Queues, stationarity, and stabilisation of last passage percolation

Joint with
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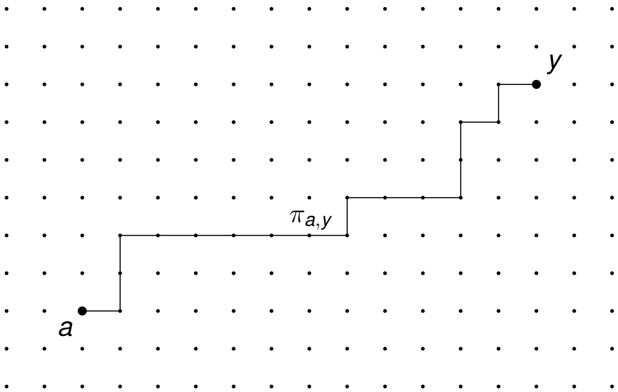
PiNE, Durham, 15 March, 2023.

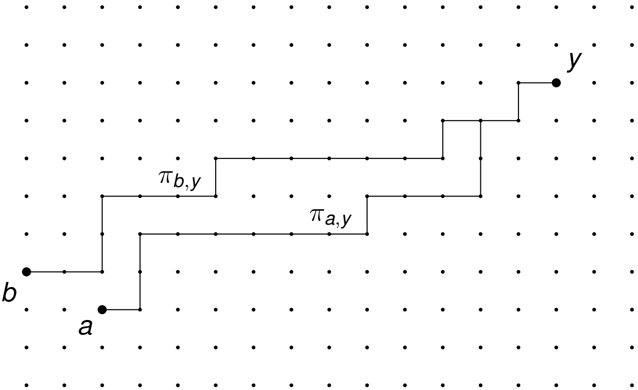
Last passage percolation

- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{a,y}$ from a to y is the a.s. unique heaviest up-right path from a to y. Its weight is $G_{a,y}$.

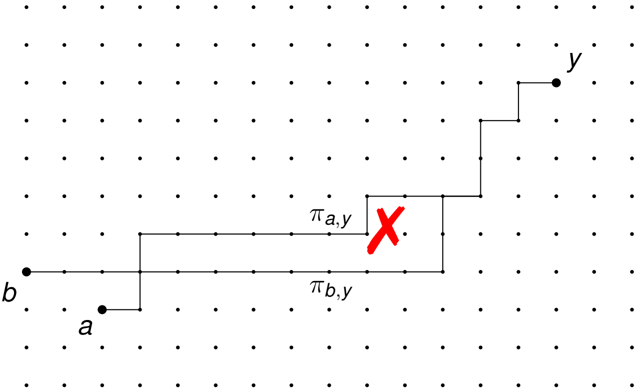


$G_{0,y}$ is the time TASEP hole y_1 swaps with particle y_2 if started from 1-0 initial condition.

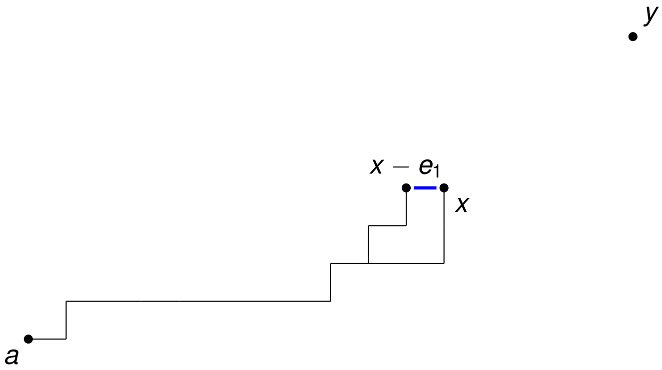
Coalescing: OK



But loops: not OK



Increments as new boundary



$$I_x = G_{a,x} - G_{a,x-e_1}$$

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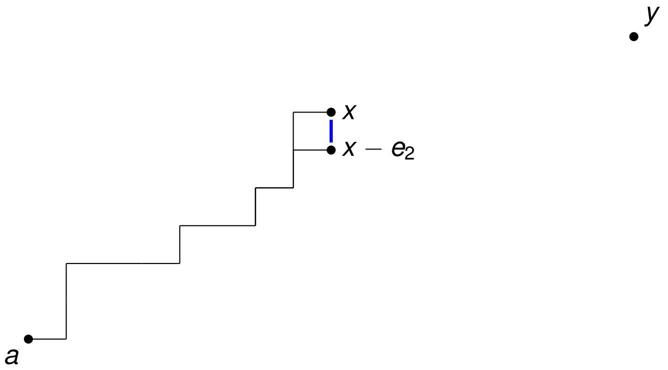

