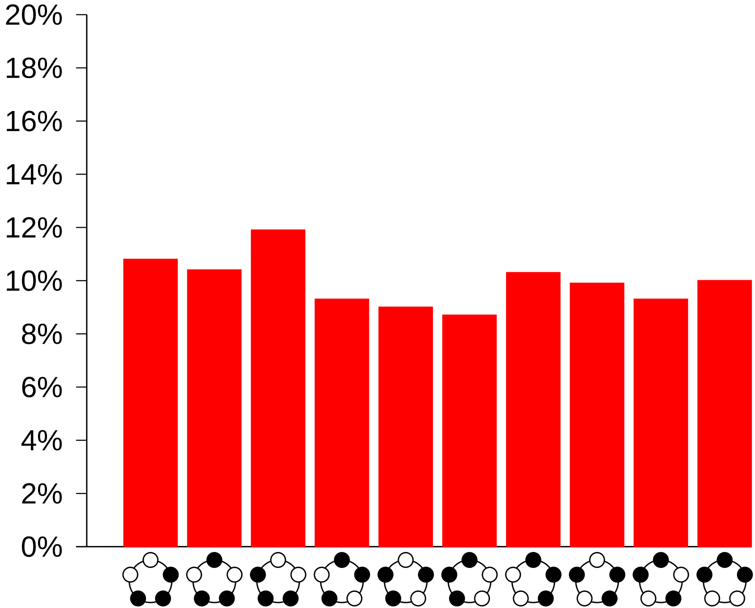

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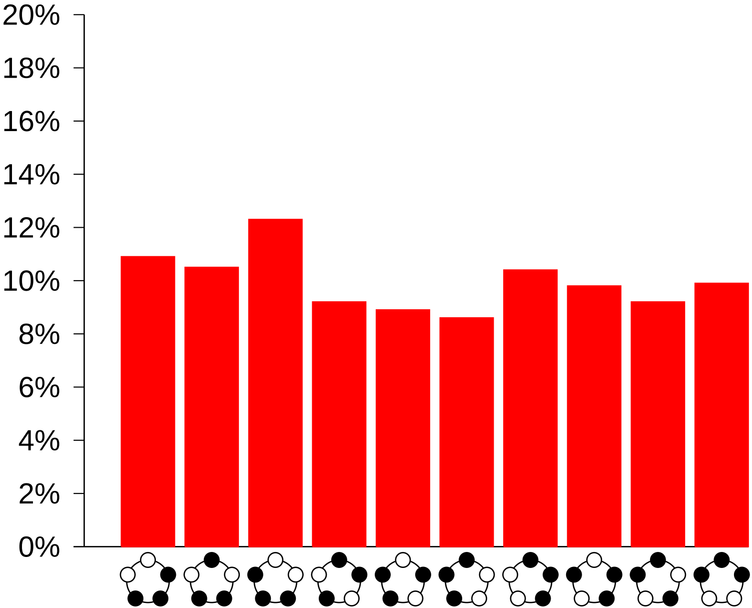

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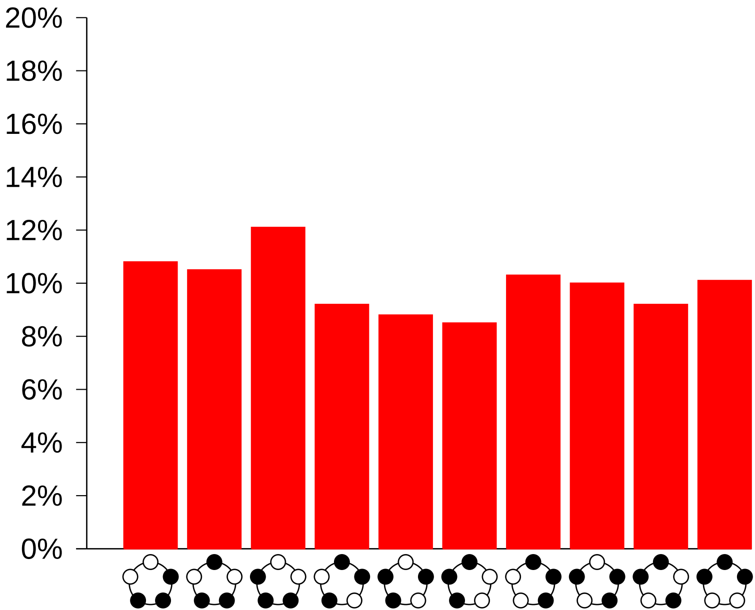

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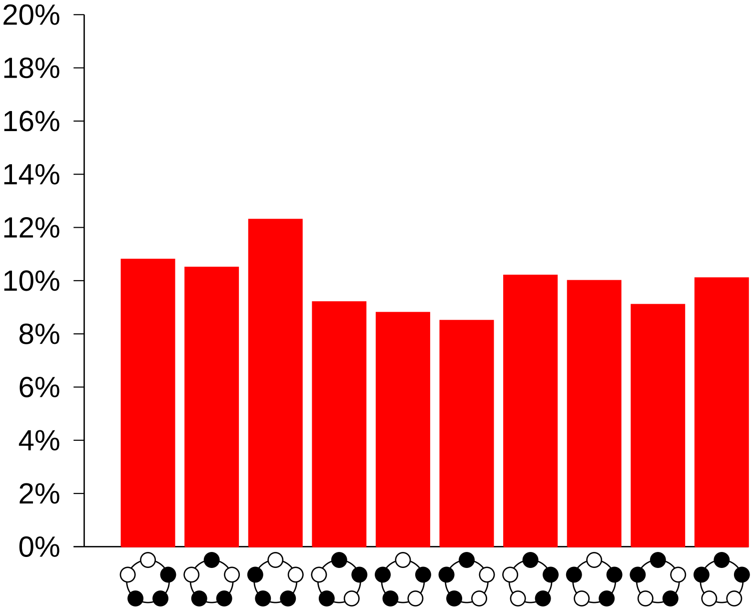

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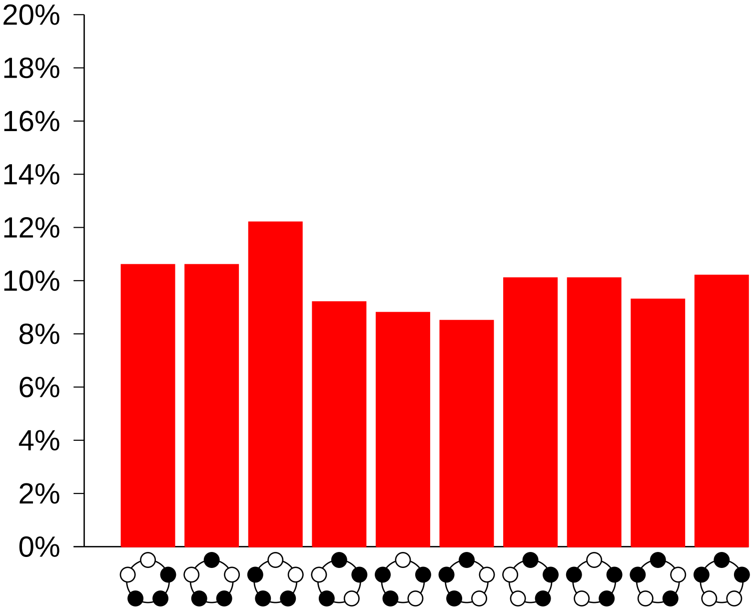

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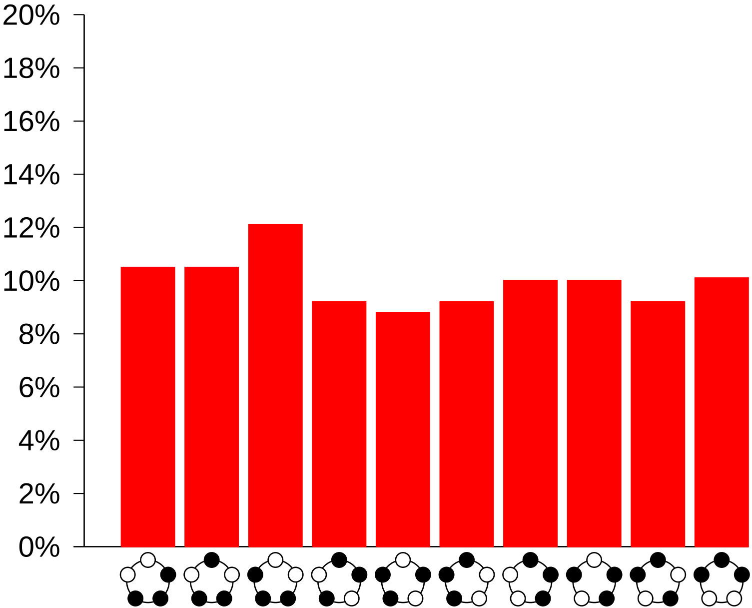

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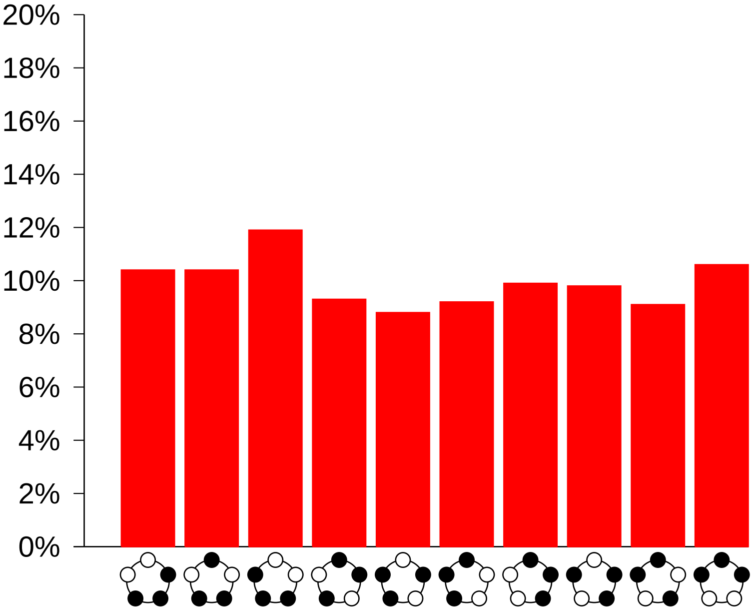

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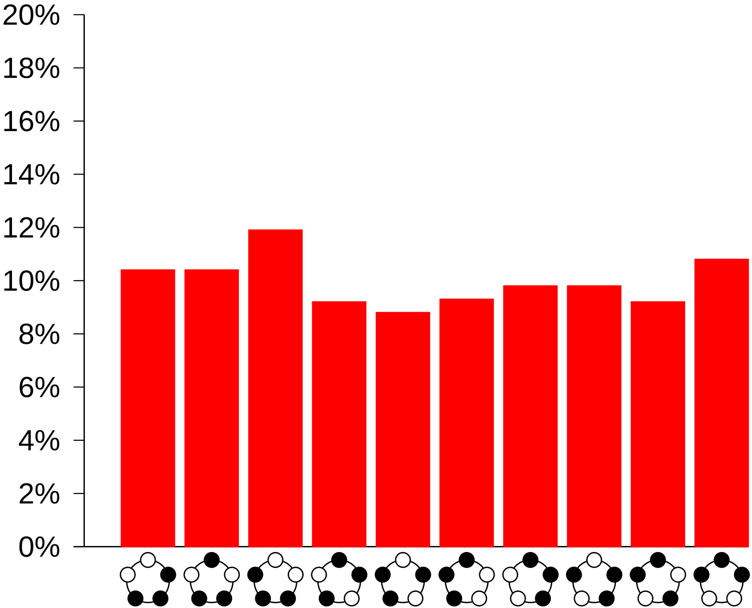

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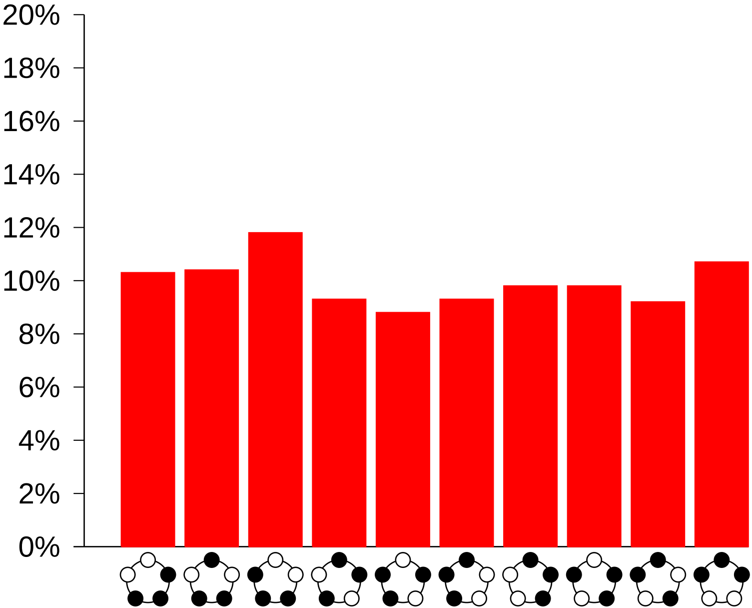

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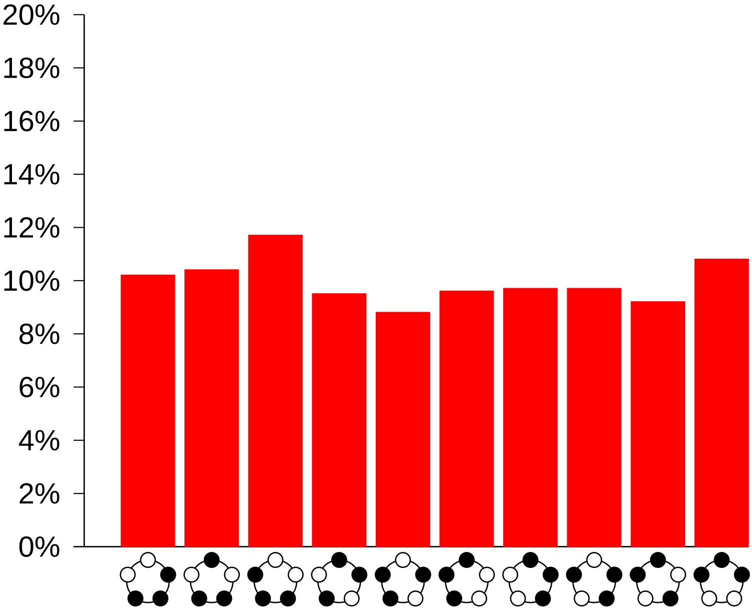

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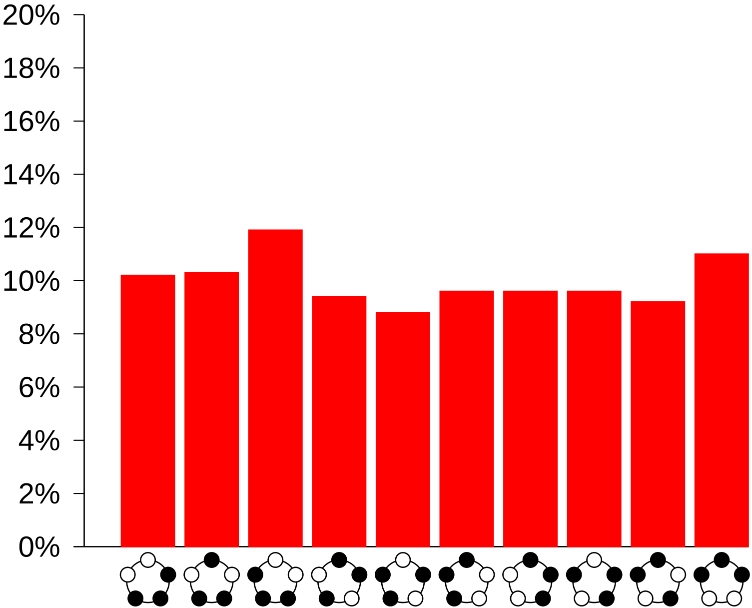

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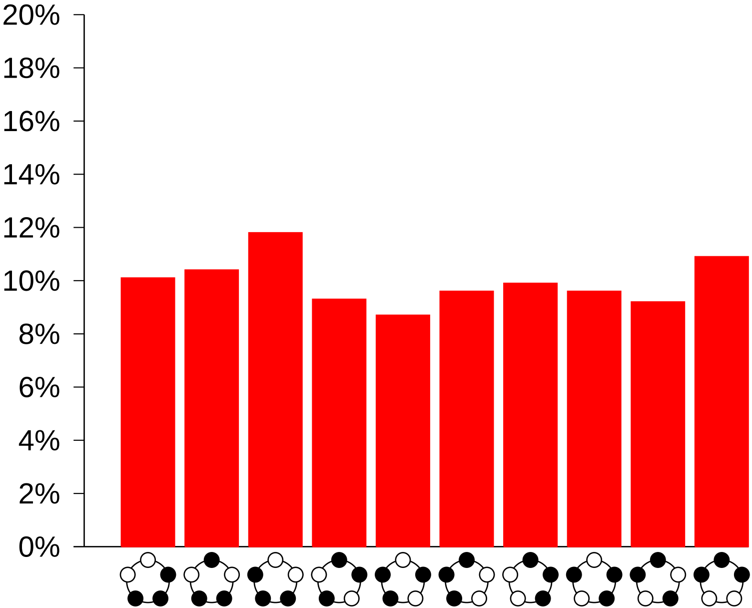

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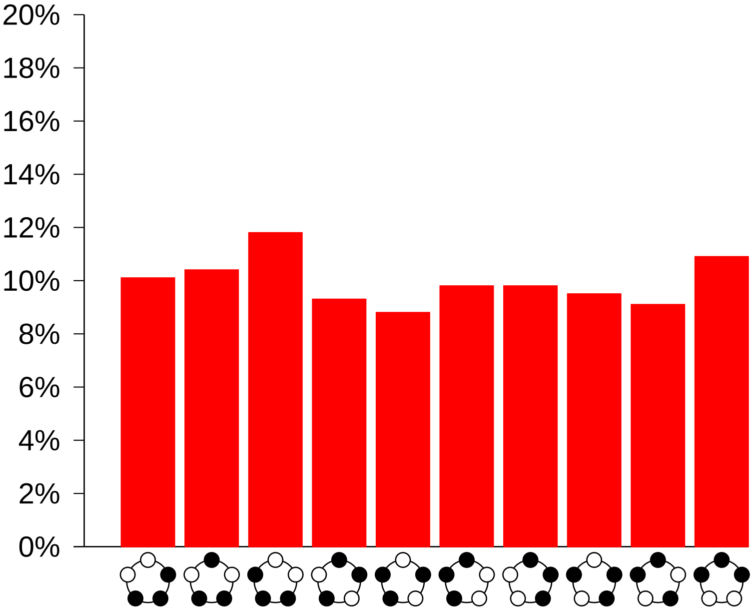

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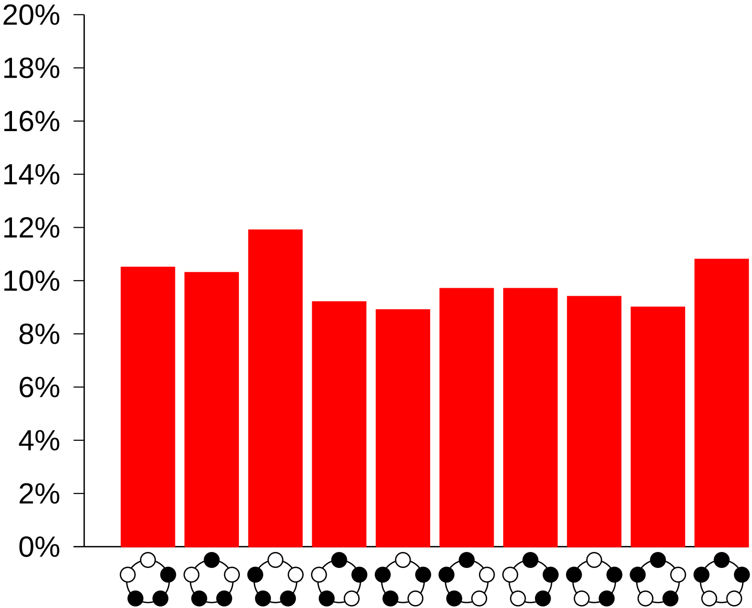

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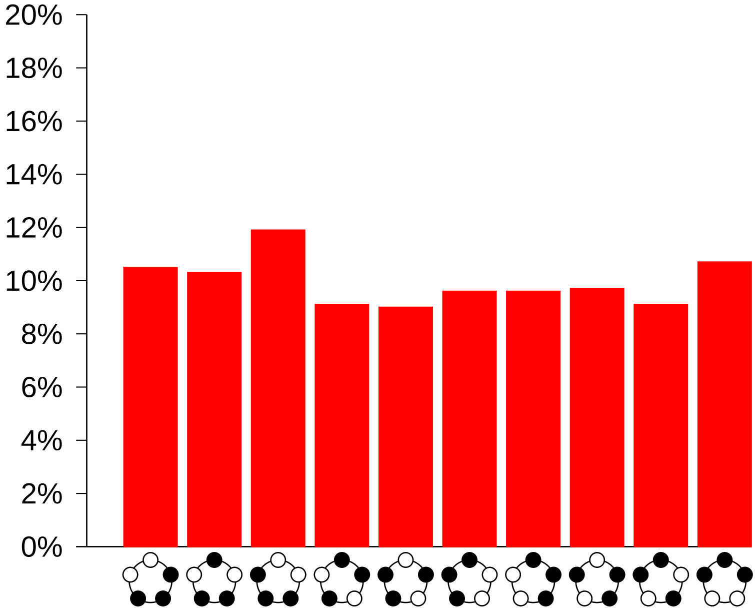

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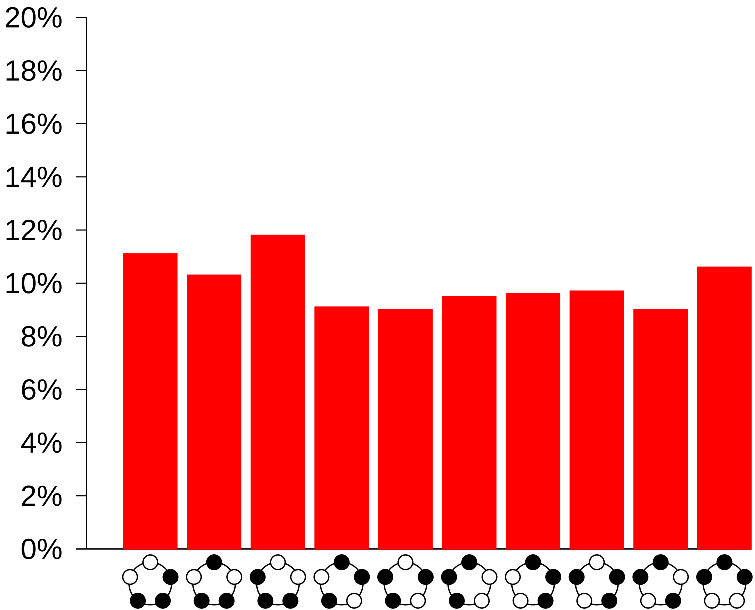

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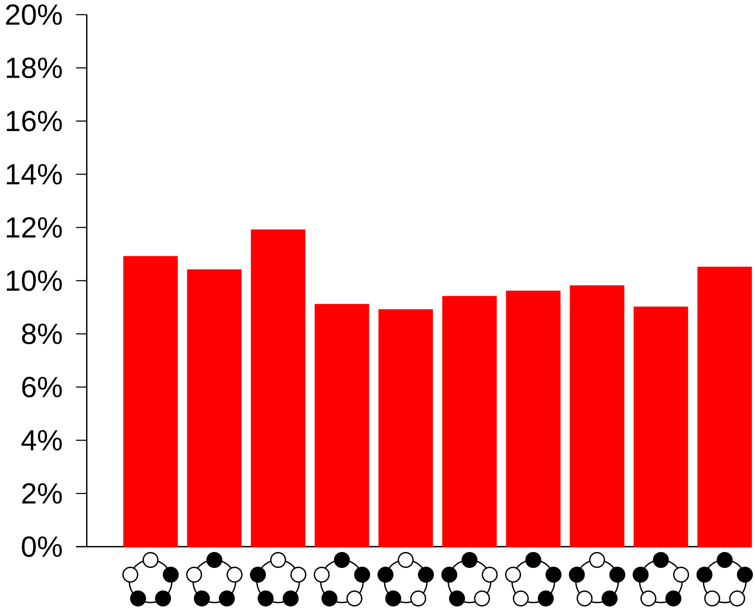

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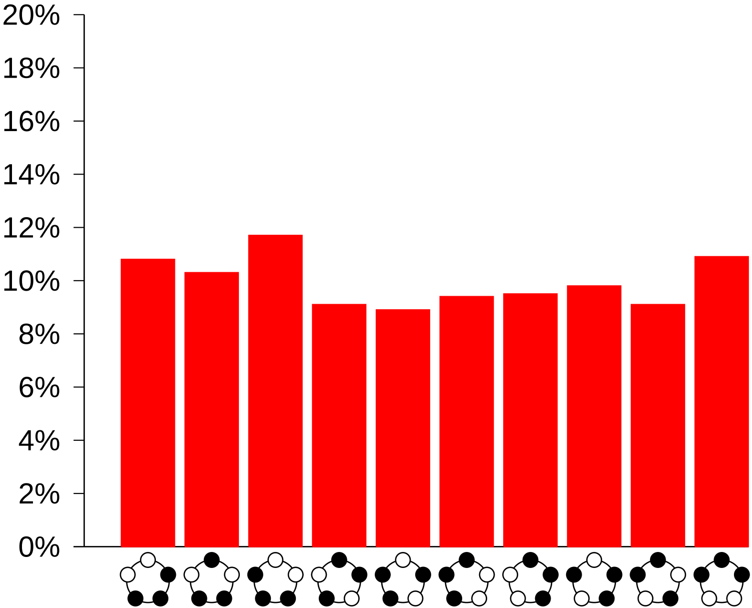

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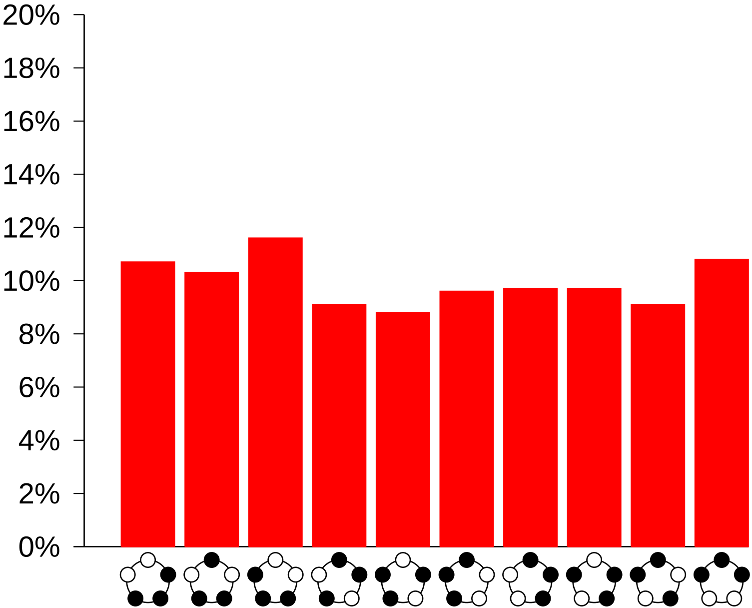

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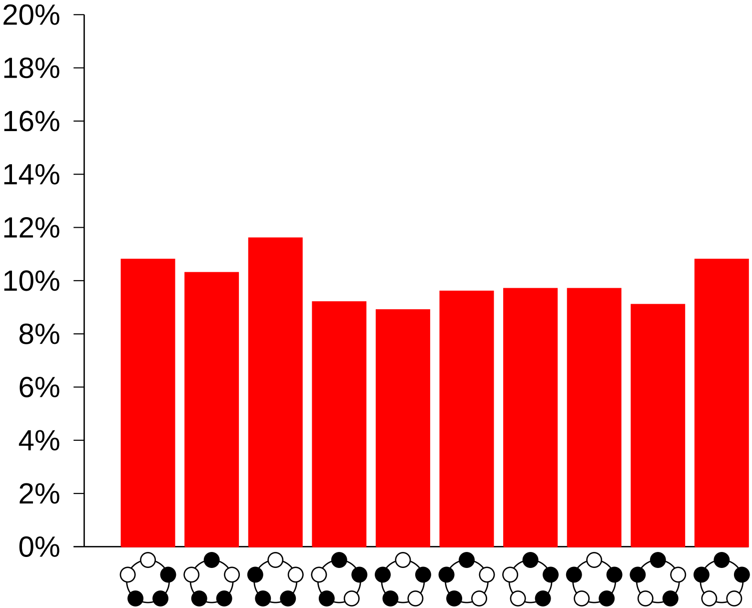

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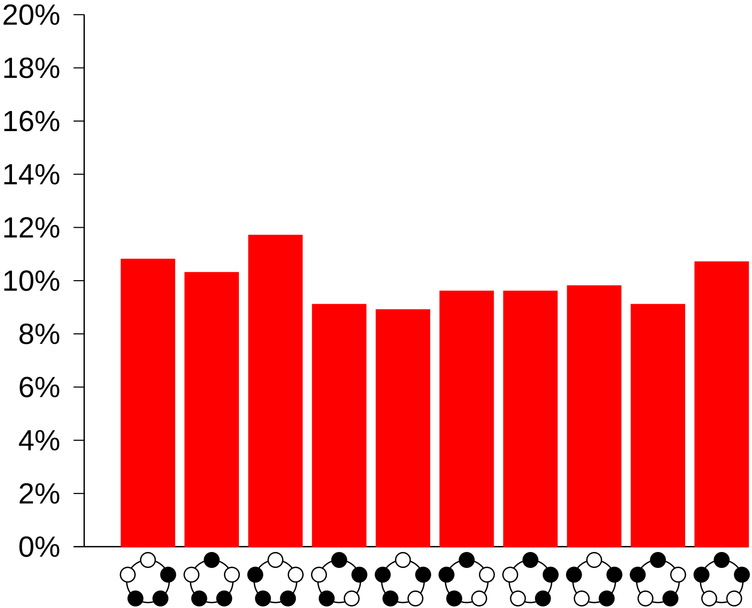

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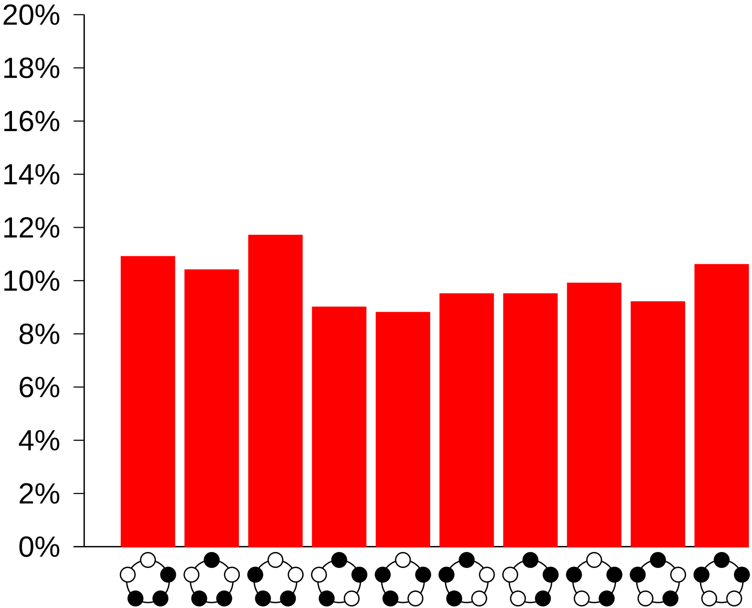

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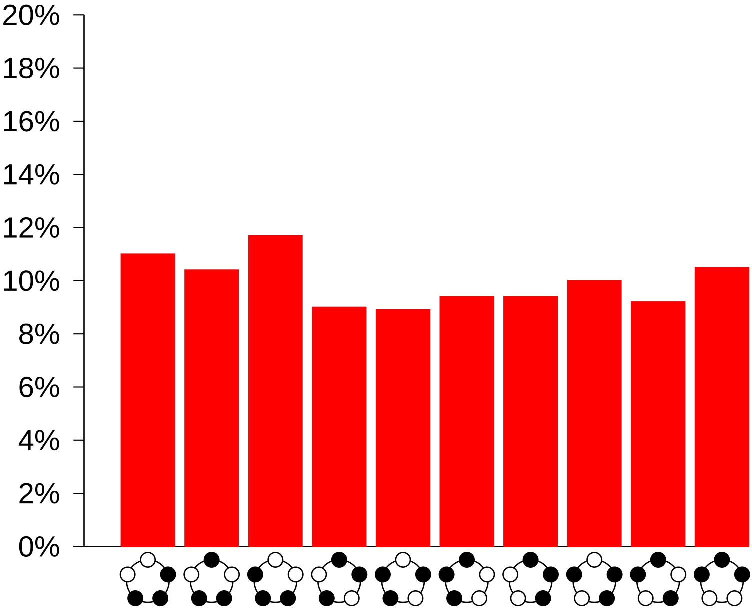

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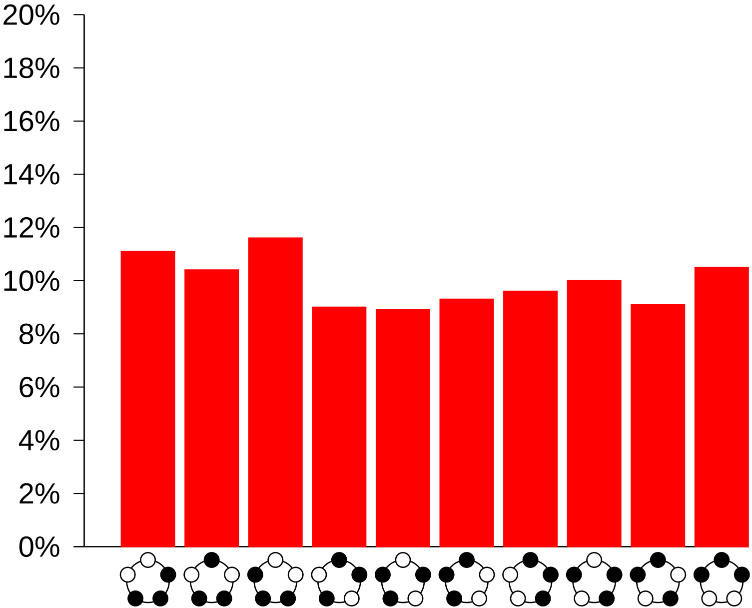

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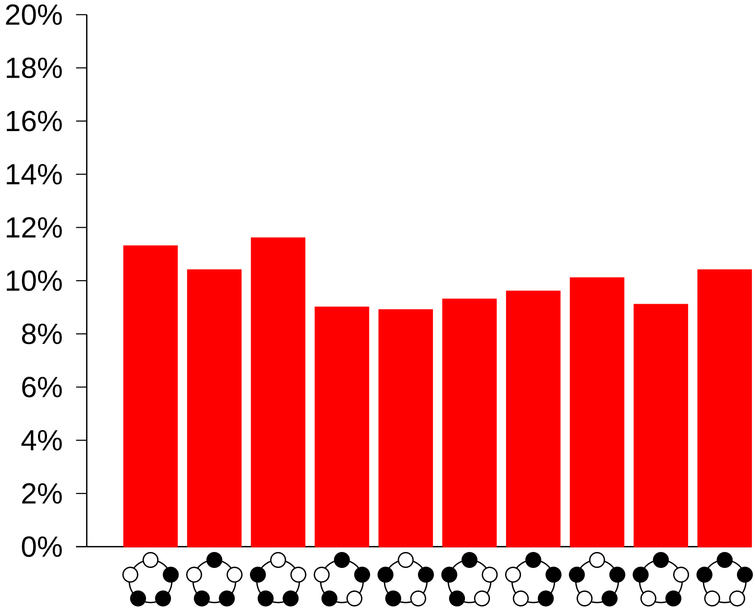

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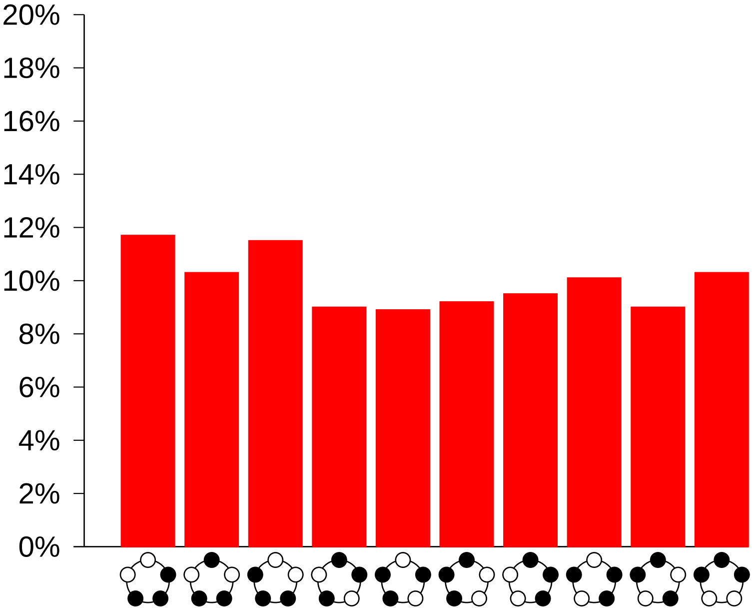

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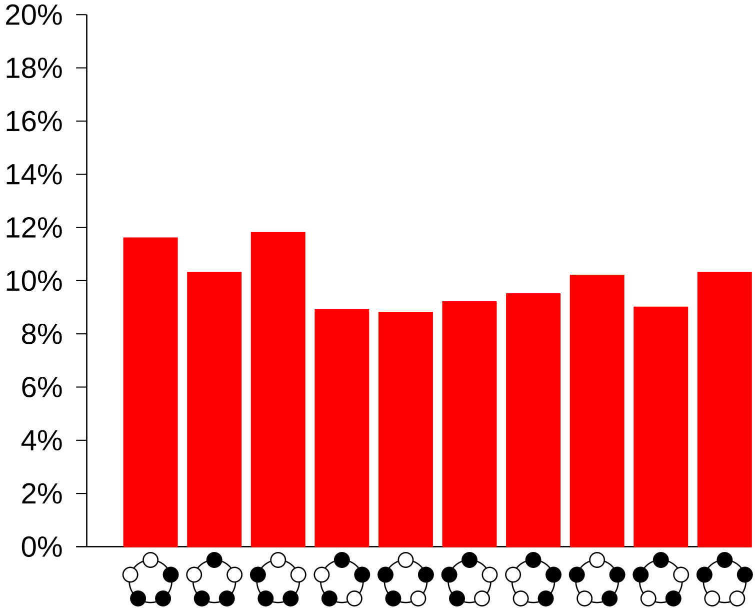

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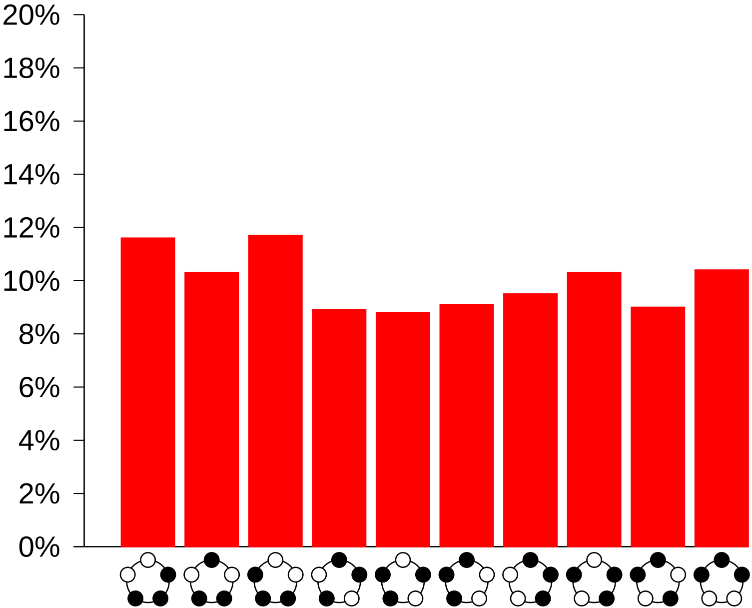

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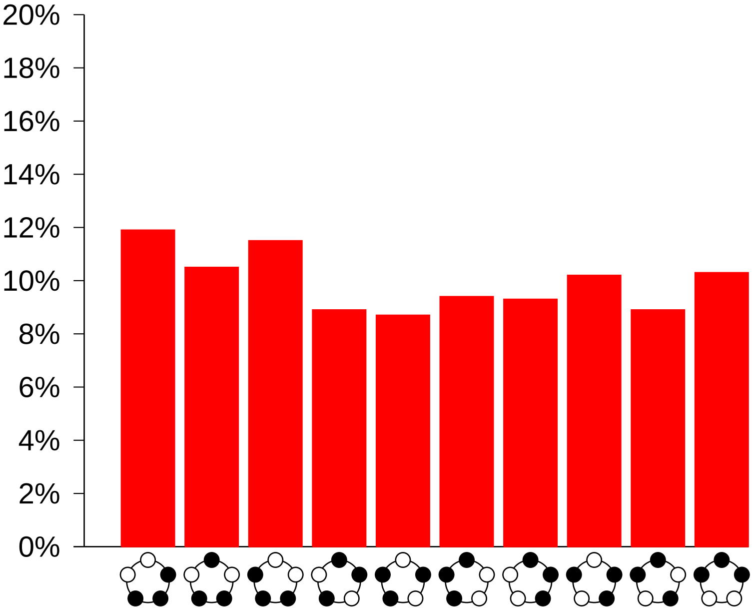

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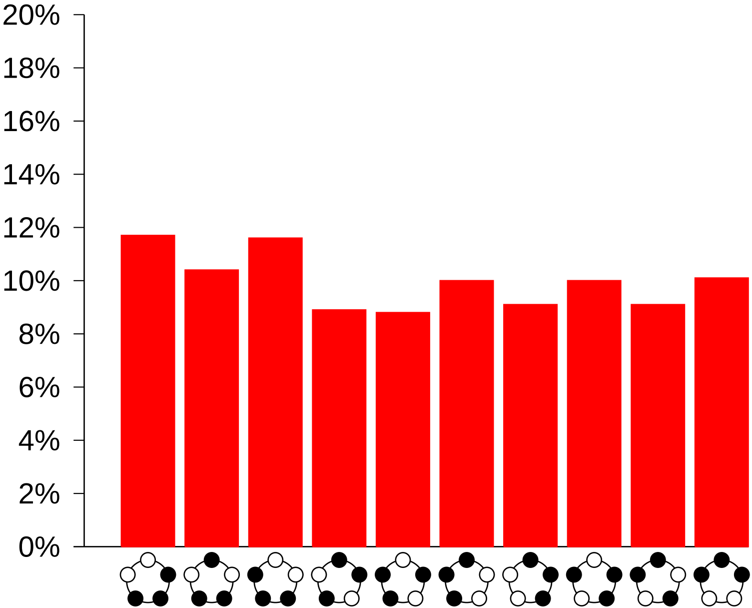

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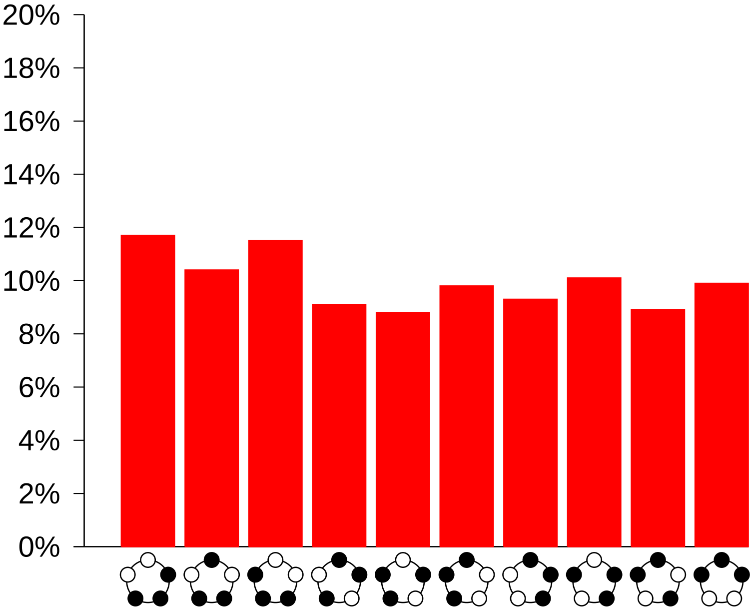

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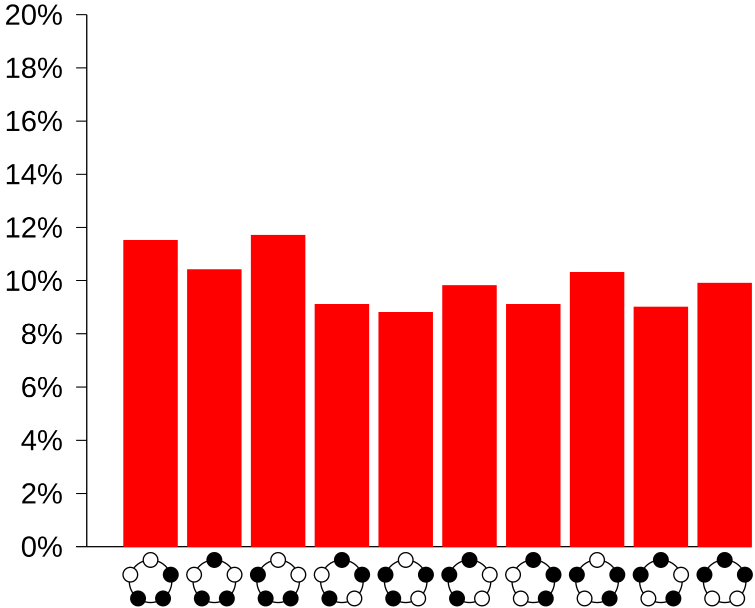

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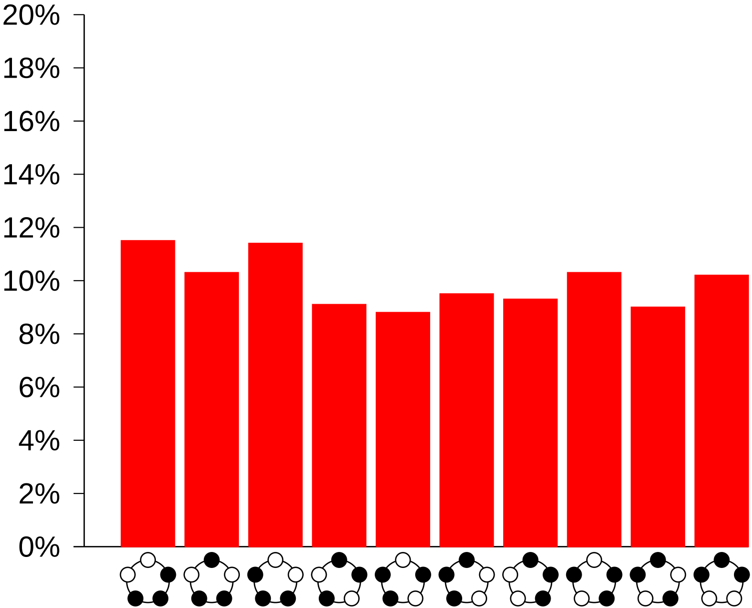

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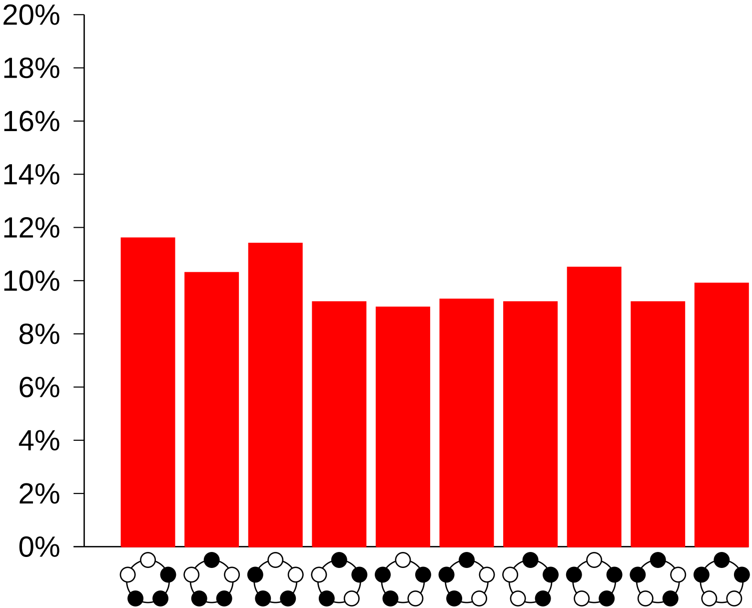