$$I_x = G_{a,x} - G_{a,x-e_1} \qquad J_x = G_{a,x} - G_{a,x-e_2}$$

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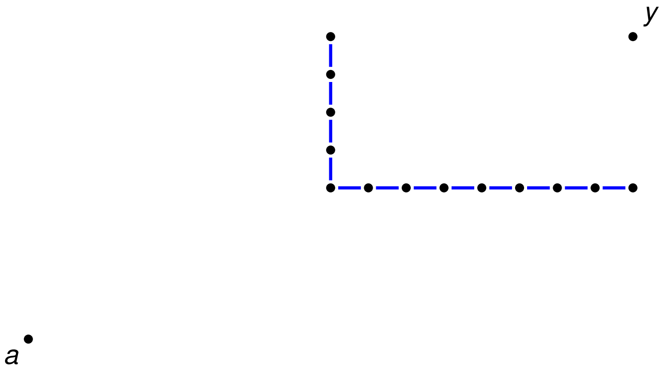

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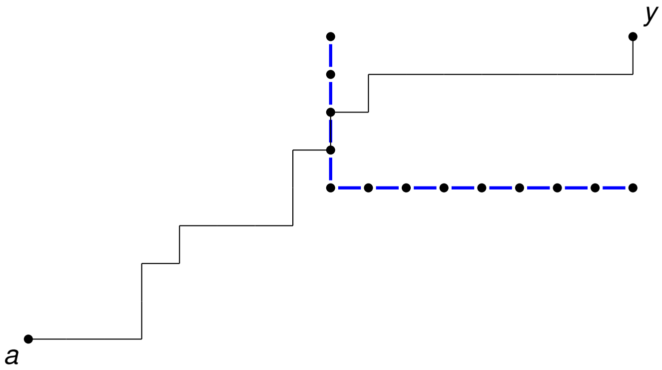

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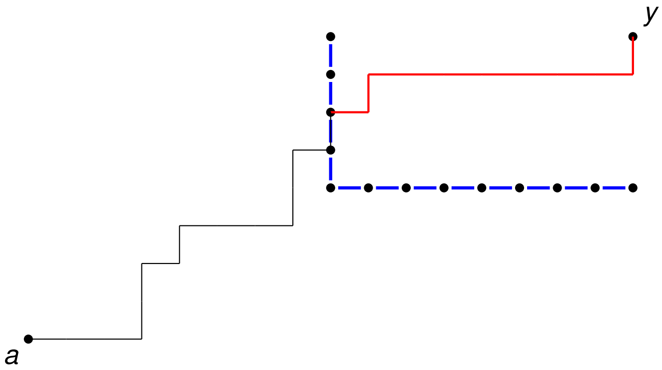

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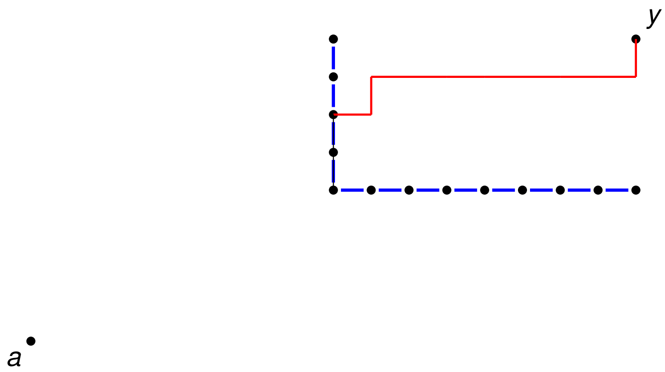
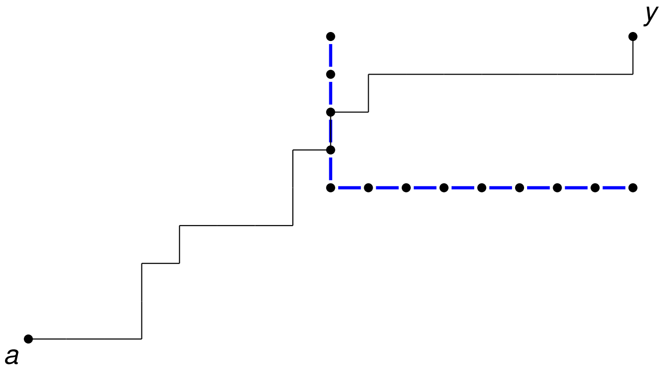


$$I_x = G_{a,x} - G_{a,x-e_1} \qquad J_x = G_{a,x} - G_{a,x-e_2}$$

$\rightsquigarrow$ Act as boundary weights for a smaller, embedded model.

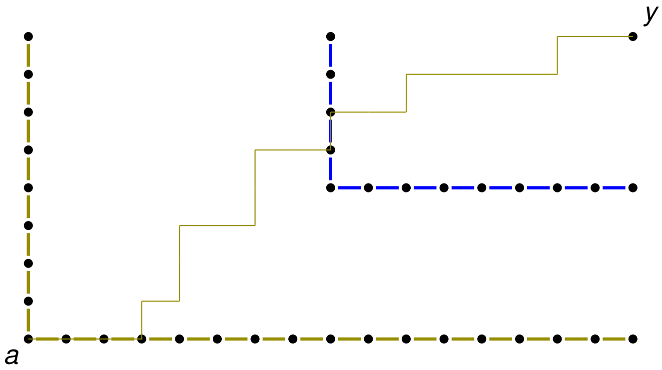
Stationary LPP



$$I_x = G_{a,x} - G_{a,x-e_1} \qquad J_x = G_{a,x} - G_{a,x-e_2}$$

Stationary LPP

Replace the boundary to | $\sim \text{Exp}(\varrho)$, — $\sim \text{Exp}(1-\varrho)$ independent.



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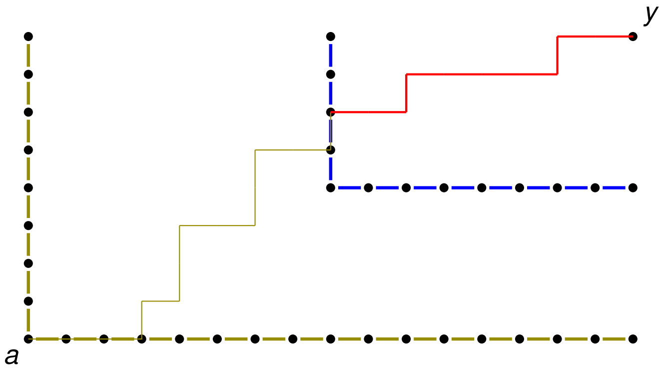

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The embedded model has the same structure.

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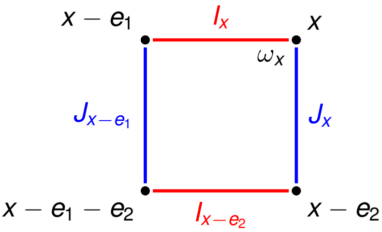

$$I_x = G_{a,x} - G_{a,x-e_1} \qquad J_x = G_{a,x} - G_{a,x-e_2}$$

The recursion:

$$G_{a,x} = (G_{a,x-e_1} \vee G_{a,x-e_2}) + \omega_x$$

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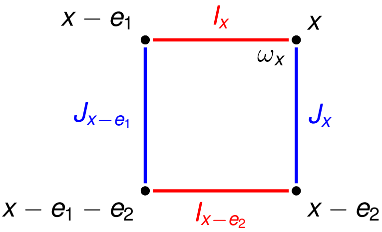

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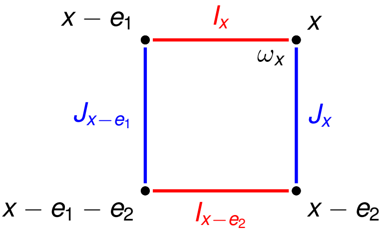

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$$I_x = (I_{x-e_2} - J_{x-e_1})^+ + \omega_x$$

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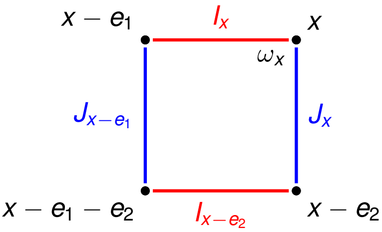