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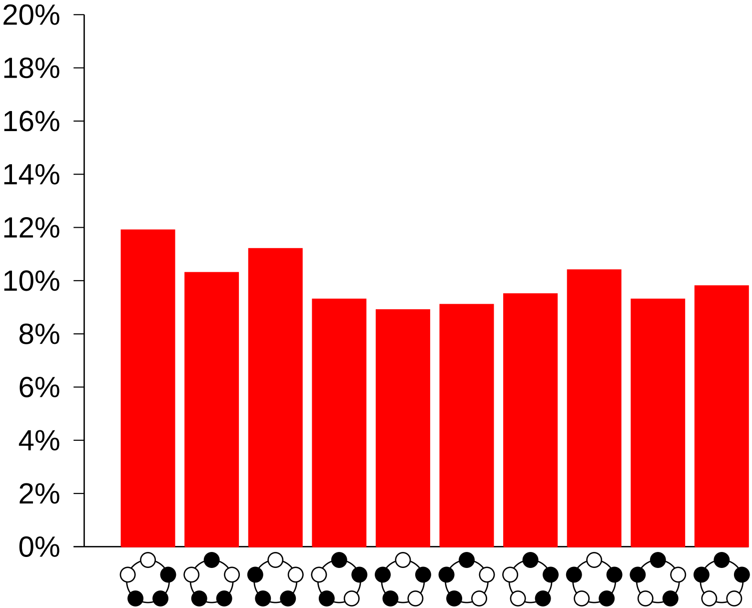

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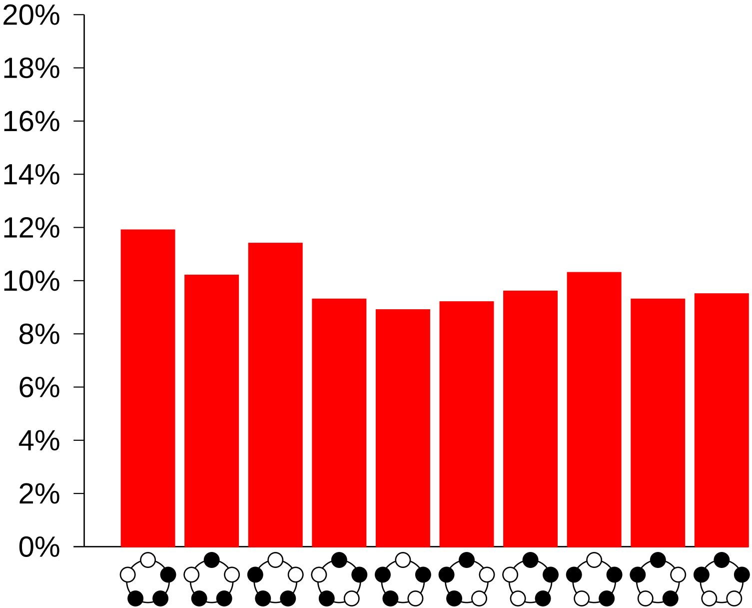

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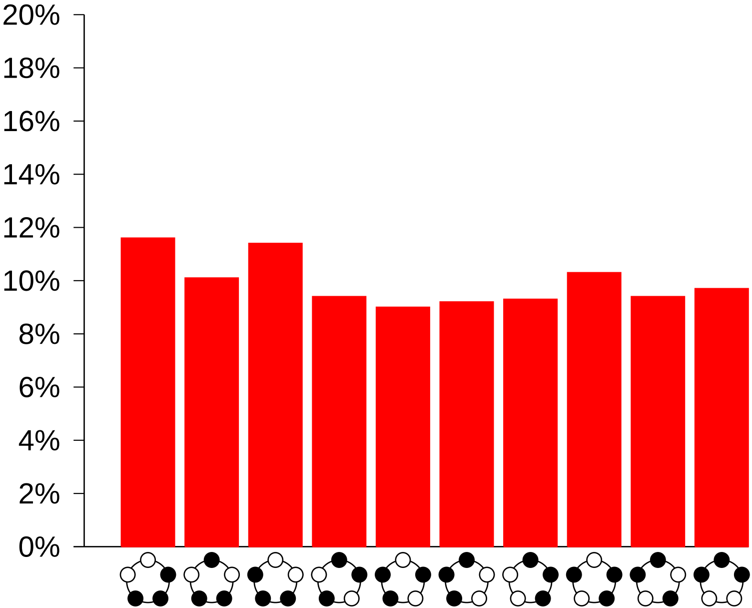

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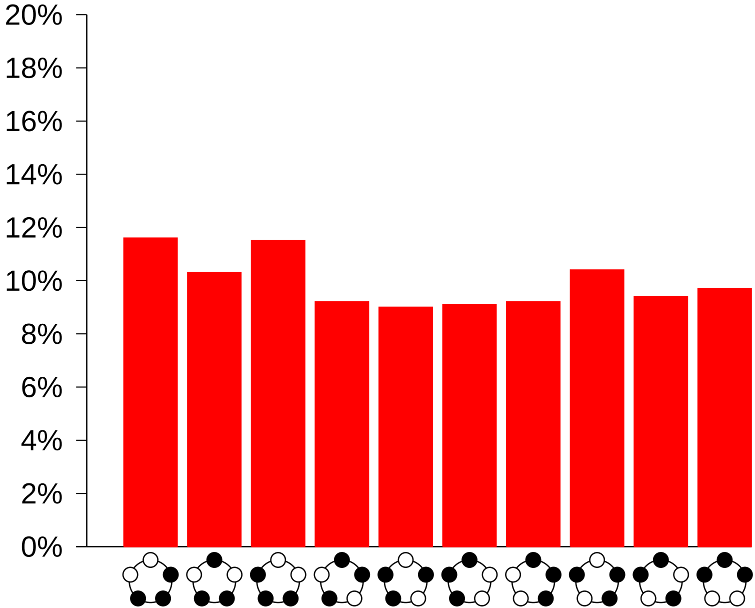

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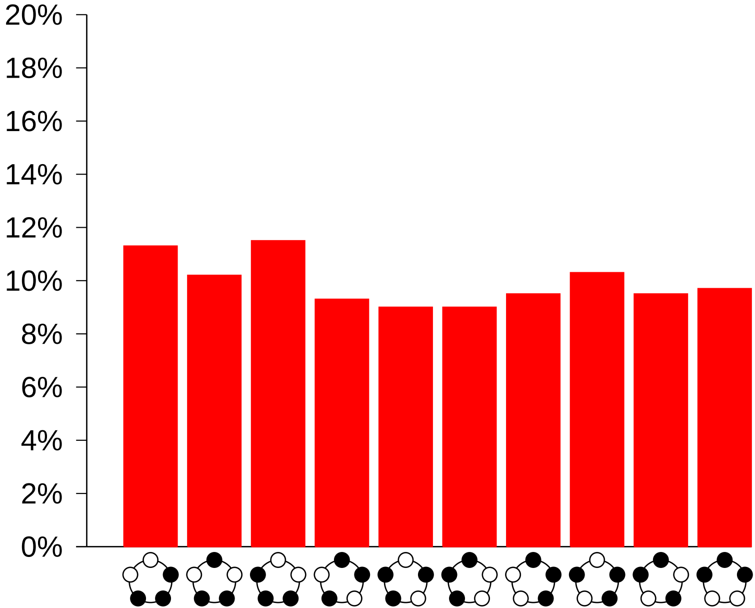

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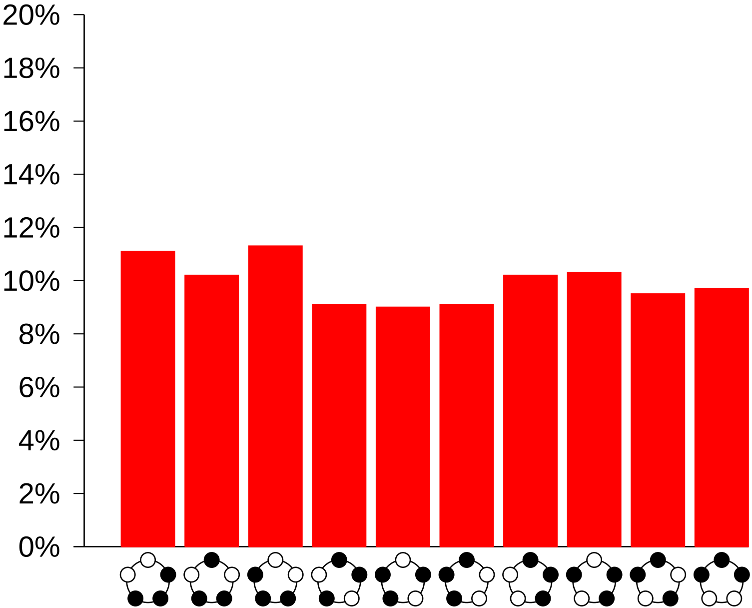

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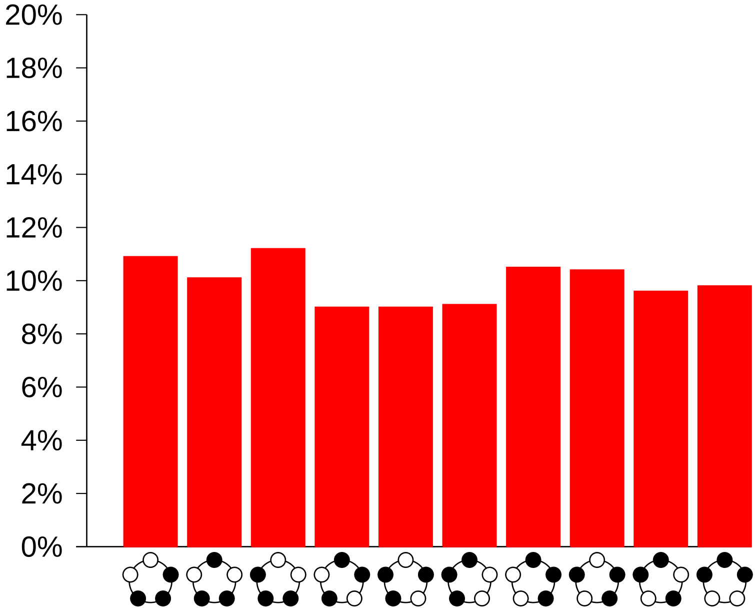

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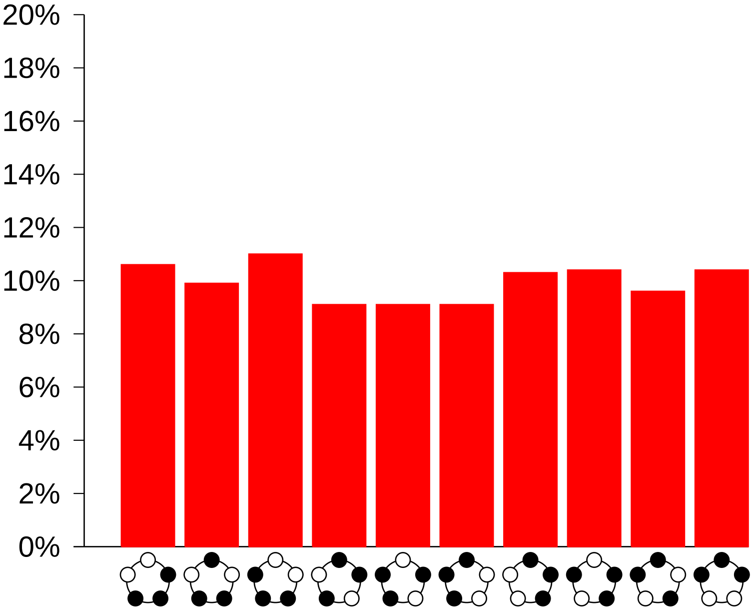

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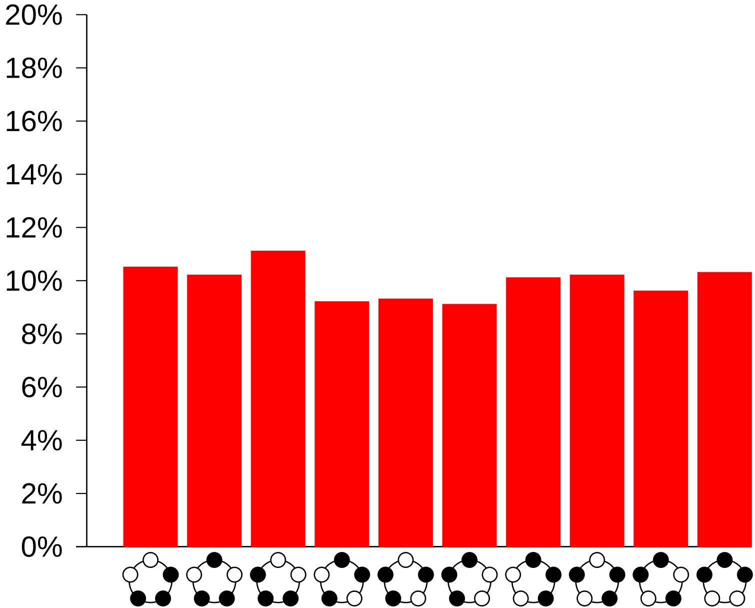

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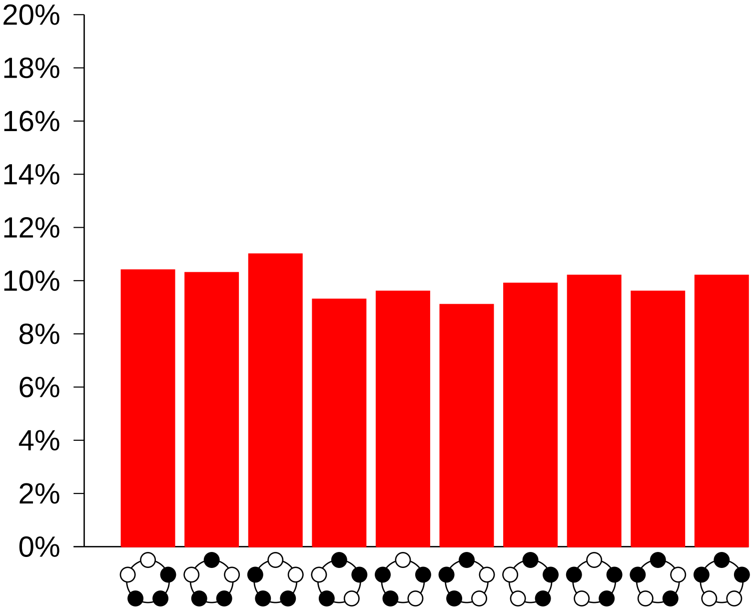

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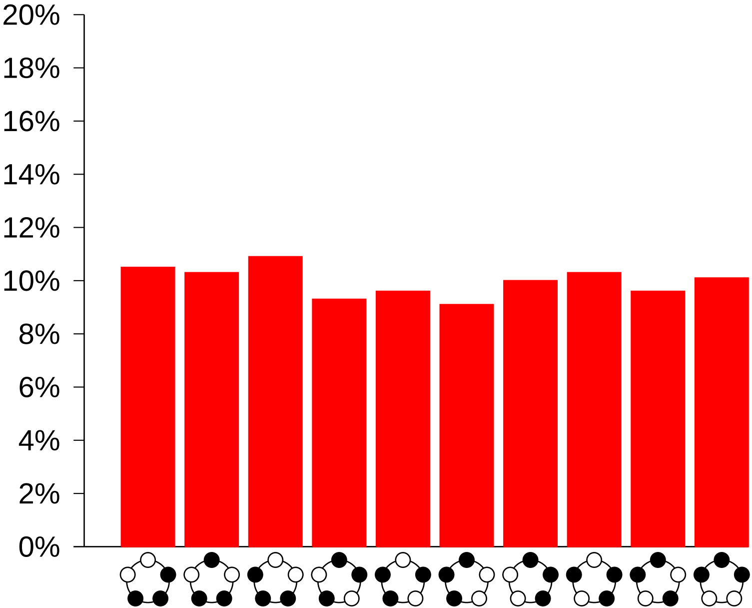

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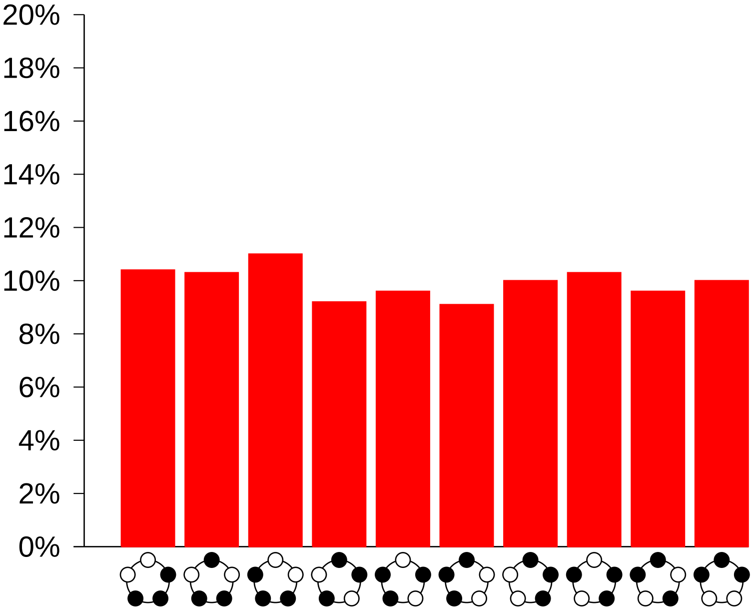

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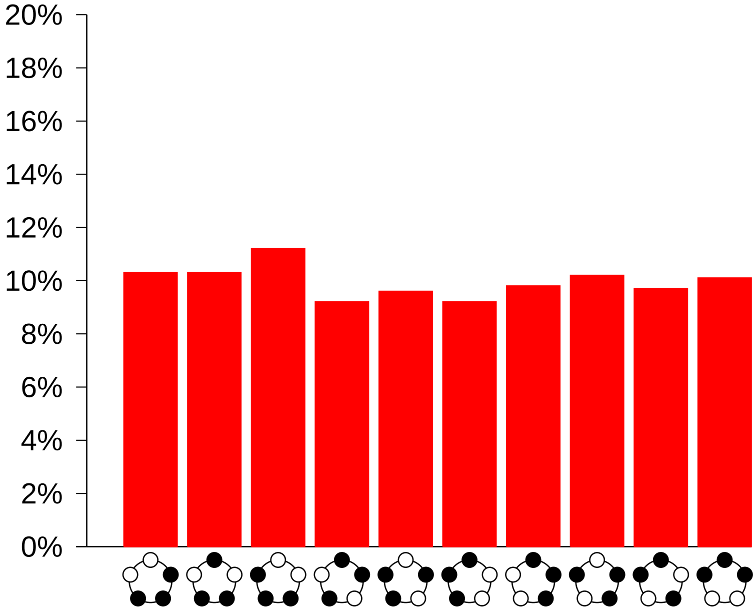

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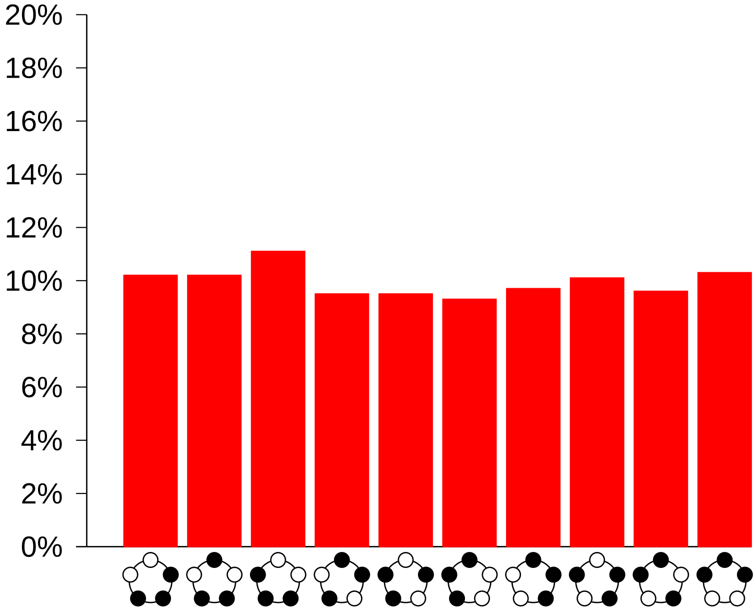

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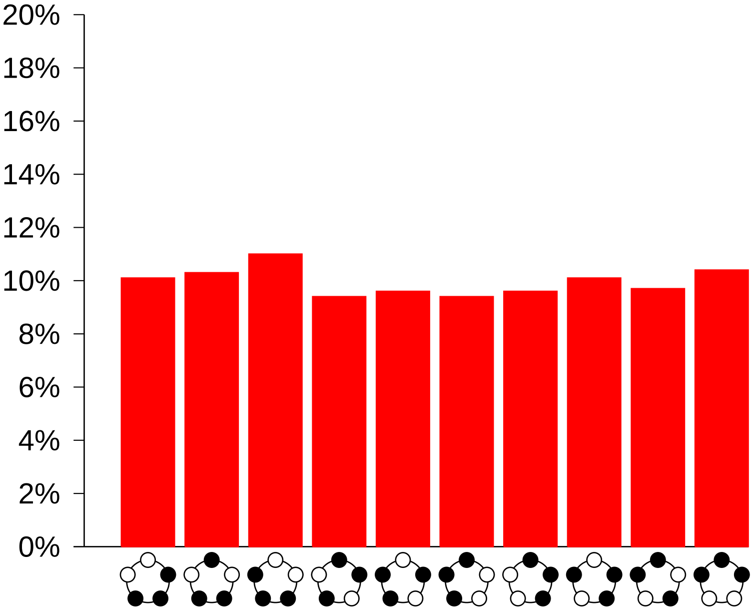

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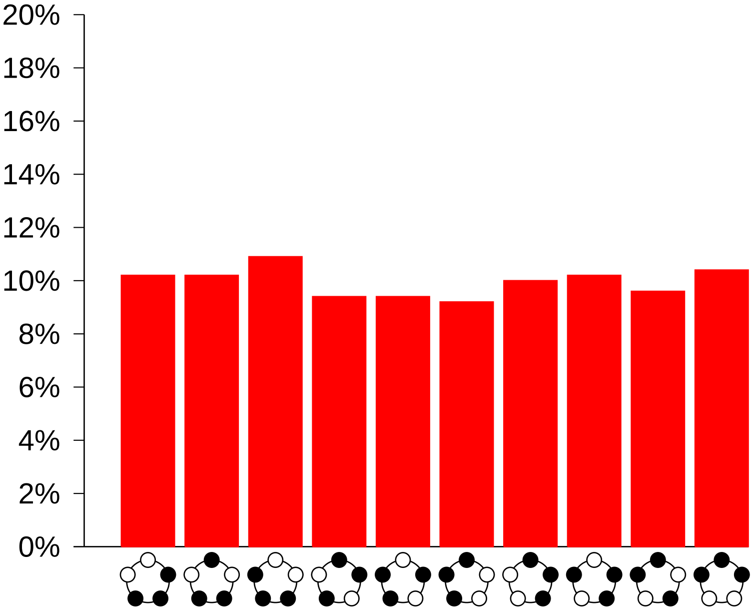

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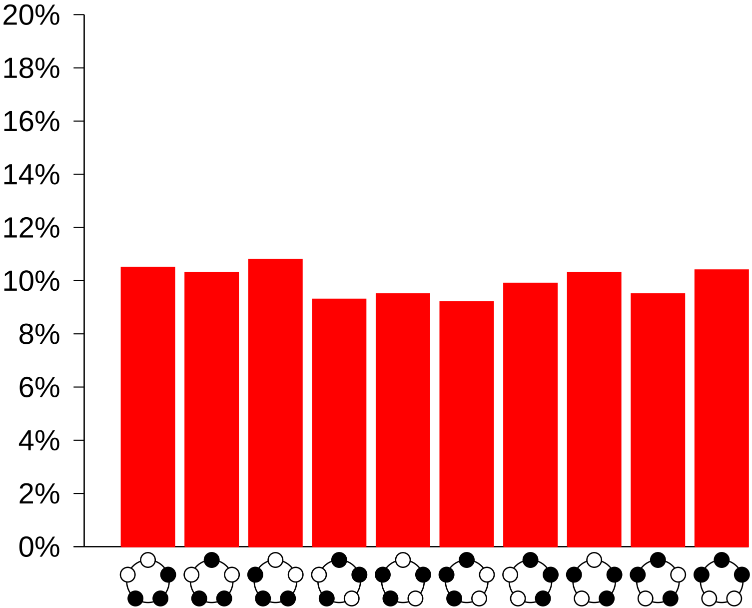