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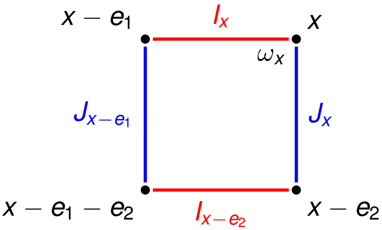
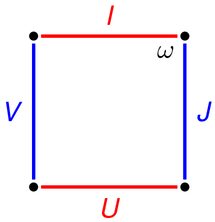
G_{a,x} &= (G_{a,x-e_1} \vee G_{a,x-e_2}) + \omega_x \\
G_{a,x} - G_{a,x-e_1} &= (G_{a,x-e_2} - G_{a,x-e_1})^+ + \omega_x \\
I_x &= (I_{x-e_2} - J_{x-e_1})^+ + \omega_x \\
J_x &= (J_{x-e_1} - I_{x-e_2})^+ + \omega_x.
\end{aligned}$$

Stationary LPP



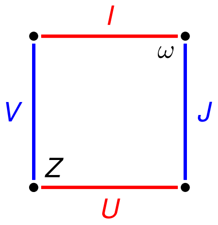
$$I_x = (I_{x-e_2} - J_{x-e_1})^+ + \omega_x \qquad J_x = (J_{x-e_1} - I_{x-e_2})^+ + \omega_x$$

Stationary LPP



$$I = (U - V)^+ + \omega \qquad J = (V - U)^+ + \omega$$

Stationary LPP



$$I = (U - V)^+ + \omega \qquad J = (V - U)^+ + \omega$$

Proposition

$$\left.\begin{array}{l} U \sim Exp(1-\varrho), \\ V \sim Exp(\varrho), \\ \omega \sim Exp(1), \end{array}\right\} \textit{indep.} \quad \Rightarrow \quad \left.\begin{array}{l} I \sim Exp(1-\varrho), \\ J \sim Exp(\varrho), \\ Z := U \wedge V \sim Exp(1), \end{array}\right\} \textit{indep.}$$

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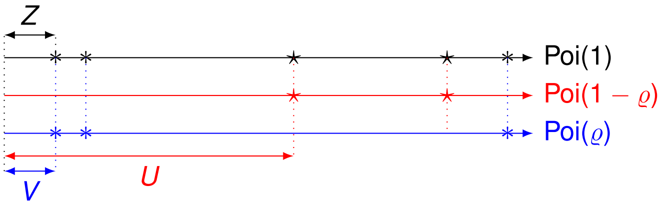

$U - V$ and Z are independent.

Stationary LPP

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$$(\omega,\ (U - V),\ Z) \stackrel{\text{d}}{=} (Z,\ (U - V),\ \omega)$$

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Hence

$$\begin{aligned} &\big((U - V)^+ + \omega,\ (V - U)^+ + \omega,\ Z\big) \\ \stackrel{\text{d}}{=} &\big((U - V)^+ + Z,\ (V - U)^+ + Z,\ \omega\big) \end{aligned}$$

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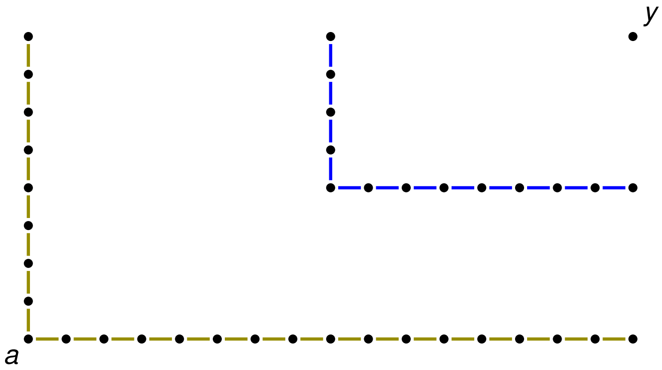
□

Stationary LPP

Replace the boundary to | $\sim$ Exp(ϱ), — $\sim$ Exp($1-\varrho$) independent.



$$I_x = G_{a,x} - G_{a,x-e_1} \qquad J_x = G_{a,x} - G_{a,x-e_2}$$

Then $J_x \sim \text{Exp}(\varrho)$, $I_x \sim \text{Exp}(1-\varrho)$, independent.

The embedded model has the same structure.

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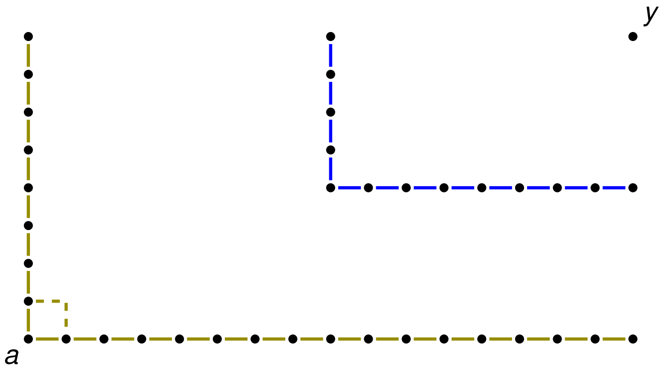

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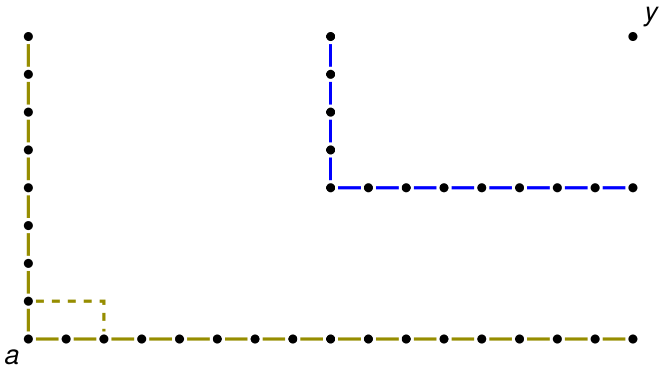

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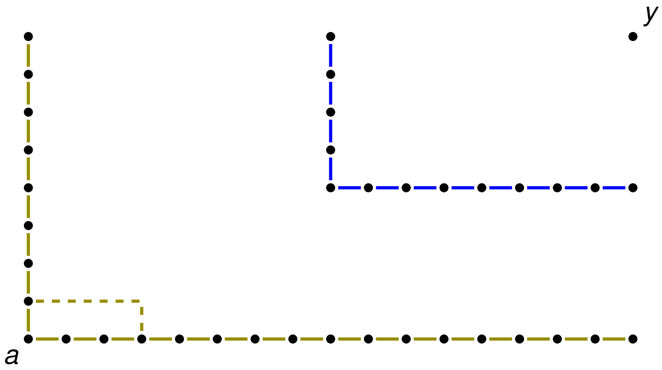

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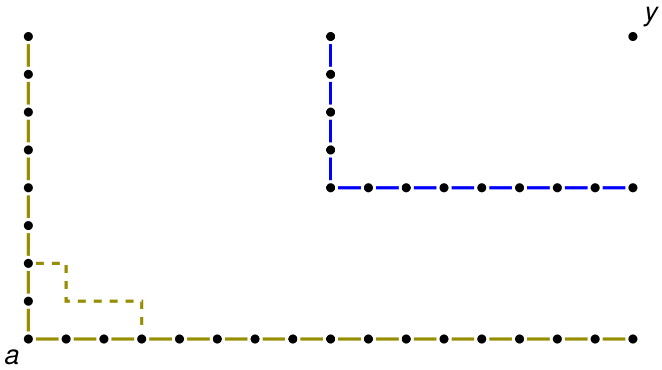

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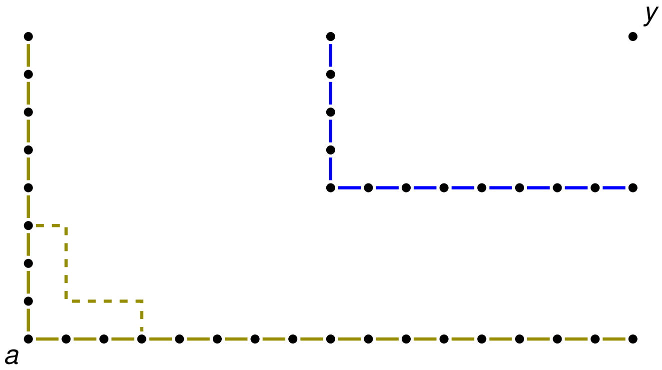