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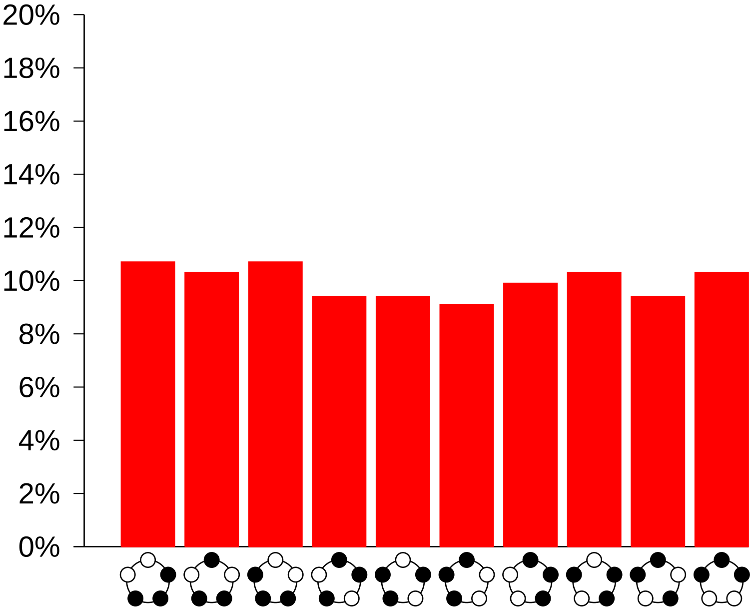

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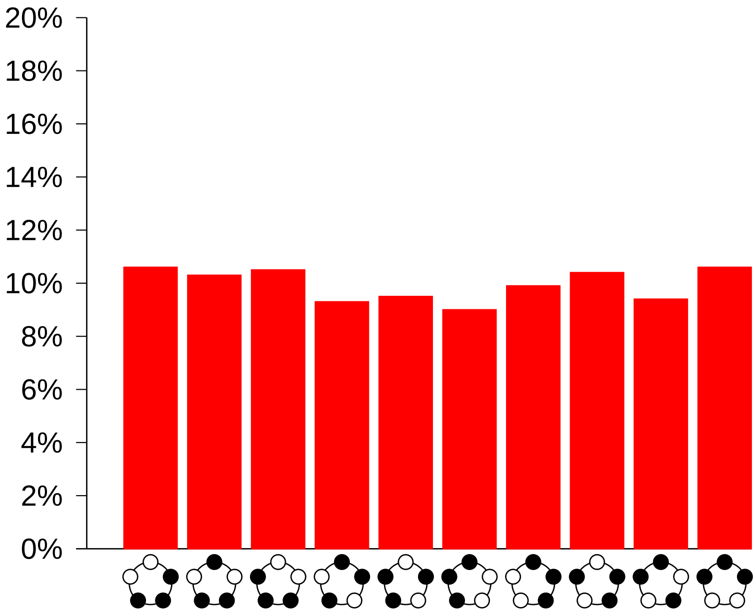

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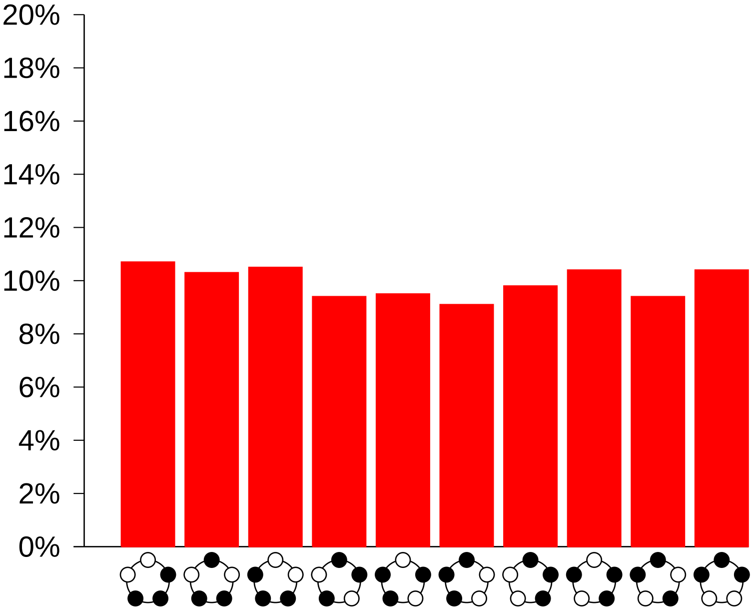

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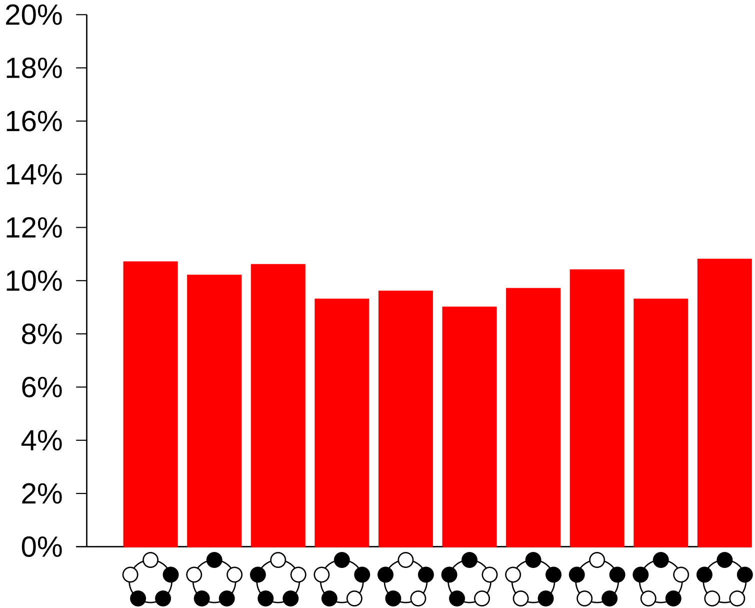

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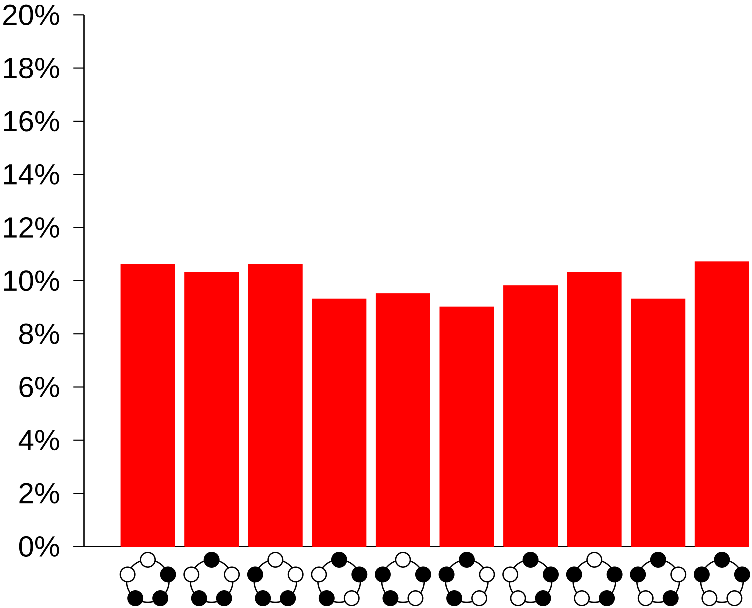

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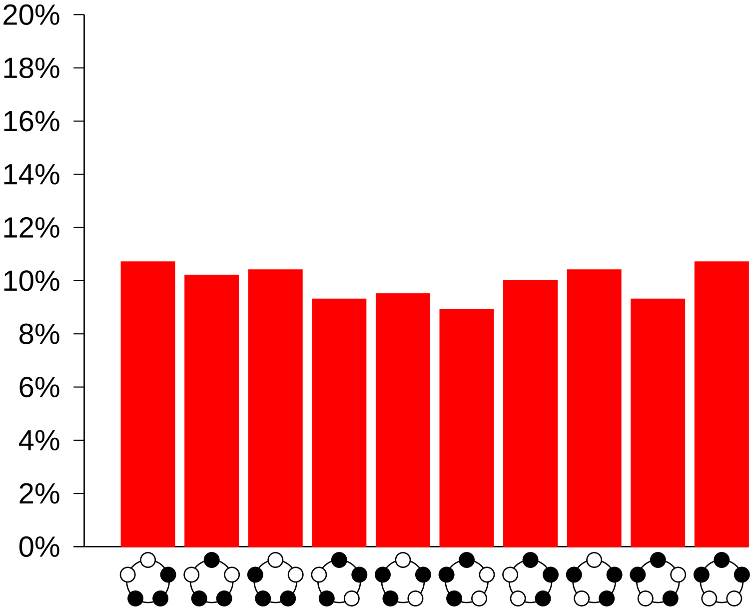

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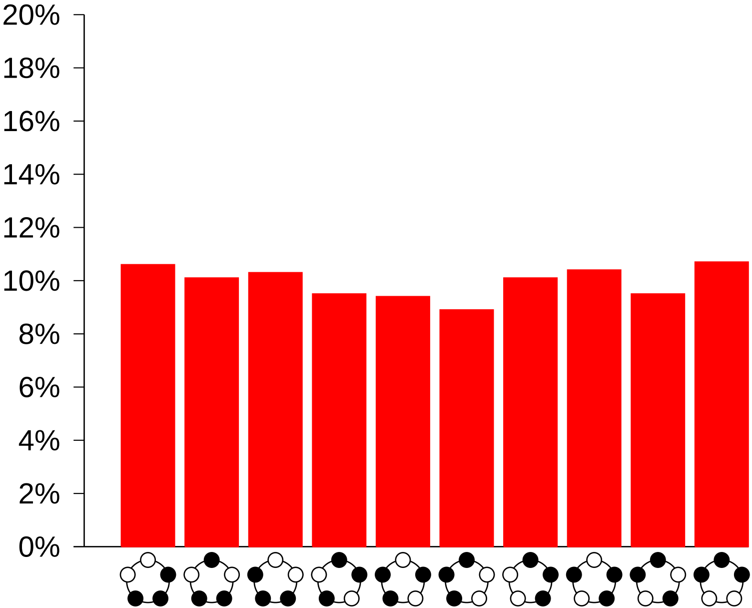

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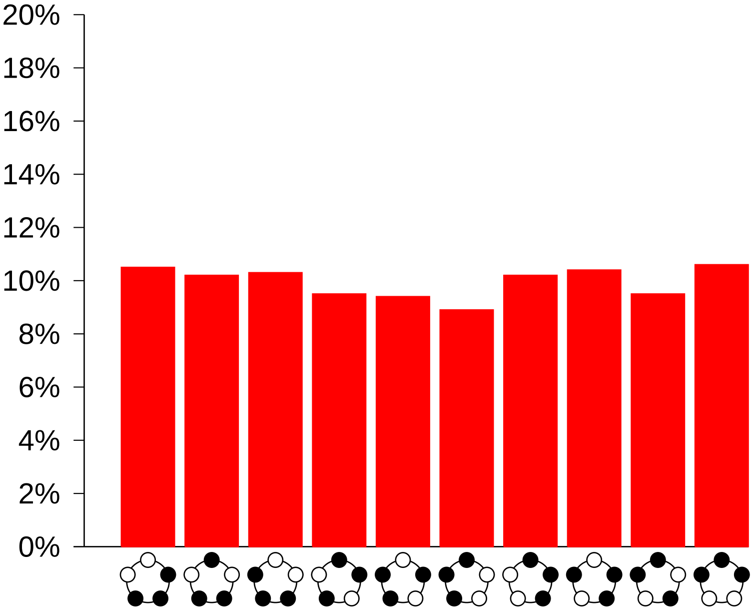

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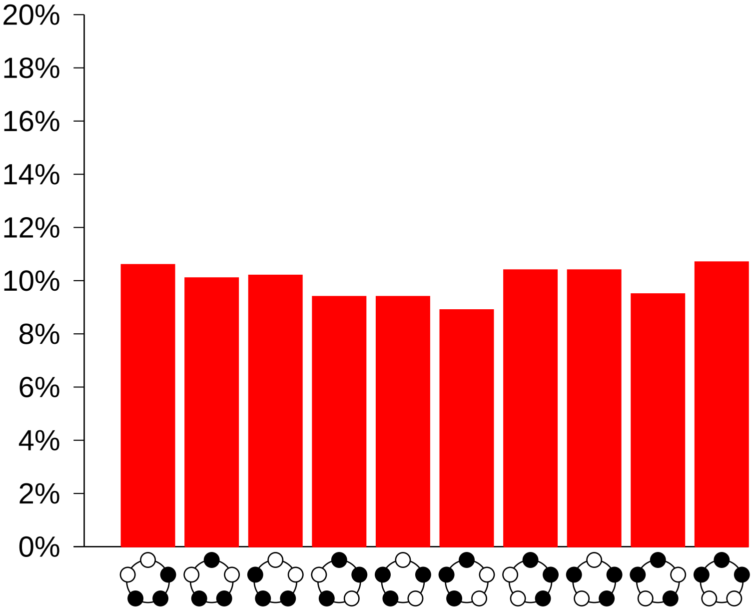

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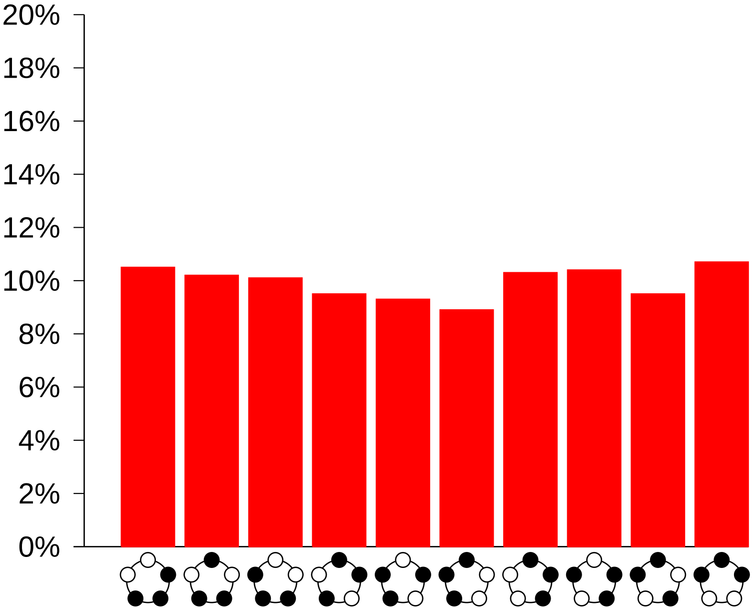

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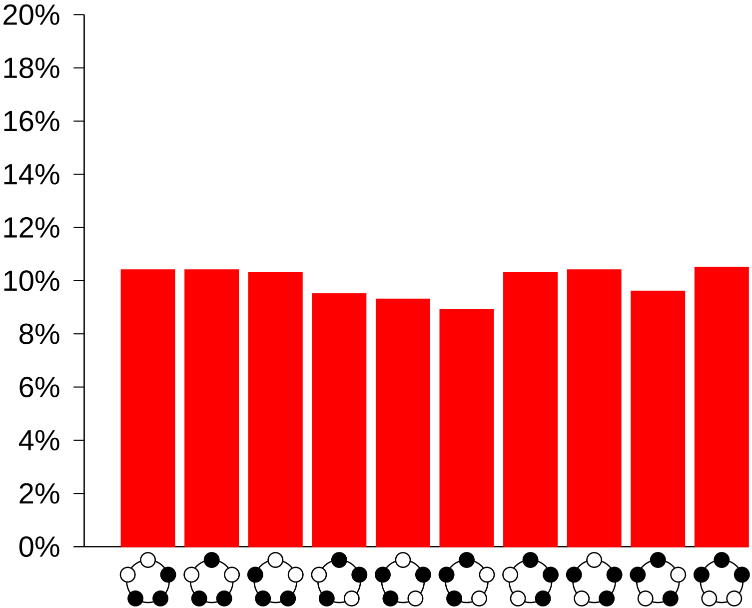

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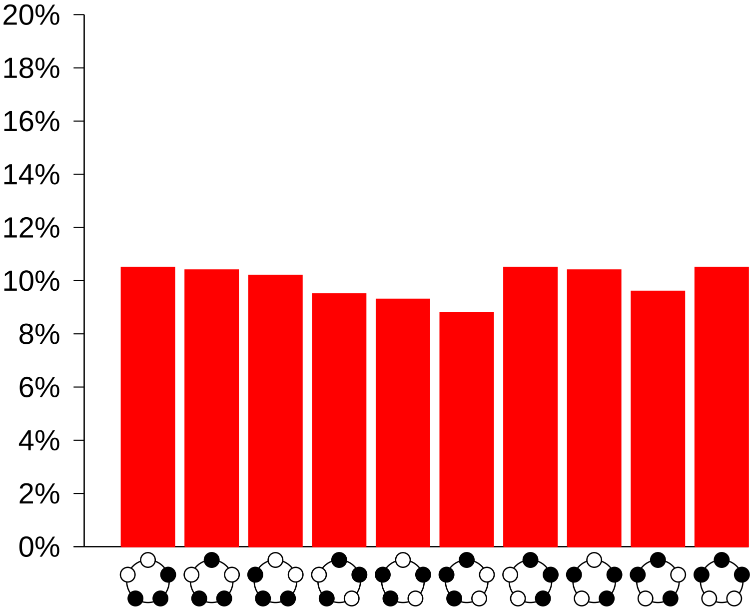

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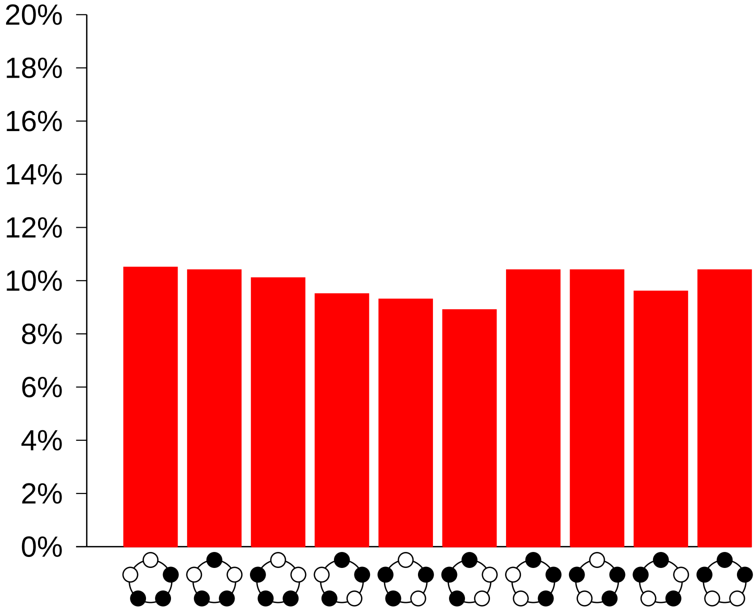

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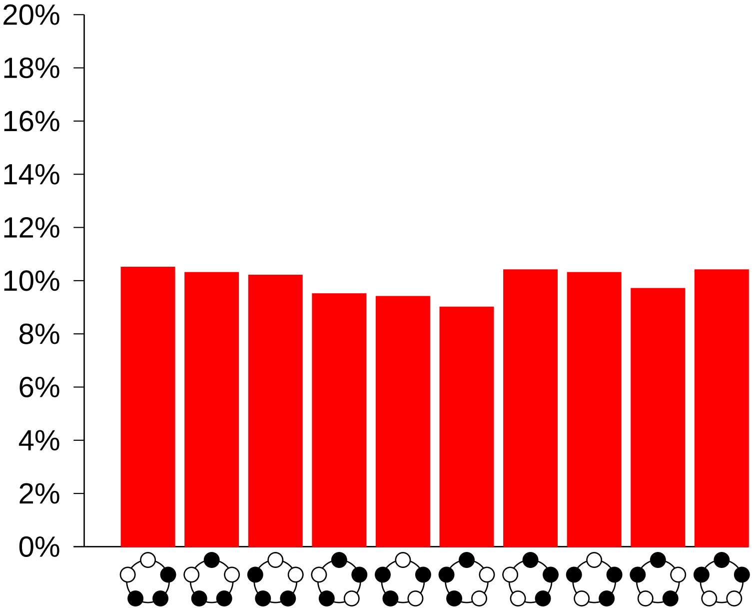

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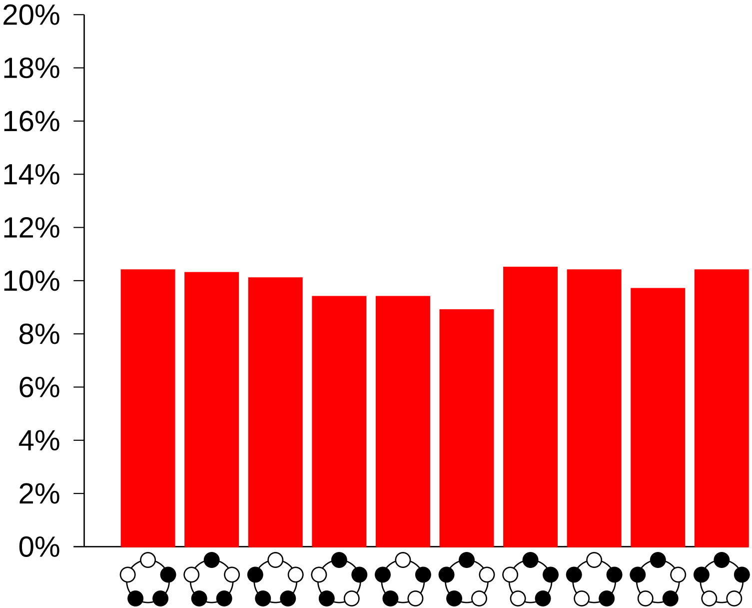

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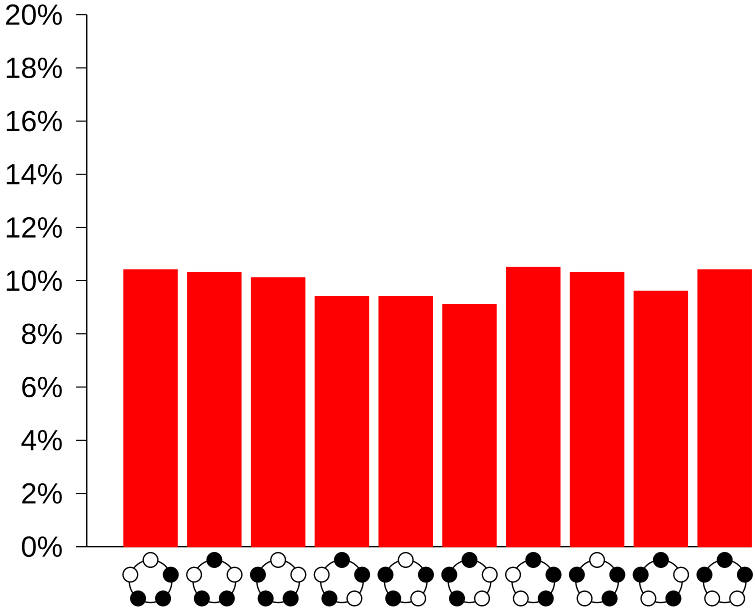

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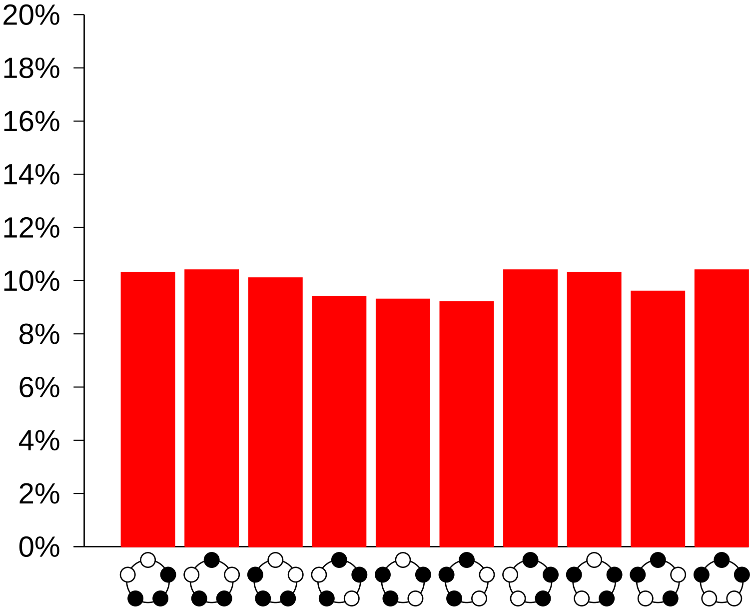