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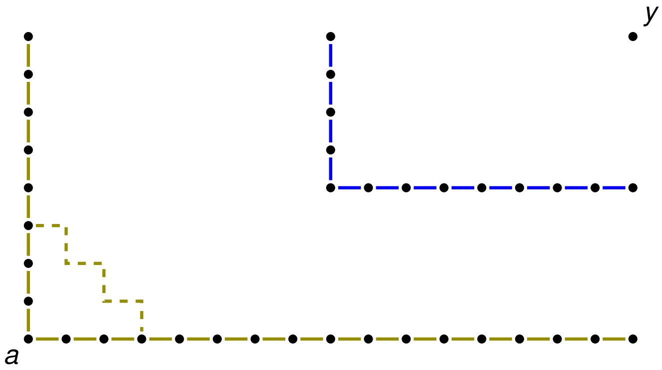

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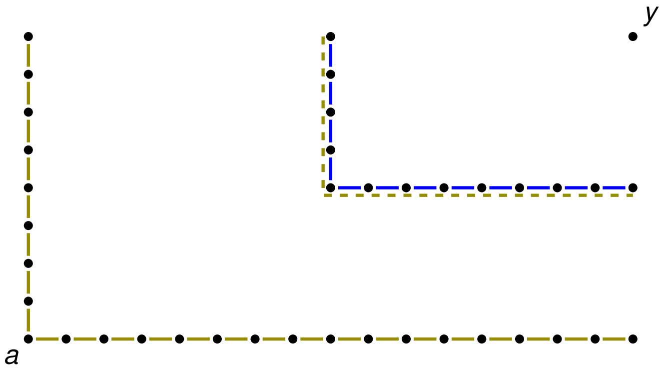

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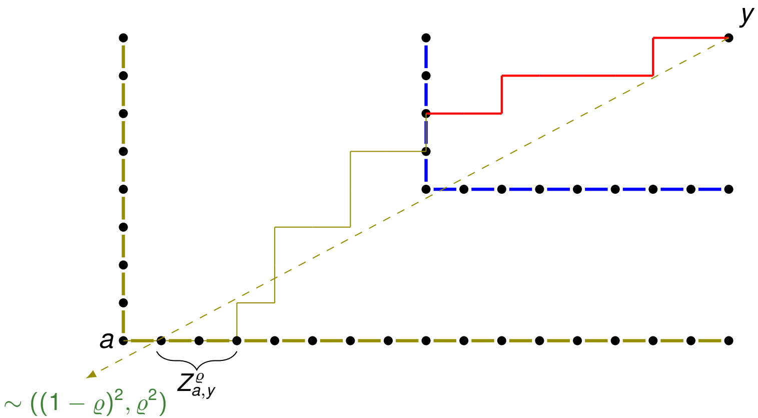
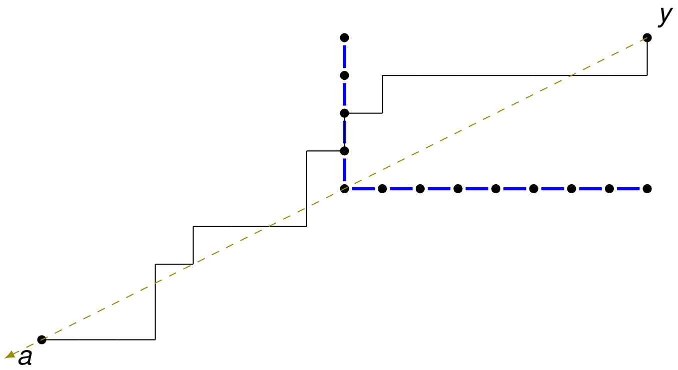


B., Cator, Seppäläinen '06: $\mathbb{P}\{|Z^{\varrho}_{a,y}| \geq \ell\} \leq \text{box}^2/\ell^3$, *good directional control*.

Infinite geodesics

Even without the boundary:
$J \underset{a\to-\infty}{\longrightarrow}$ i.i.d. Exp(ϱ), $I \underset{a\to-\infty}{\longrightarrow}$ i.i.d. Exp($1-\varrho$), independent.