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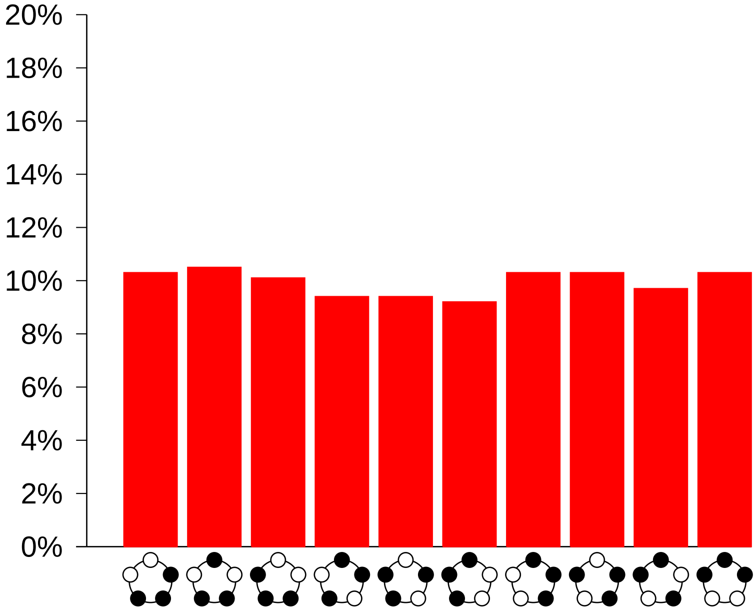

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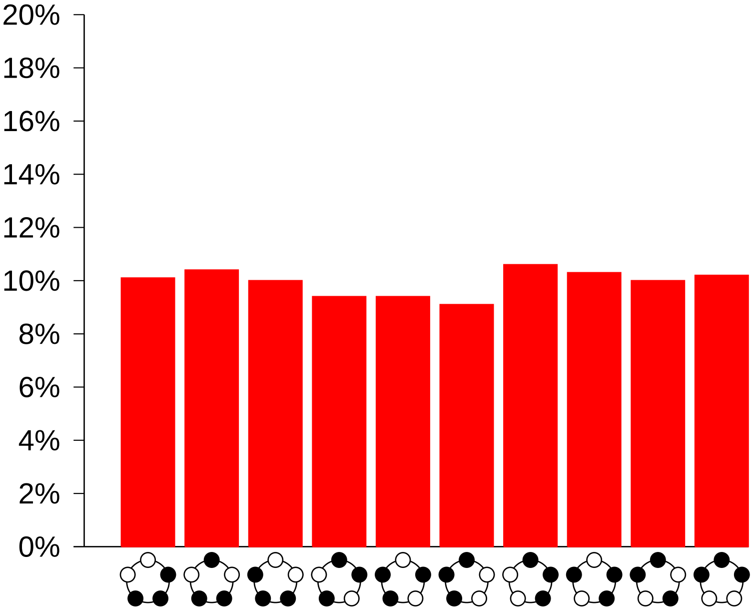

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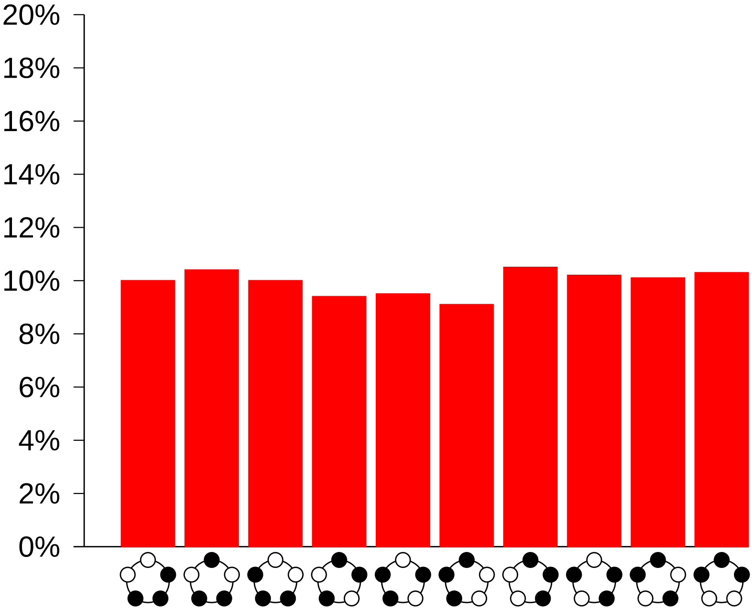

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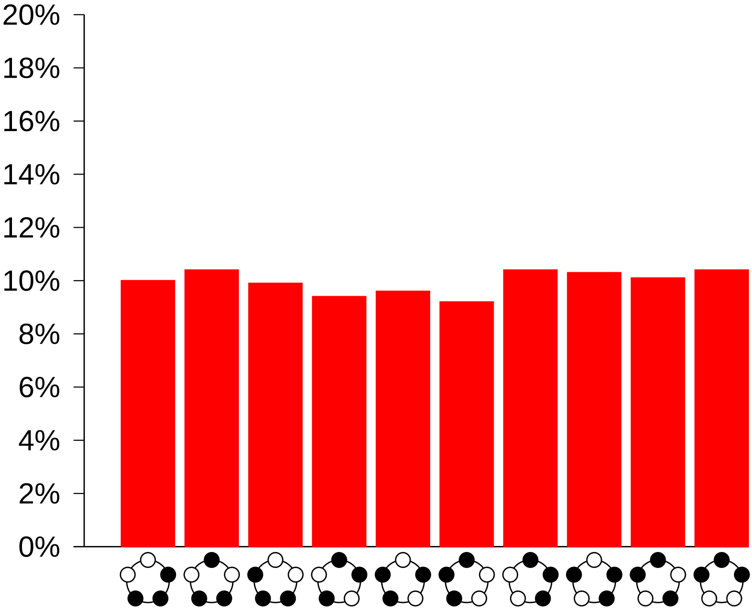

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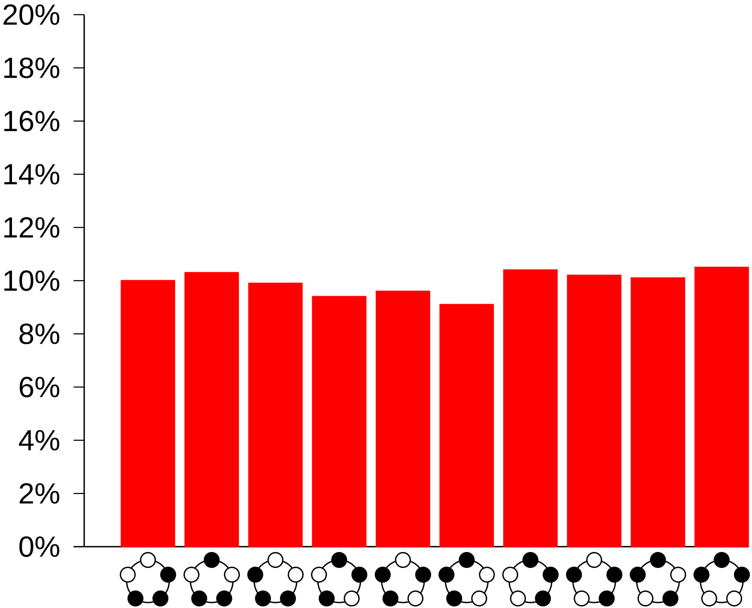

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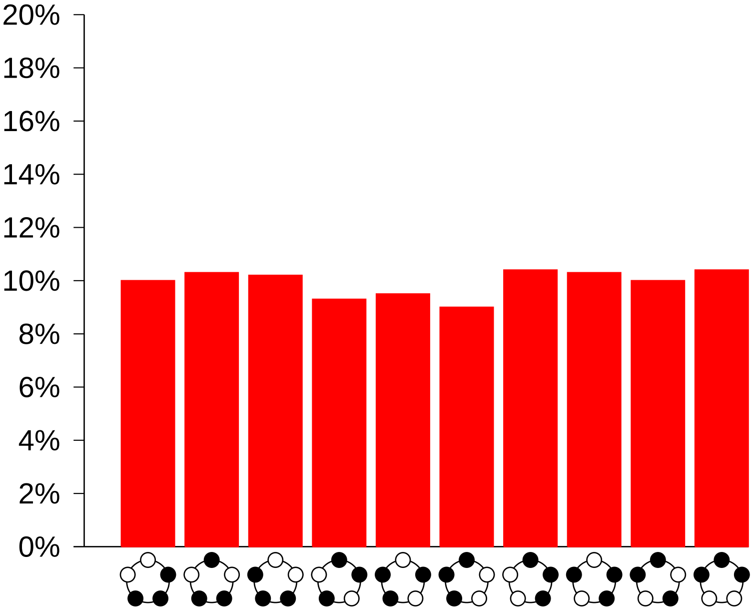

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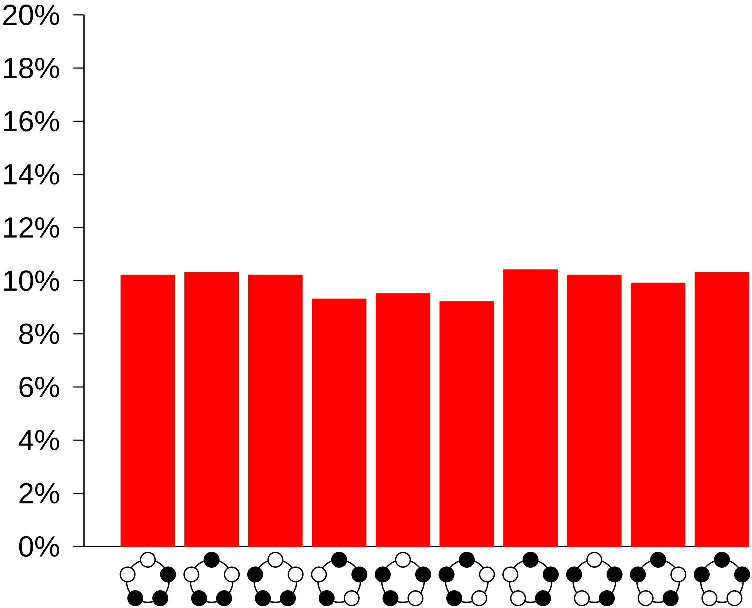

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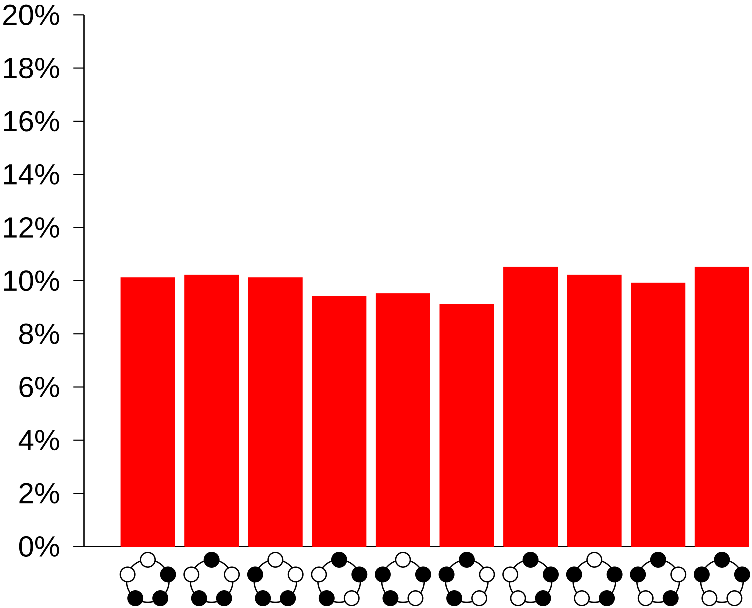

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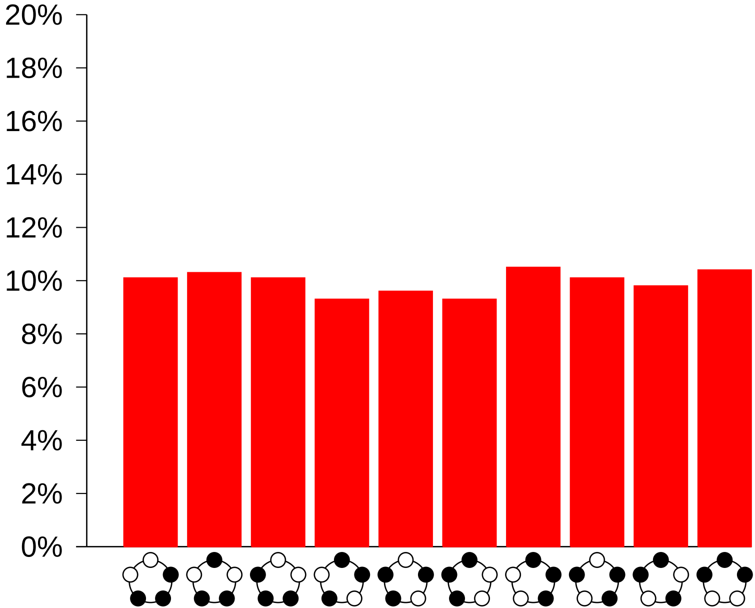

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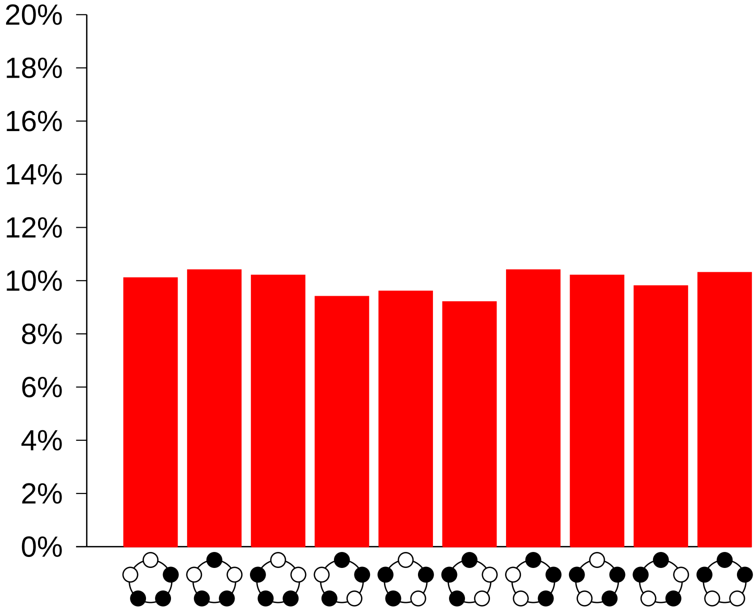

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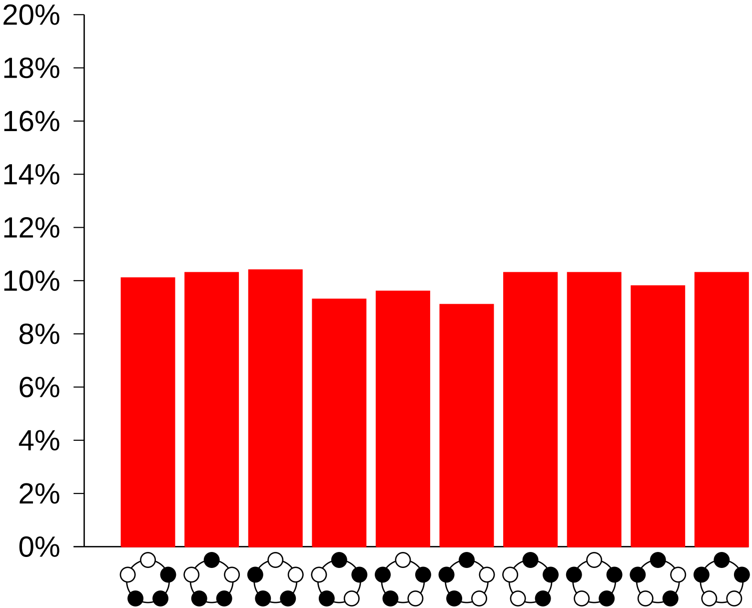

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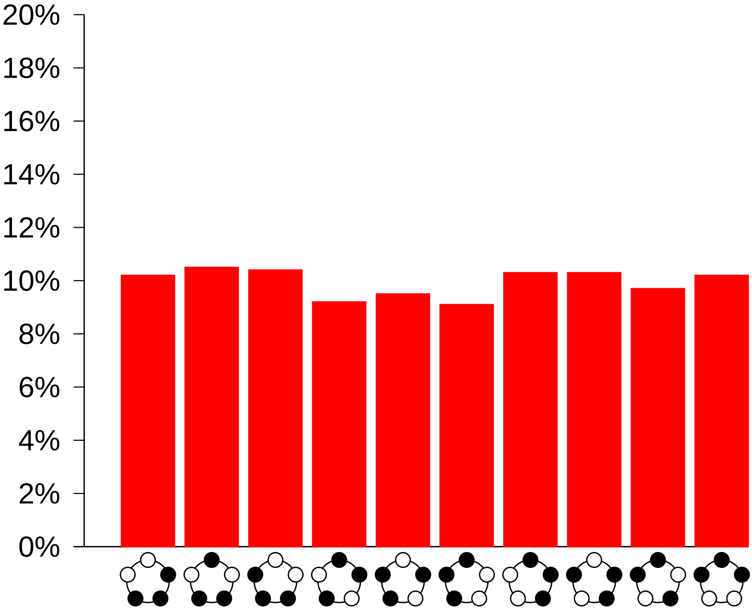

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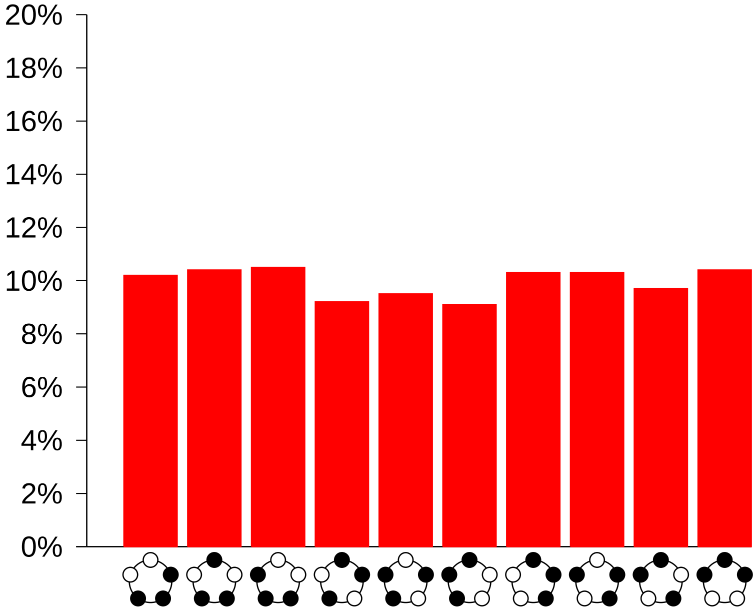

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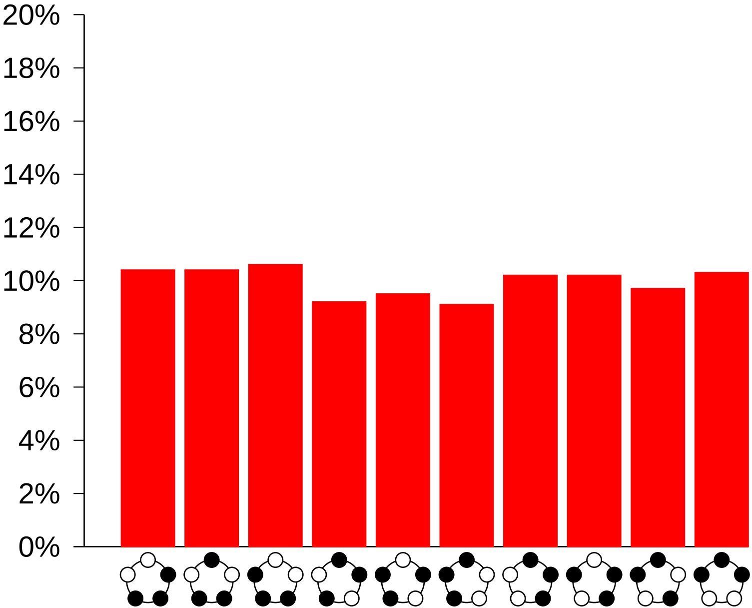

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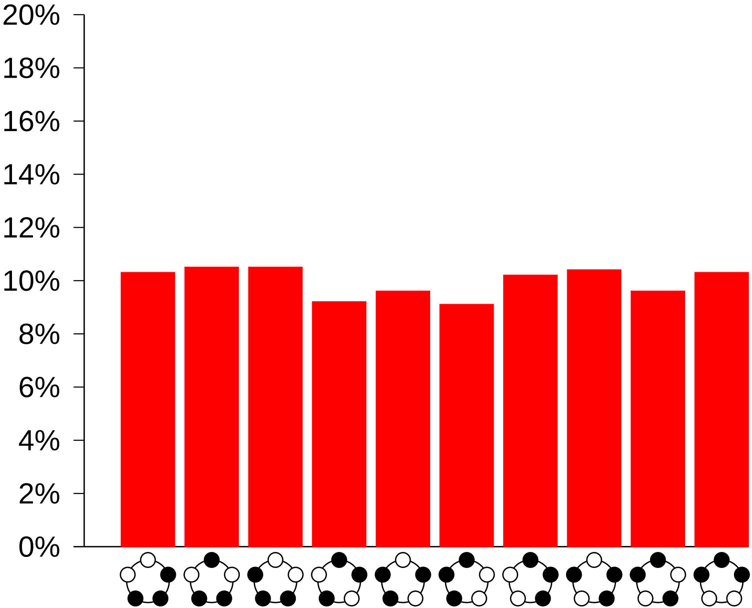

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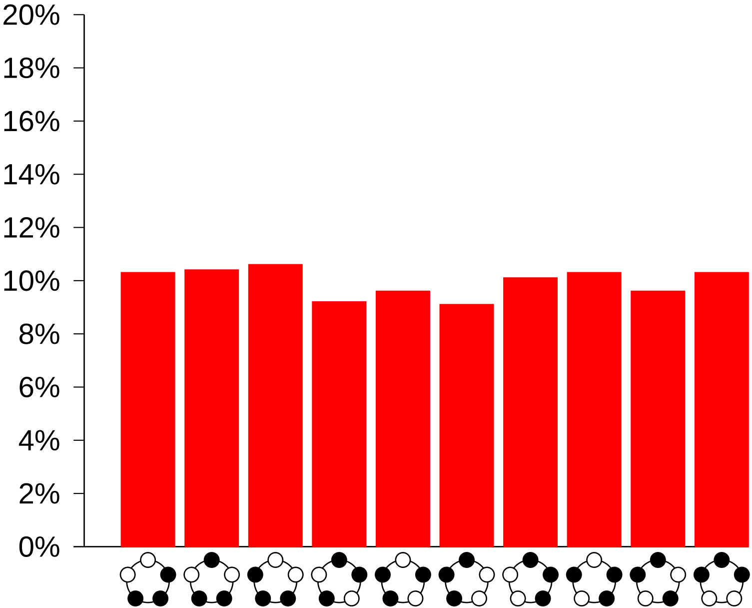

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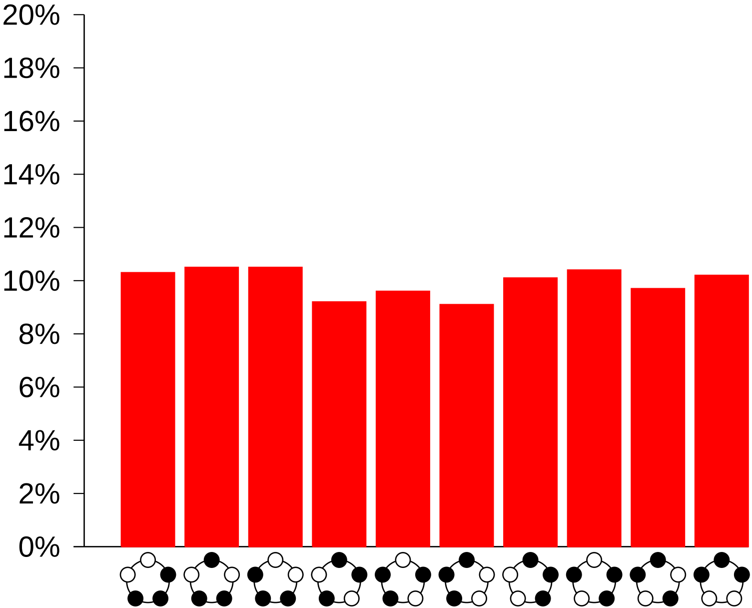

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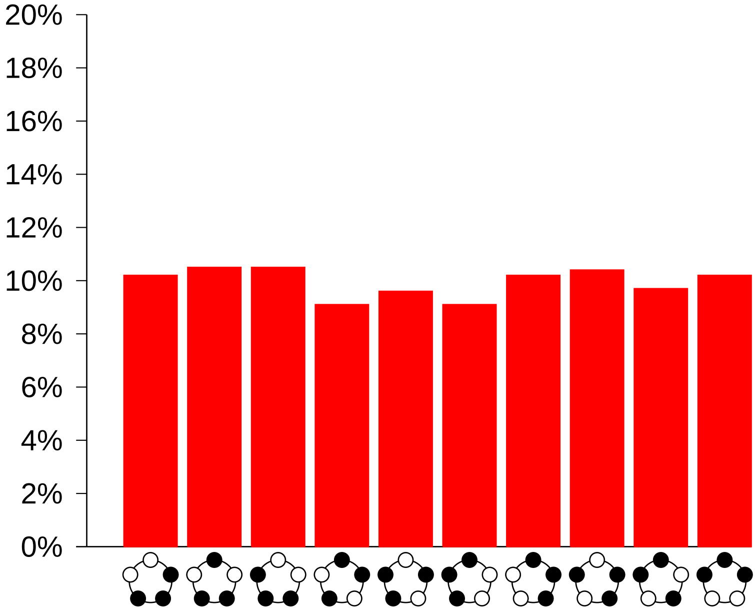

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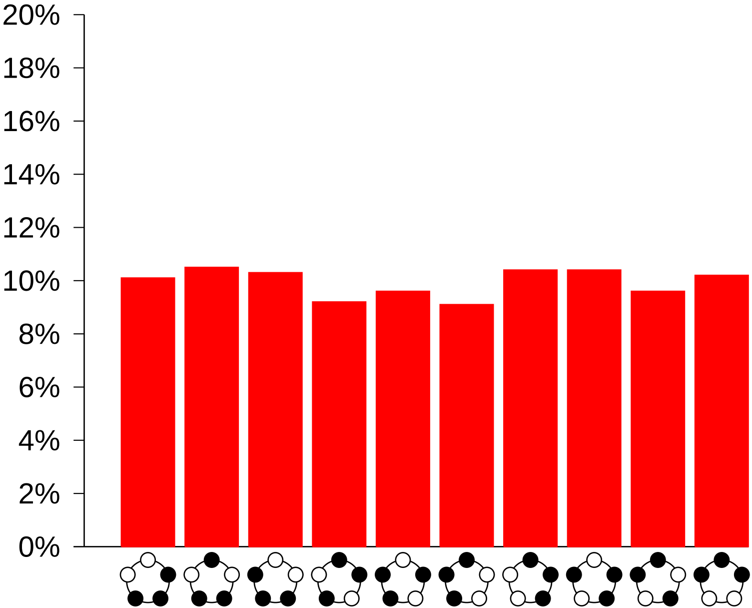

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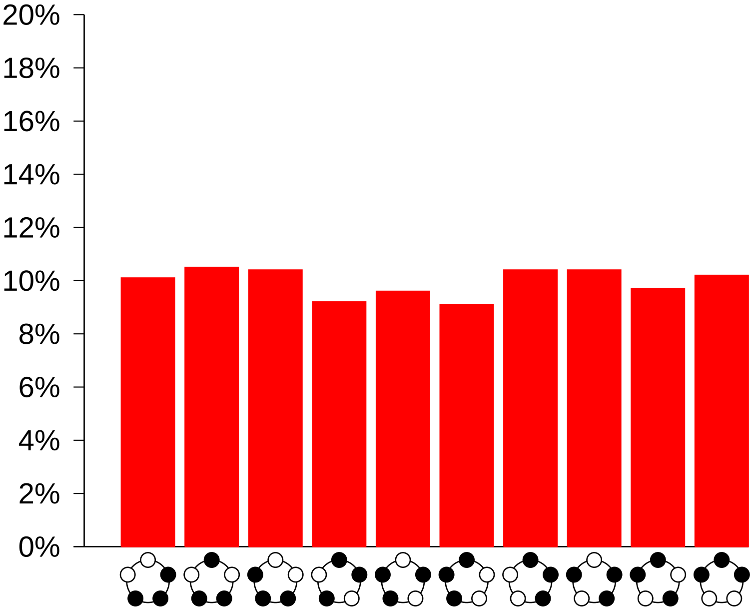

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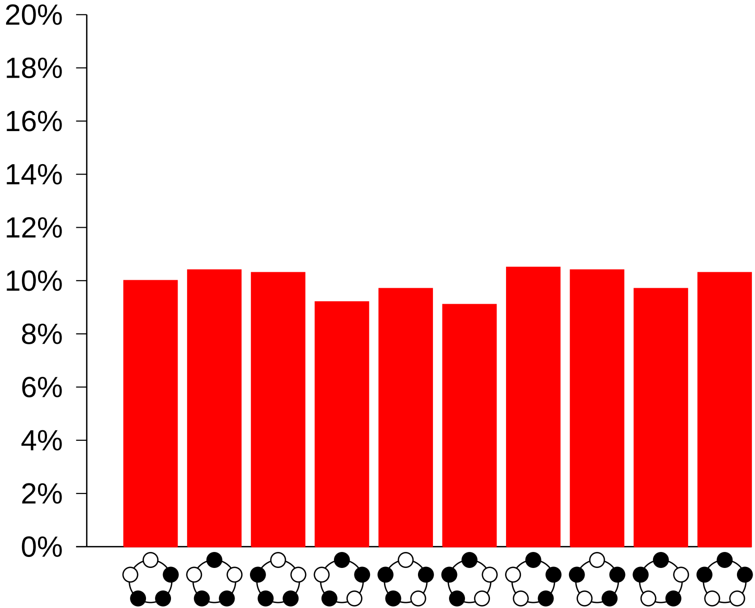

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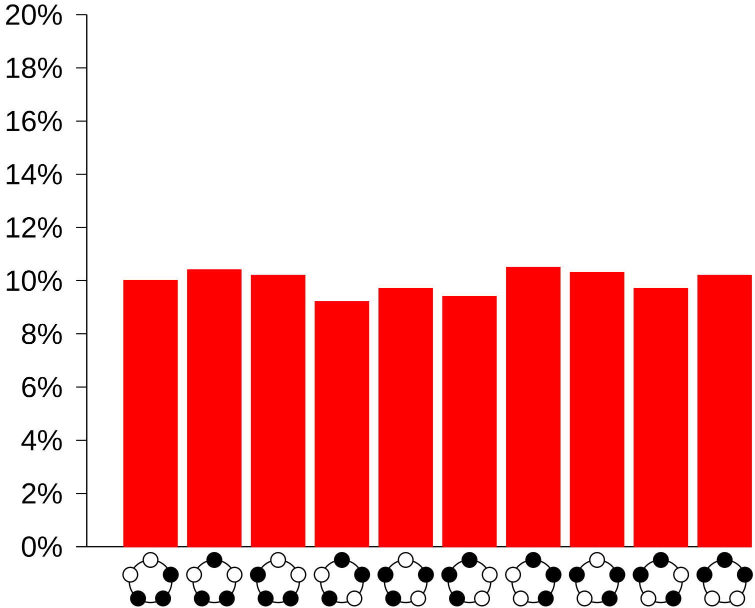

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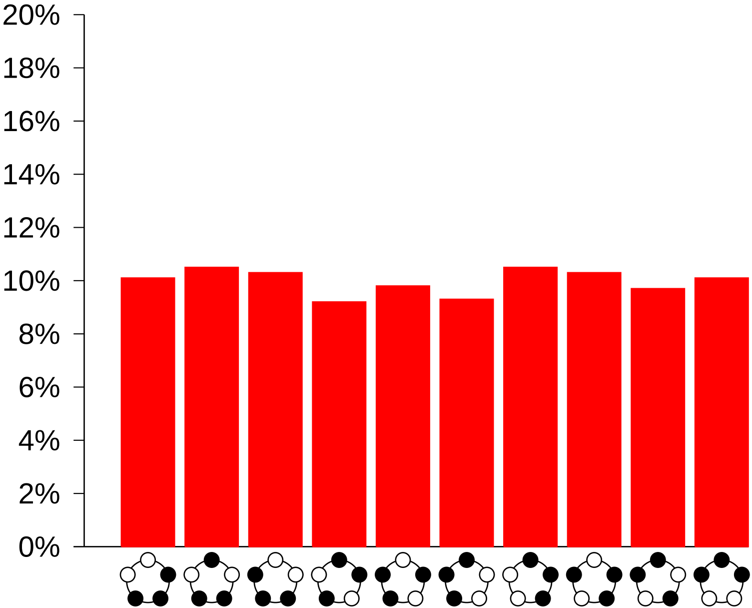

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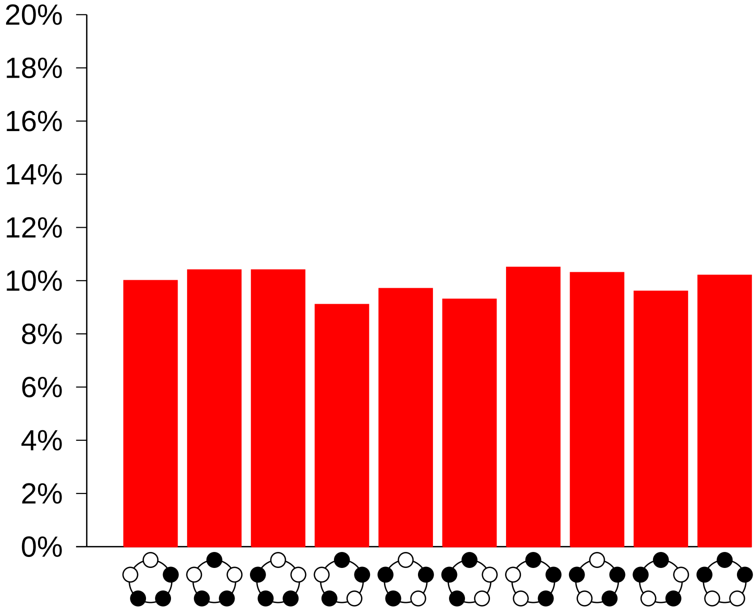

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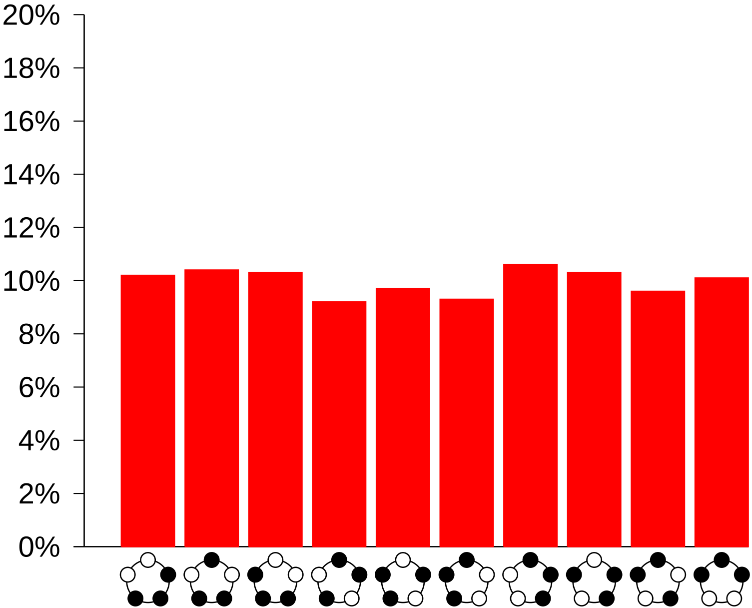
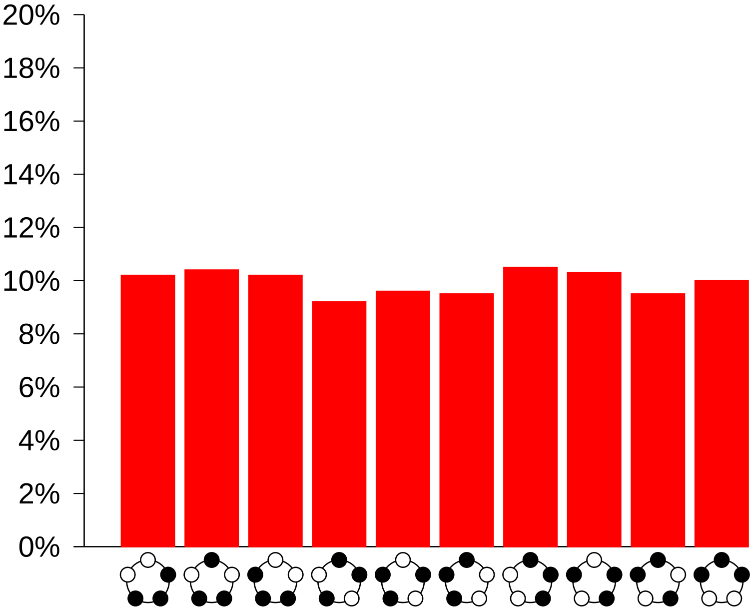


Stationary distribution



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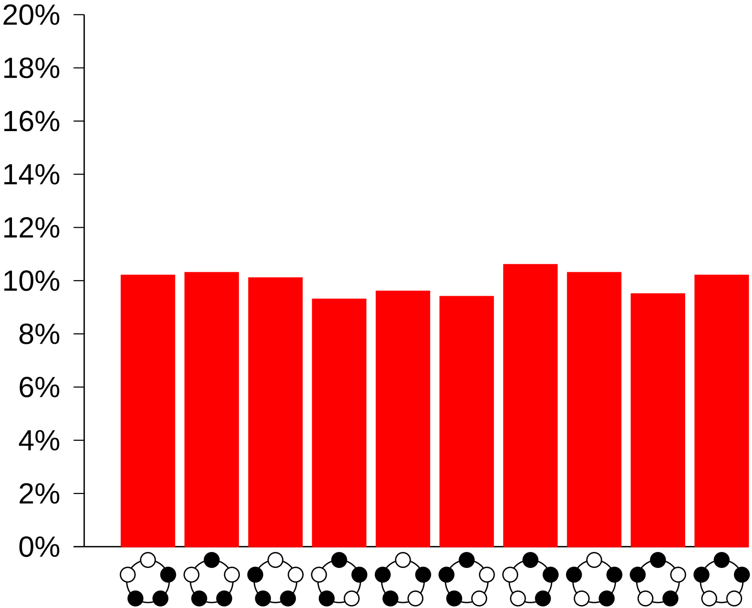

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Etc. In the limit, on $\mathbb{Z}$: ball with probability ϱ, no ball with probability $1-\varrho$, independently for each slot.

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(*Burgers eq.*).

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The characteristic velocity: $\dot{X}(T) = 1 - 2\hat{\varrho}$.

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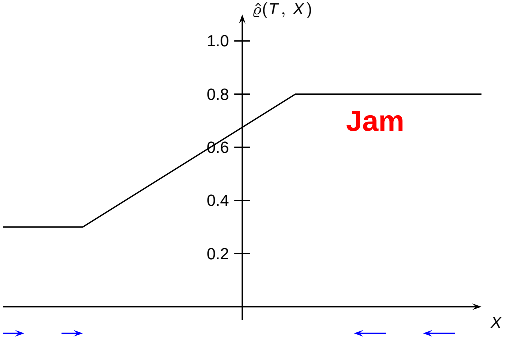

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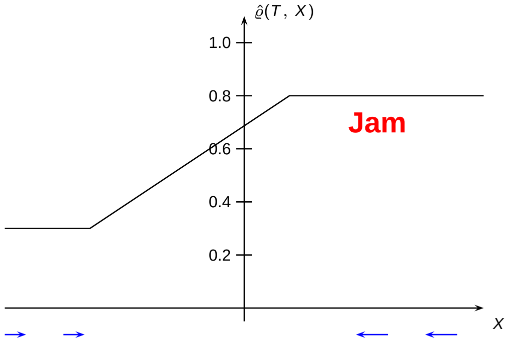

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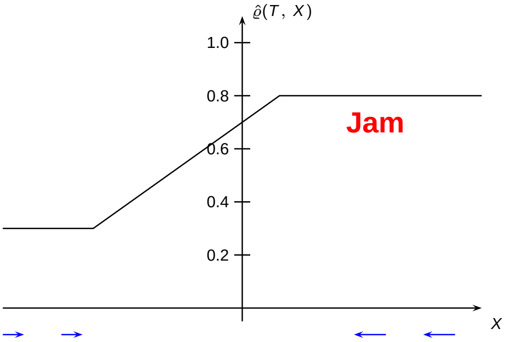

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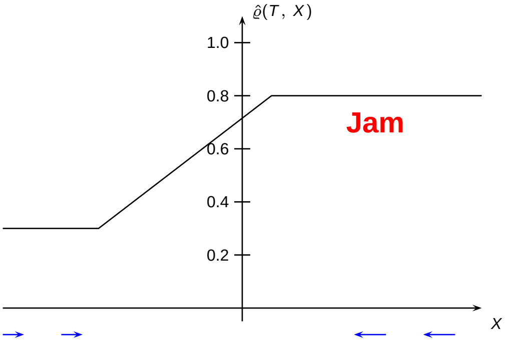

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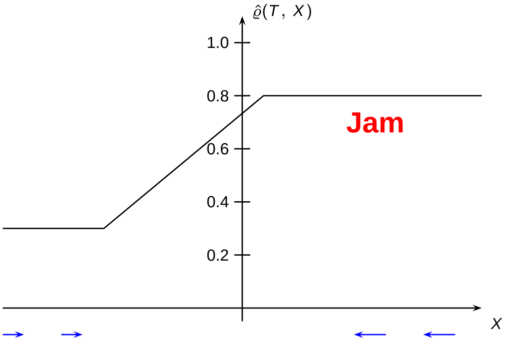

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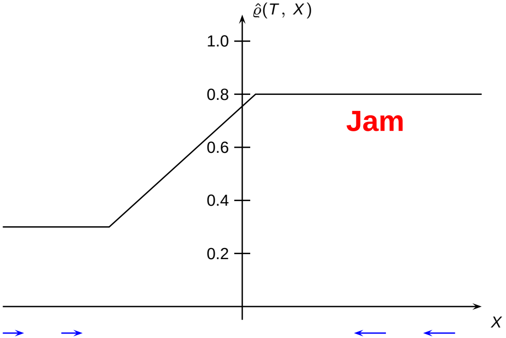

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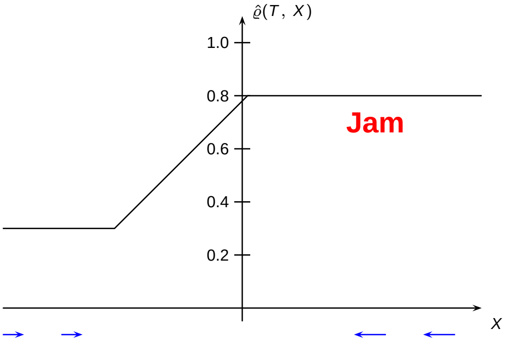

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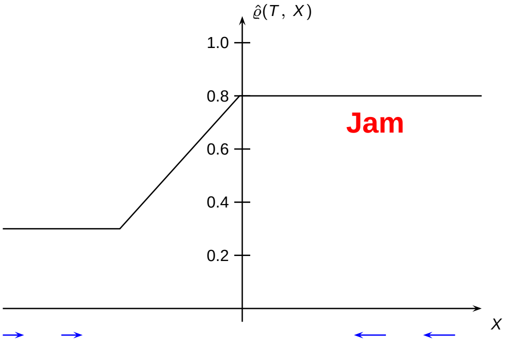

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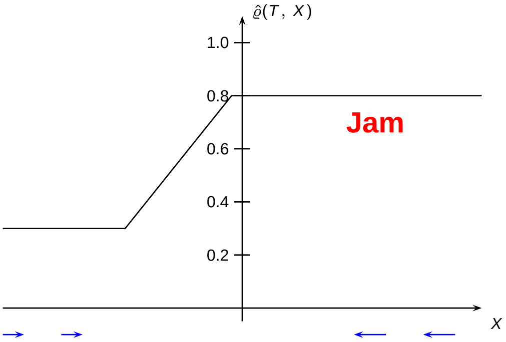

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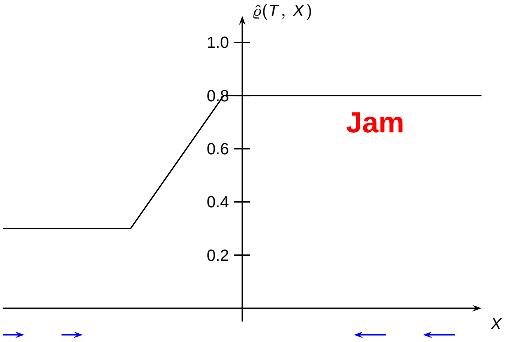

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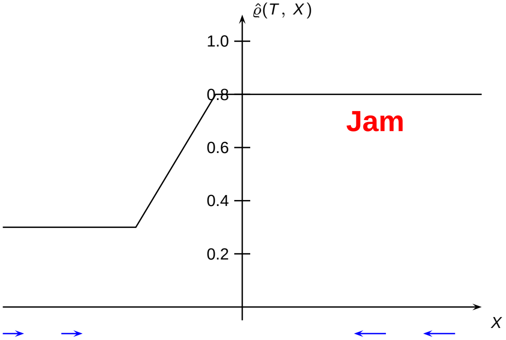

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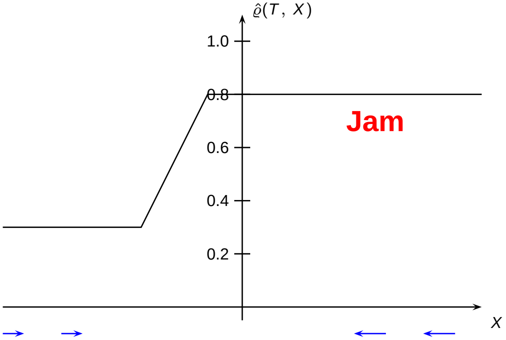

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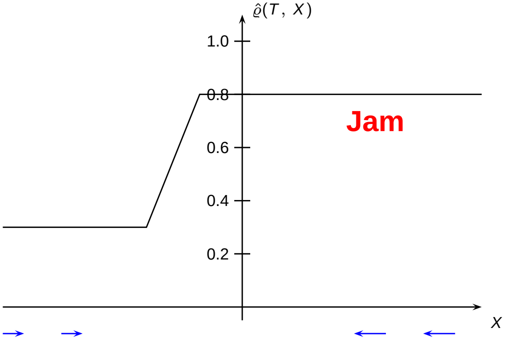

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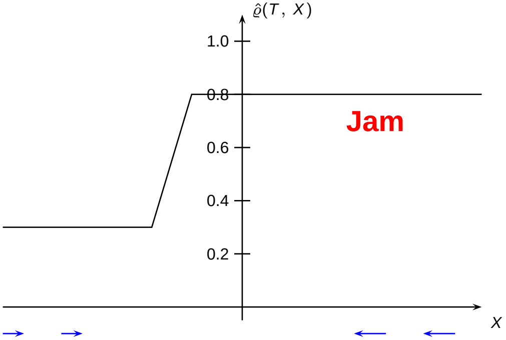

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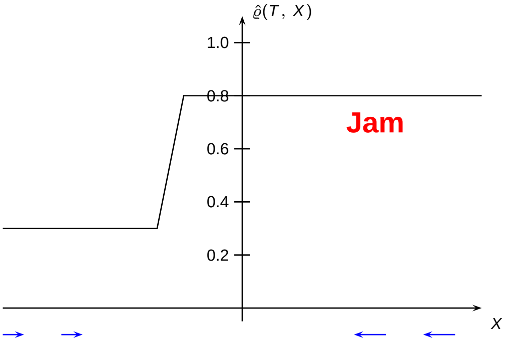

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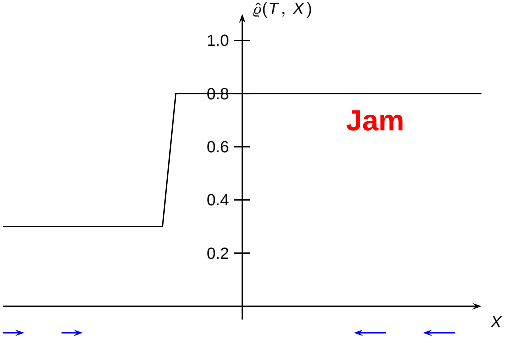

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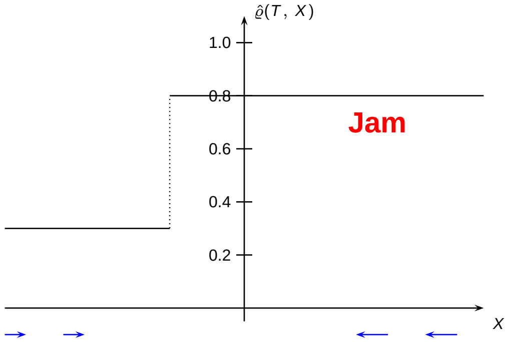

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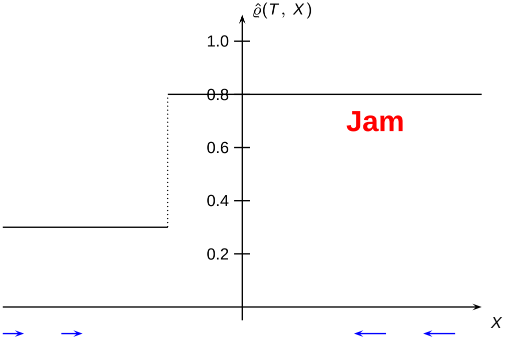

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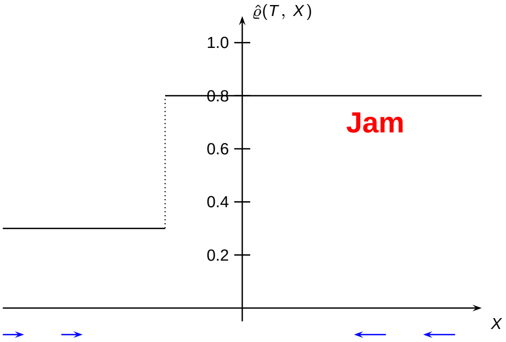

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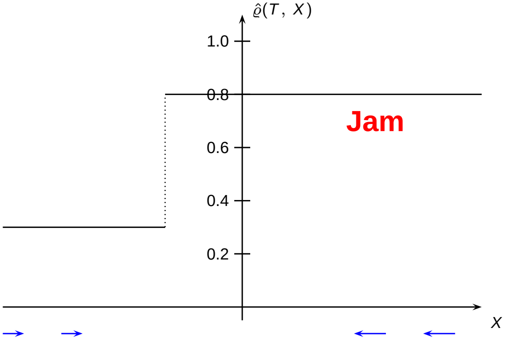

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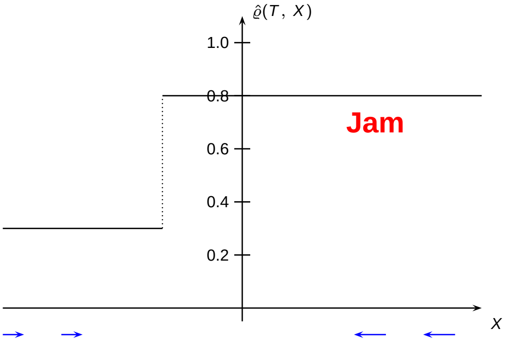

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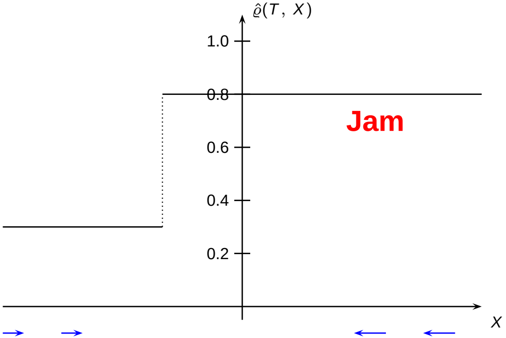

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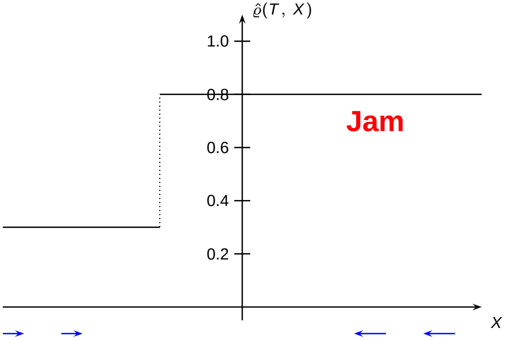

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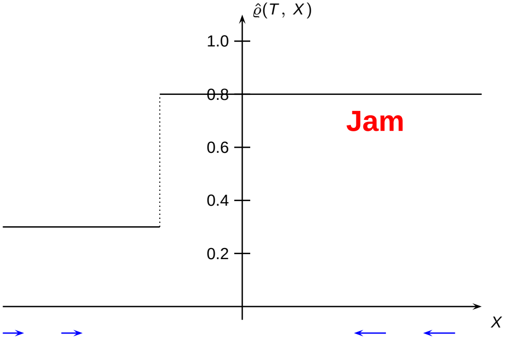

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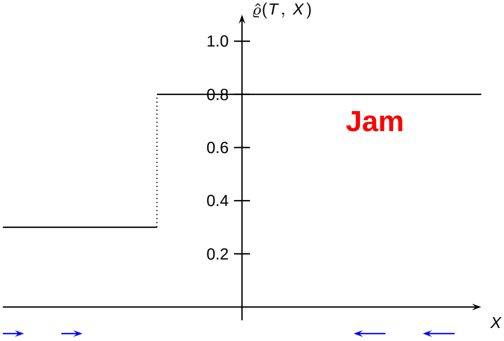

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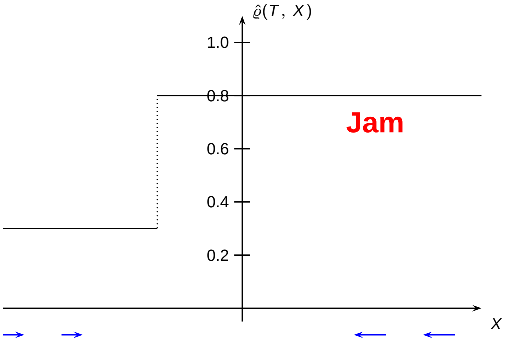

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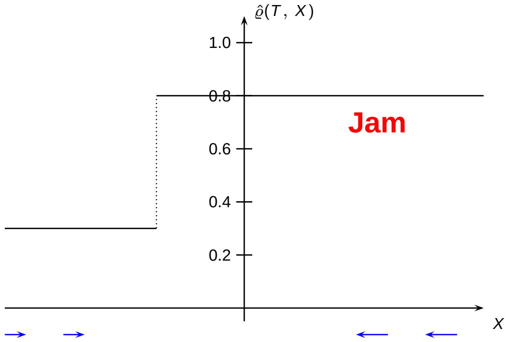

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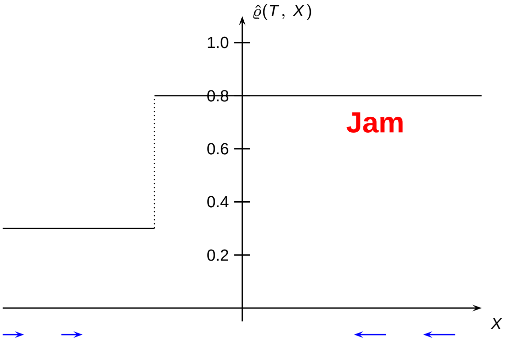

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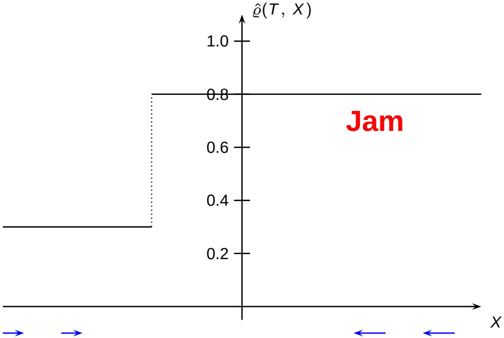

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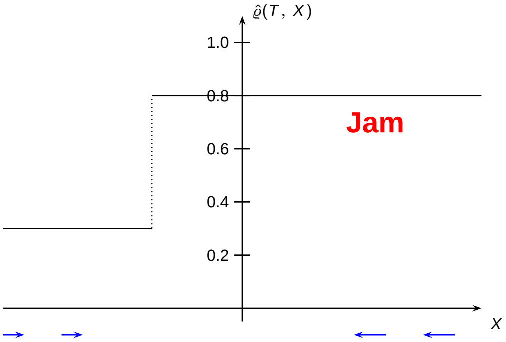

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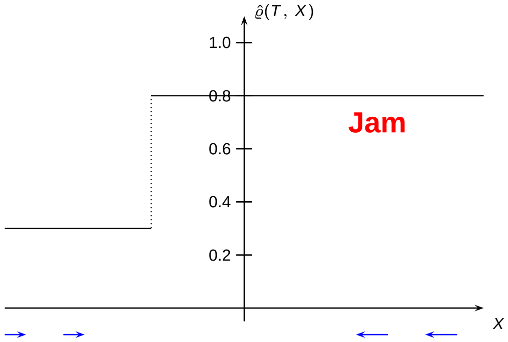

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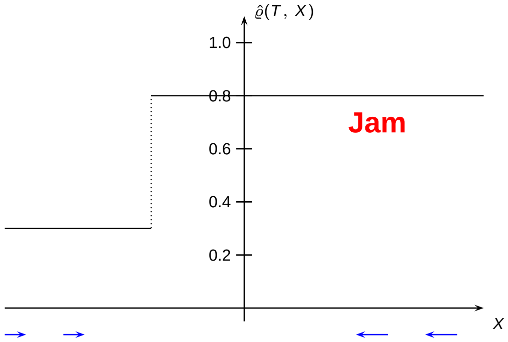

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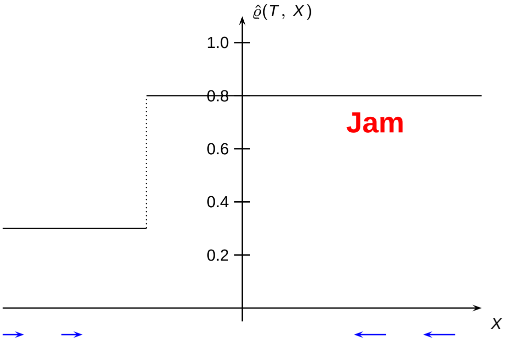

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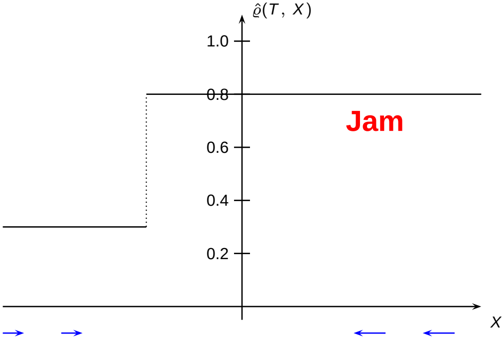

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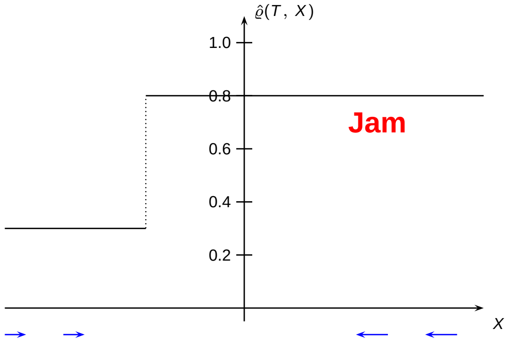

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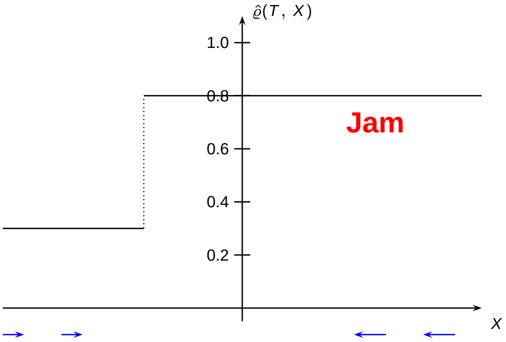

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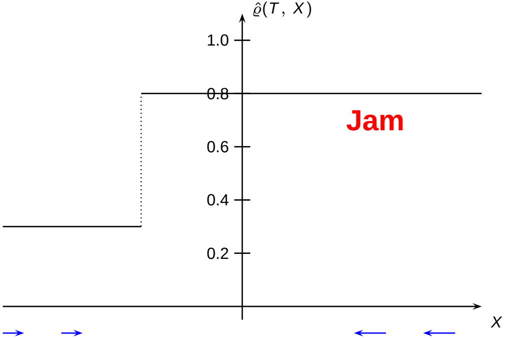

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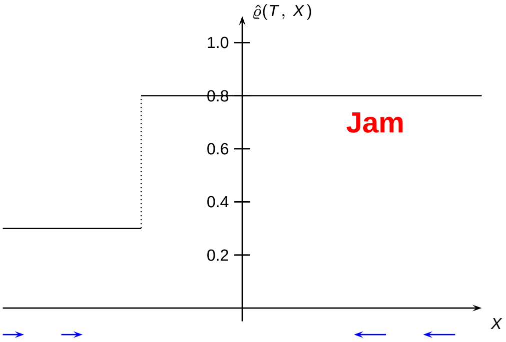
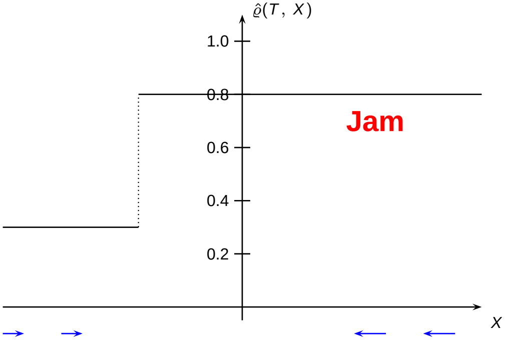


$$\dot{X}(T) = 1 - 2\hat{\varrho}$$

On large scales



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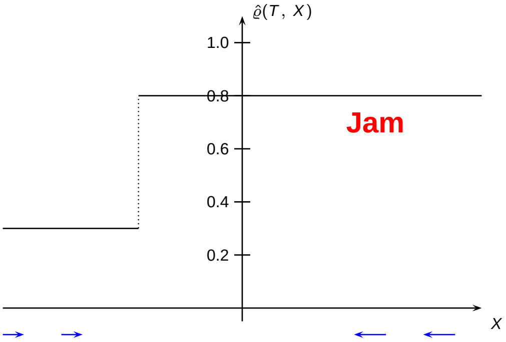

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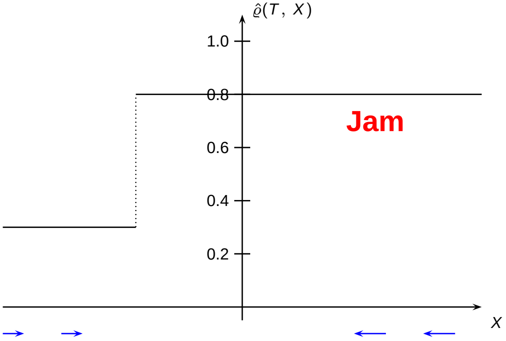

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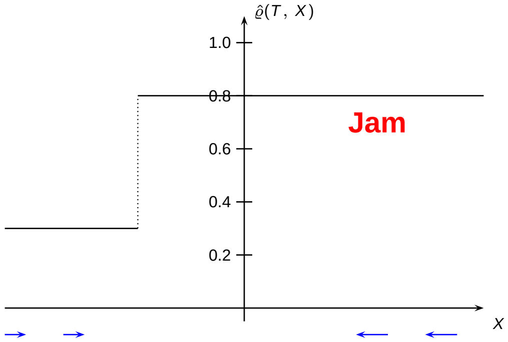

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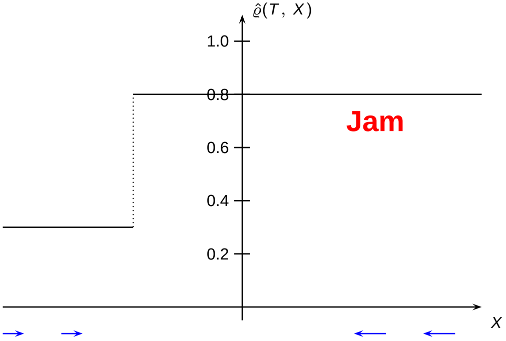

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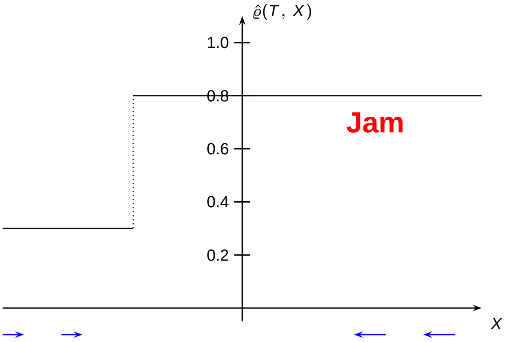

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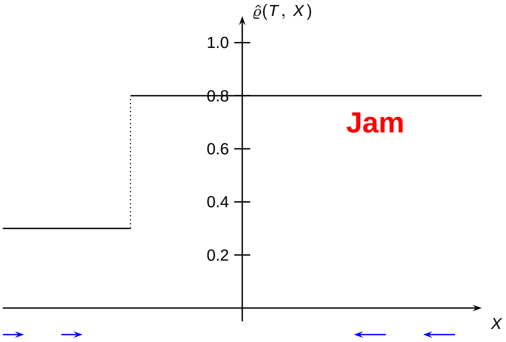

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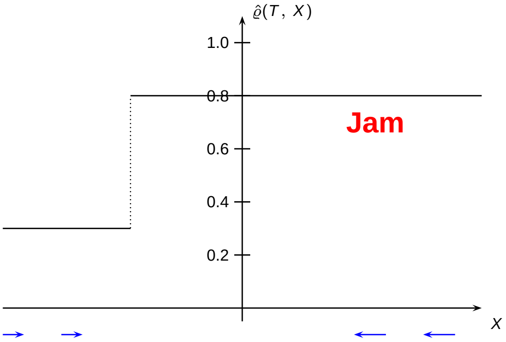

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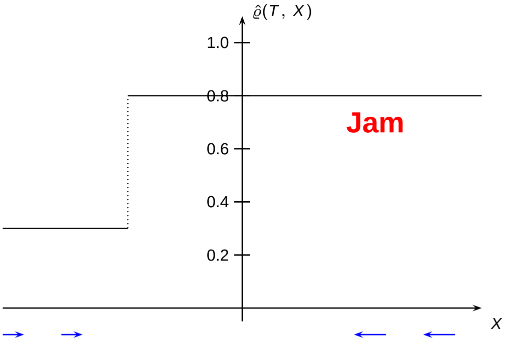

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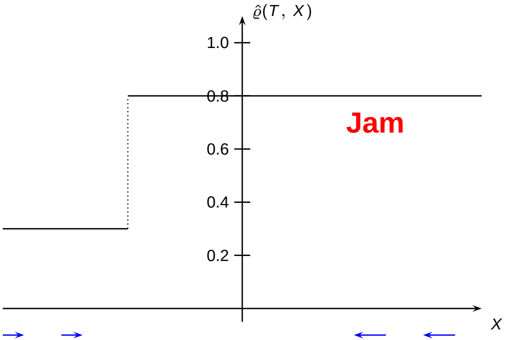

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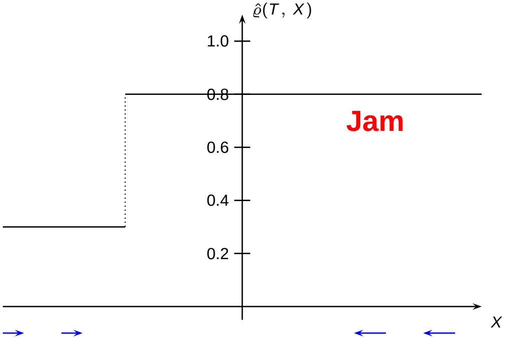

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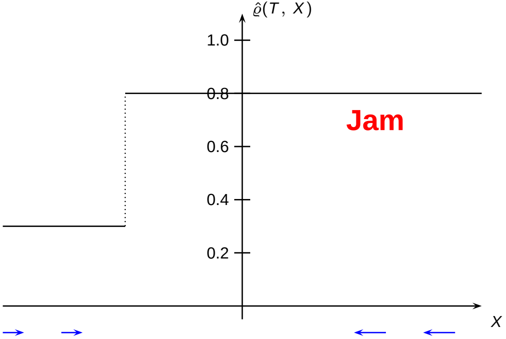

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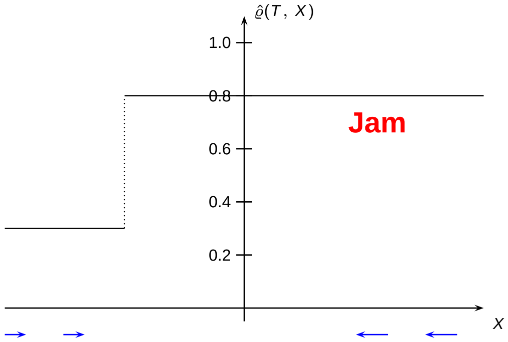

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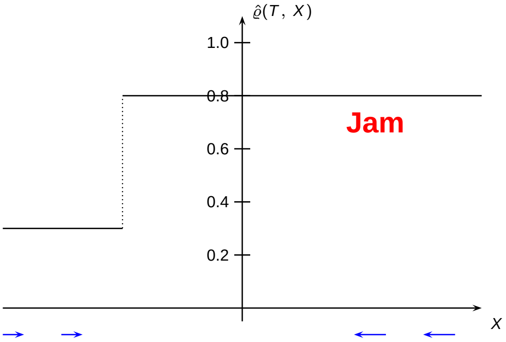

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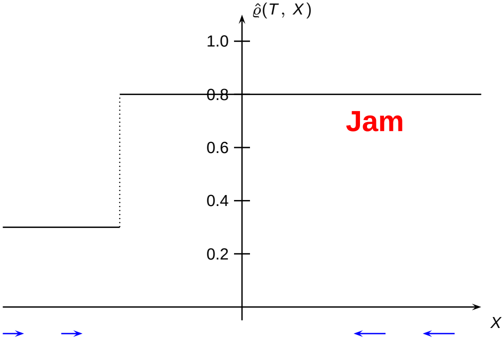

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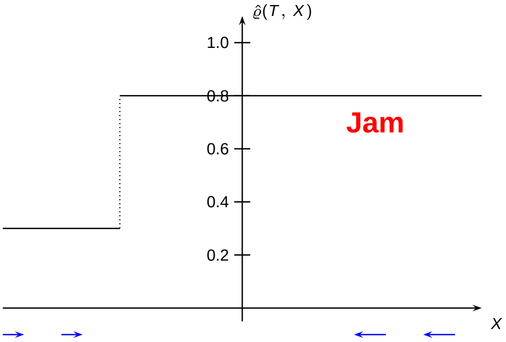

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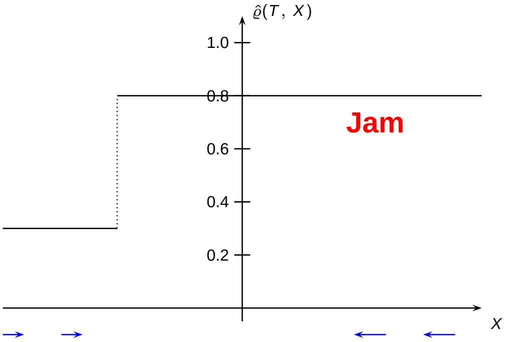

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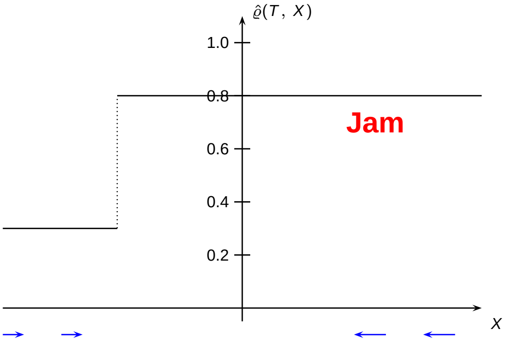

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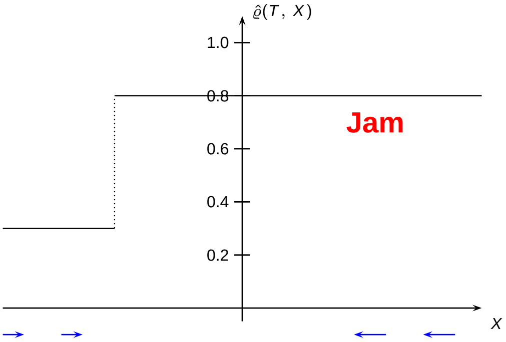

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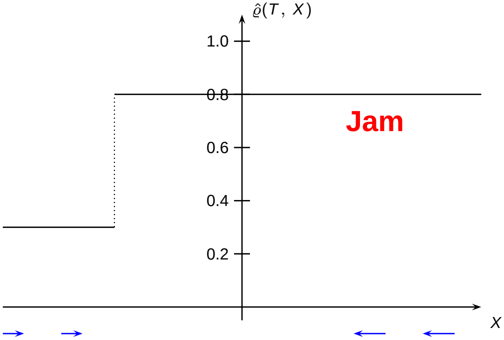

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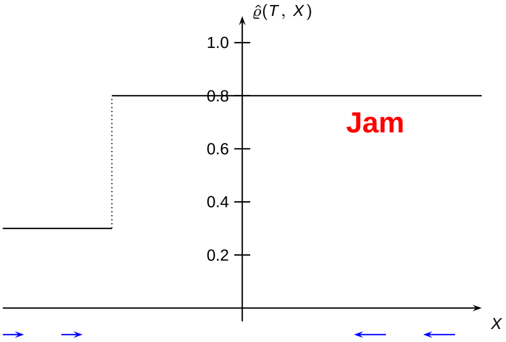

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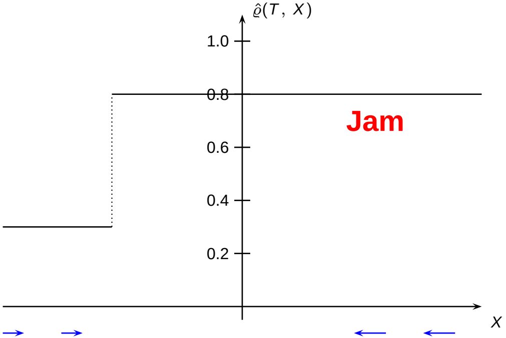

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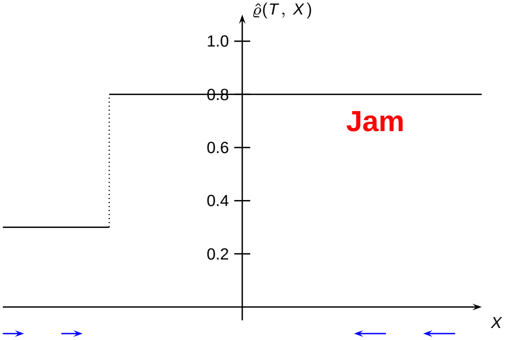

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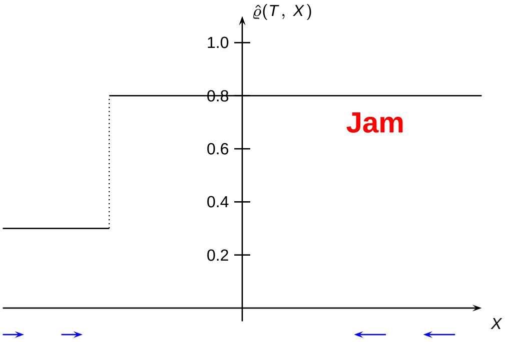

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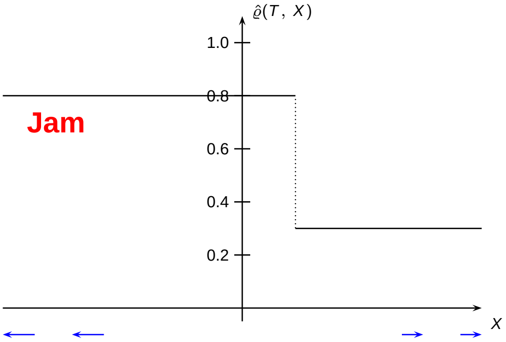
The start of the jam: sharpens.

On large scales



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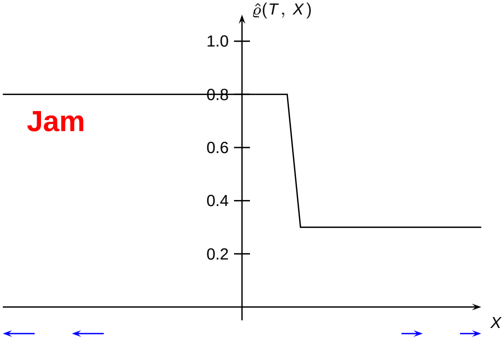

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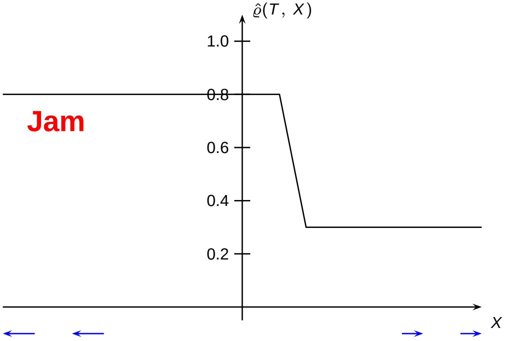

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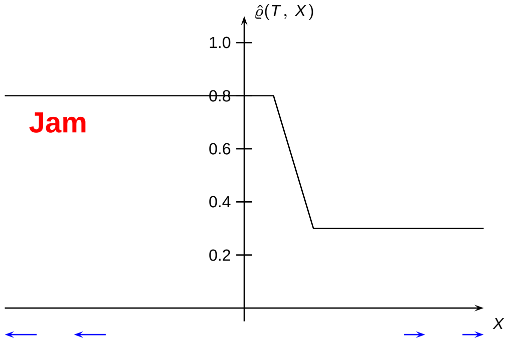

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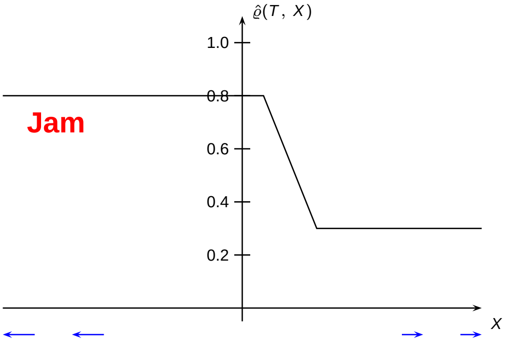

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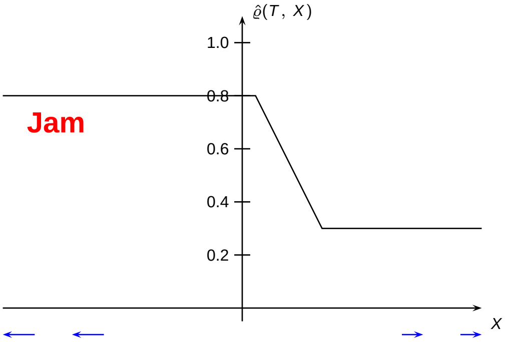

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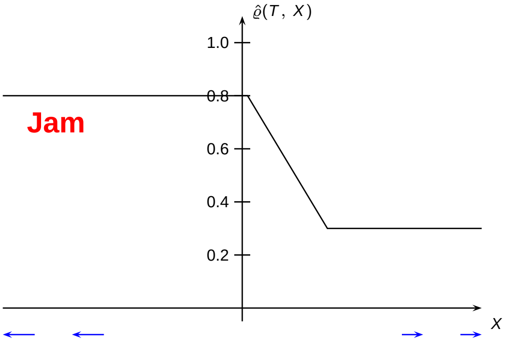

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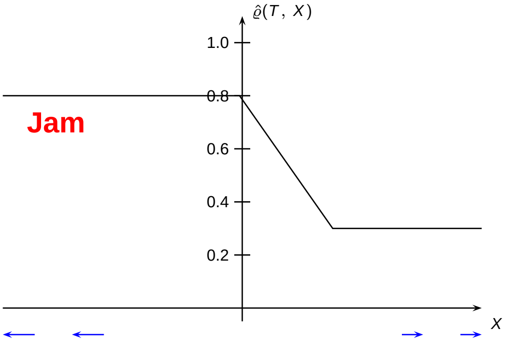

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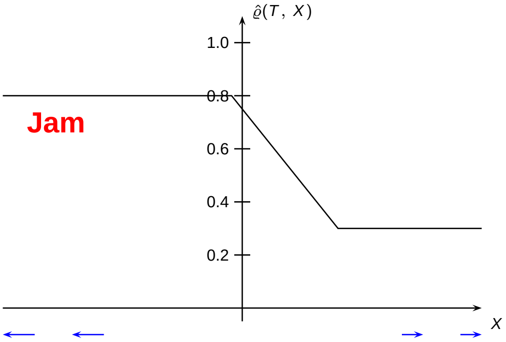

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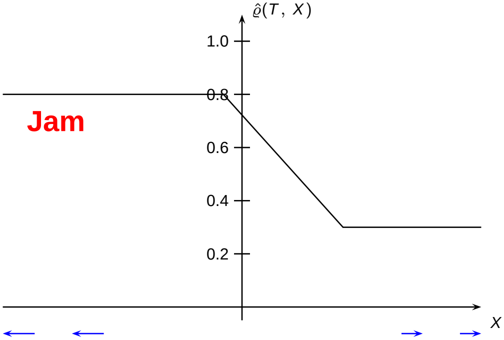

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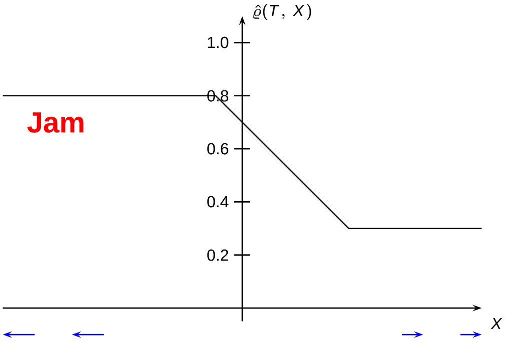

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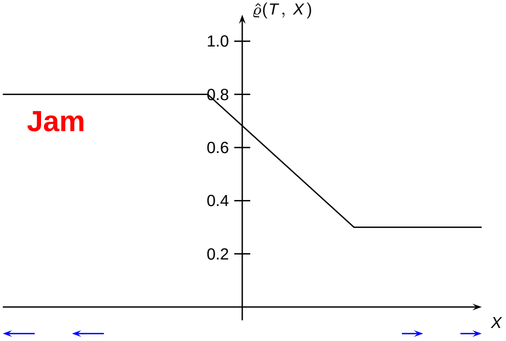

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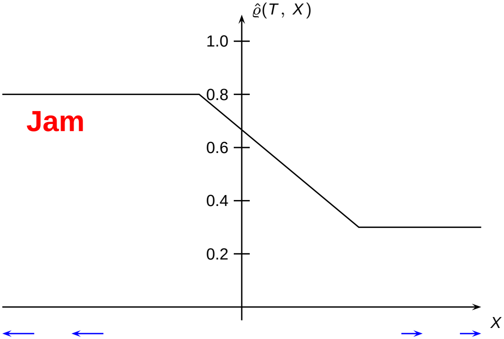

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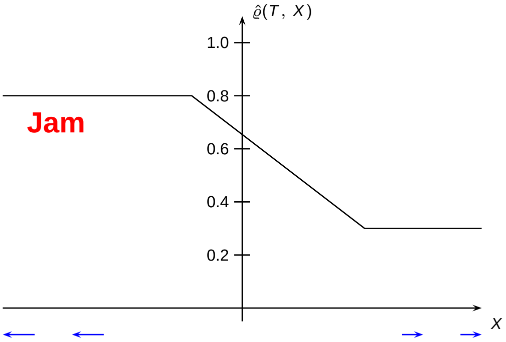

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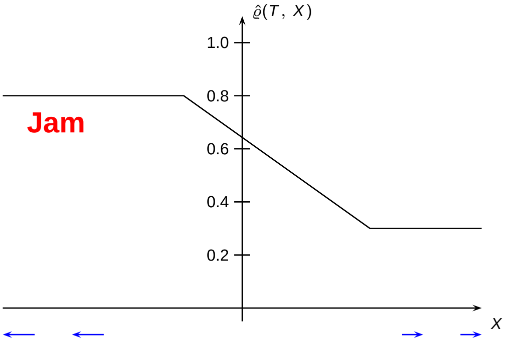

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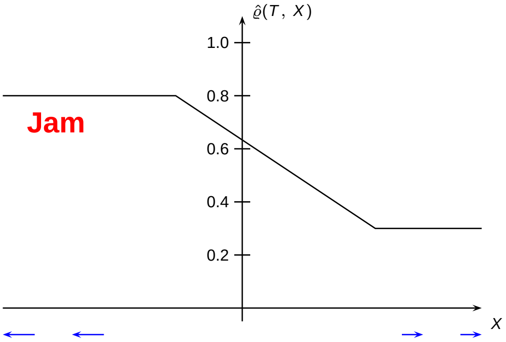

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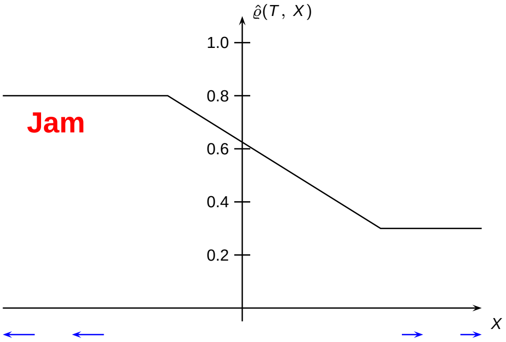

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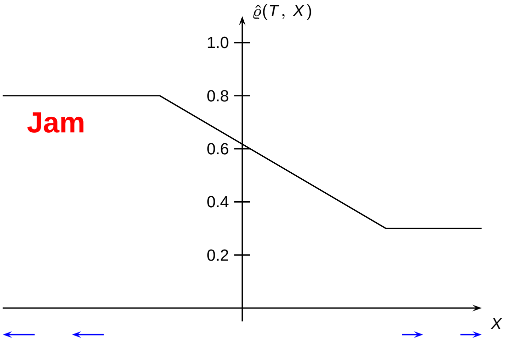

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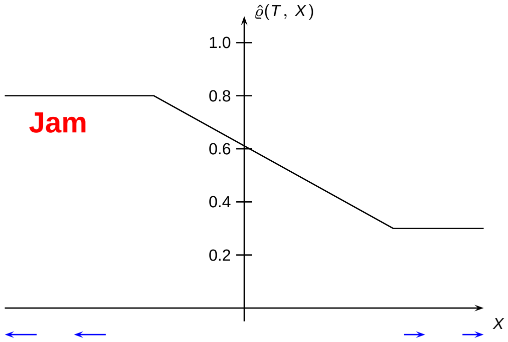

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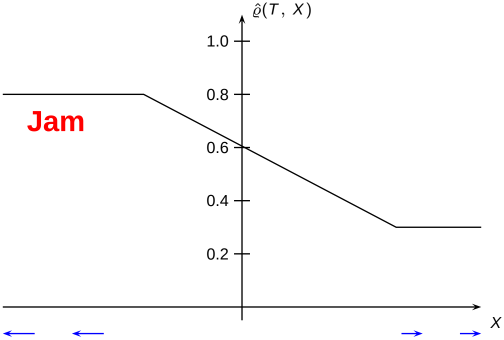

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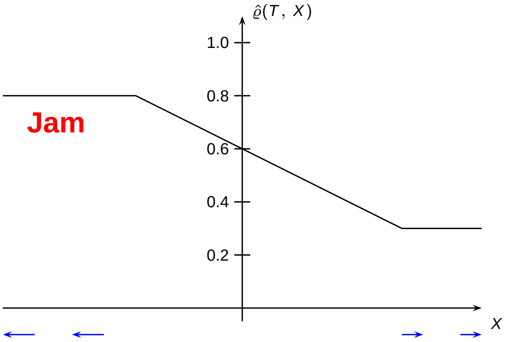

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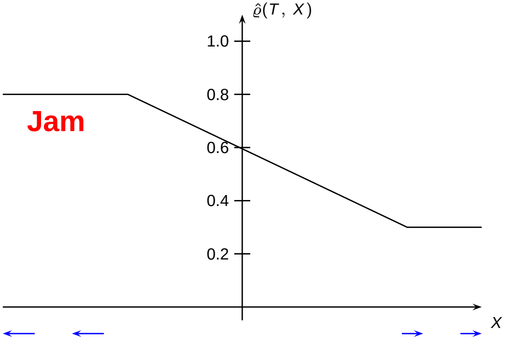

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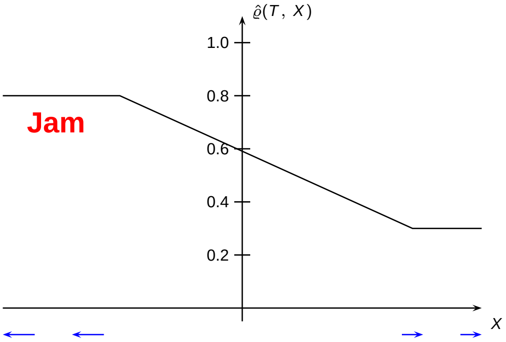

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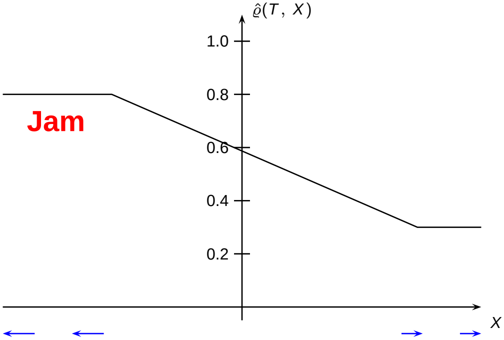

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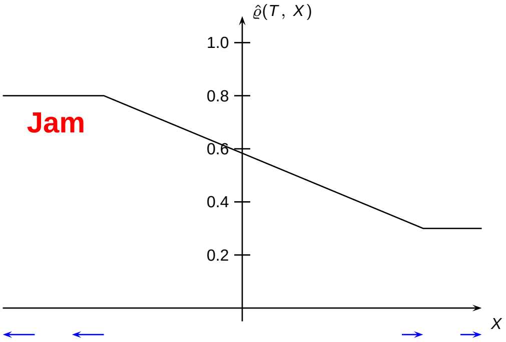

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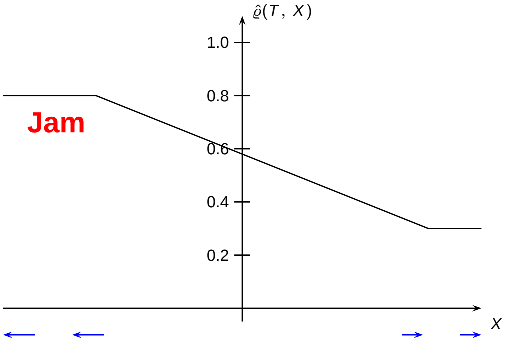

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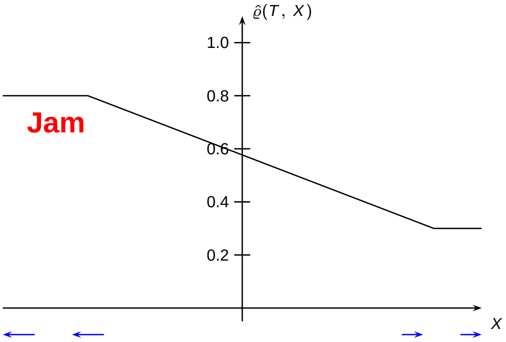

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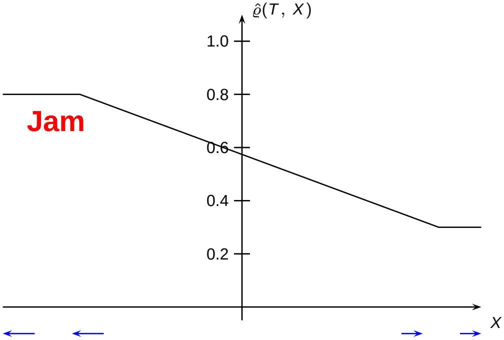

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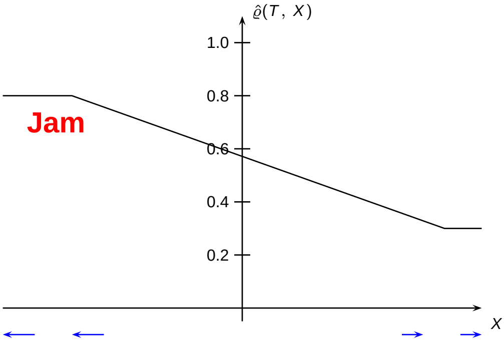

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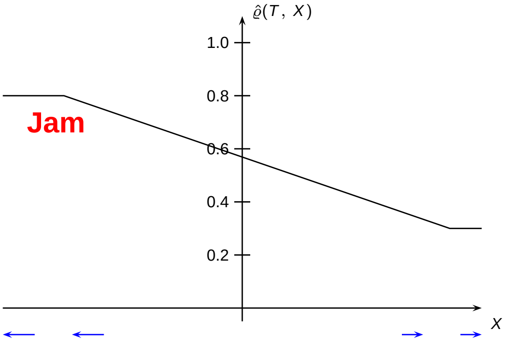

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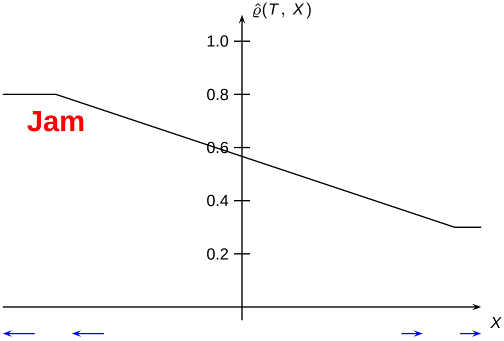

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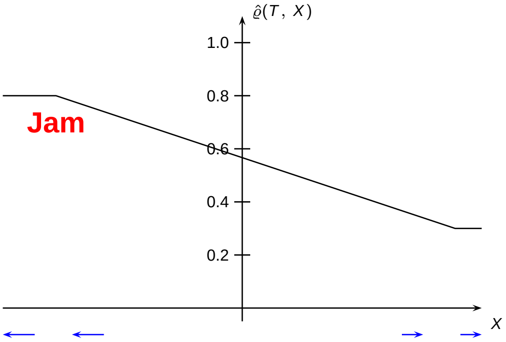

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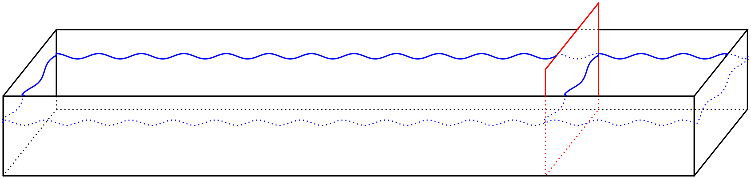
End of the jam: smoothens.

Remarks.

In general, non-linear differential equations are fun. (And difficult.)

E.g., solitary waves were discovered by John Scott Russell in 1834: he chased one along a channel for miles!



http://youtu.be/MADng1fqECY

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- But TASEP is already very interesting from the mathematics point of view, with many nice theorems and interesting open questions.

The Central Limit Theorem

Add many iid. variables Y_k (with finite second moment), rescale, and you converge to the Normal distribution:

$$\frac{Y_1 + \cdots + Y_n - n \cdot \mathbf{E} Y_1}{\sqrt{n \cdot \mathbf{Var} Y_1}} \underset{n\to\infty}{\Longrightarrow} \mathcal{N}(0, 1).$$

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Example. Take a single car on an empty road (the US still has those...), and Y_k the distance covered in the k^{th} second, $Y_1 + \cdots + Y_t$ is the position at time t.

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$\rightsquigarrow$ Y_k $\sim$Poisson(1), iid.; $\mathbf{E} Y_1 = 1$; $\mathbf{Var} Y_1 = 1$;

$$\frac{Y_1 + \cdots + Y_t - t}{\sqrt{t}} \underset{t\to\infty}{\Longrightarrow} \mathcal{N}(0, 1).$$

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$\rightsquigarrow$ $Y_k \sim$Poisson(1), iid.; $\mathbf{E} Y_1 = 1$; $\mathbf{Var} Y_1 = 1$;

$$\frac{Y_1 + \cdots + Y_t - t}{t^{1/2}} \underset{t \to \infty}{\Longrightarrow} \mathcal{N}(0, 1).$$

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He starts with $J(0) = 0$, and

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Tracy-Widom

Take an $N \times N$ Hermitian matrix with

$$M_{jk} \sim \mathcal{N}\left(0, \tfrac{1}{2}\right) + i \cdot \mathcal{N}\left(0, \tfrac{1}{2}\right), \qquad 1 \leq j < k \leq N,$$
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(all $\mathcal{N}$'s independent).

Denote the largest eigenvalue by $\lambda_{\max}$. Then

$$\frac{\lambda_{\max} - \sqrt{2N}}{\frac{1}{\sqrt{2}} \cdot N^{-1/6}} \underset{N\to\infty}{\Longrightarrow} \text{Tracy-Widom(II) distribution}.$$

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Combinatorics and difficult analysis. . .

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Probabilistic techniques (auxiliary processes, couplings)...

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Thank you.