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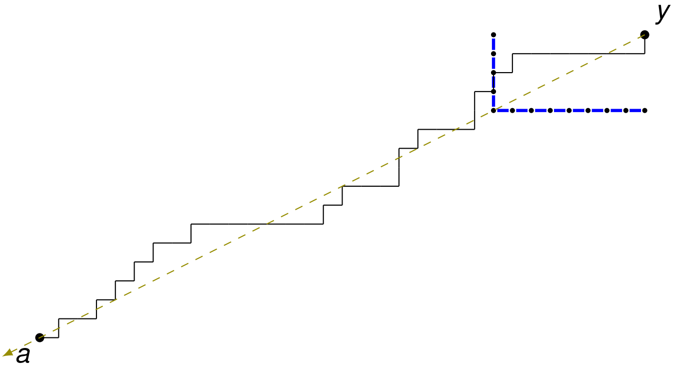

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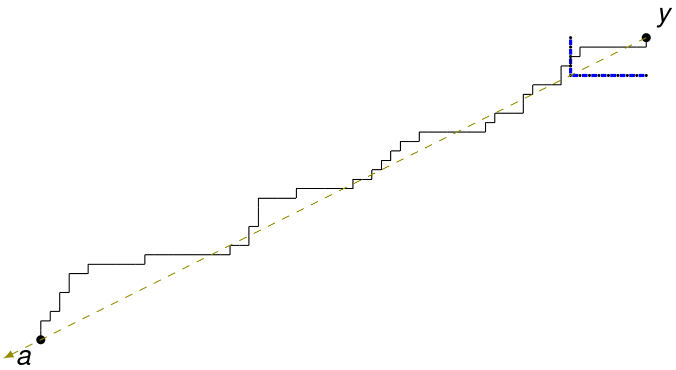

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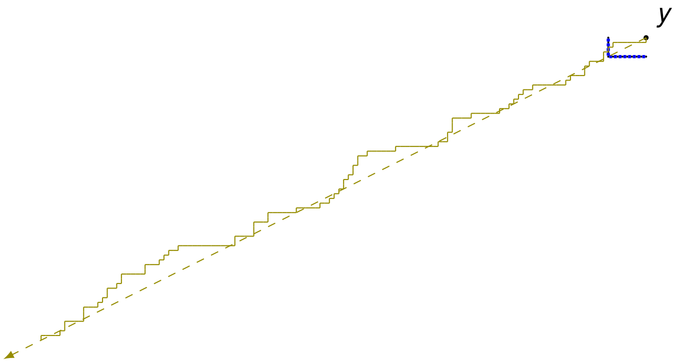
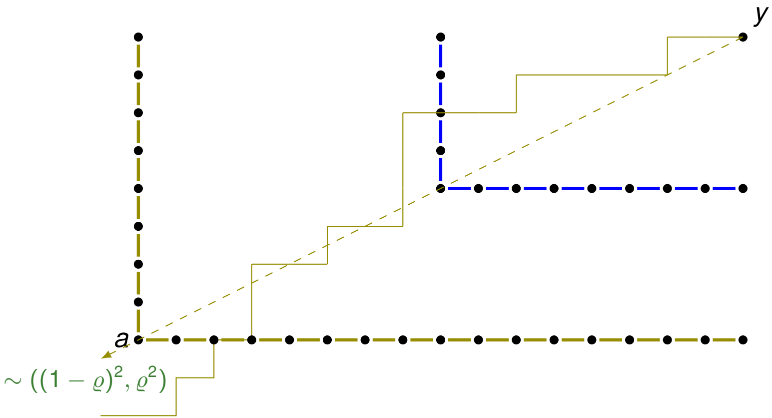


Result 1)



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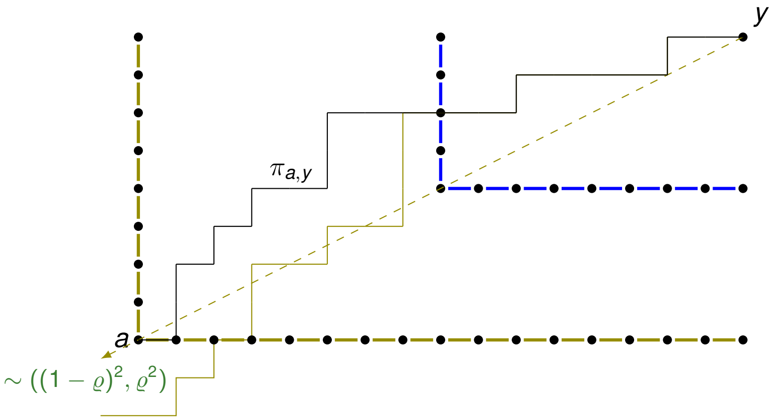

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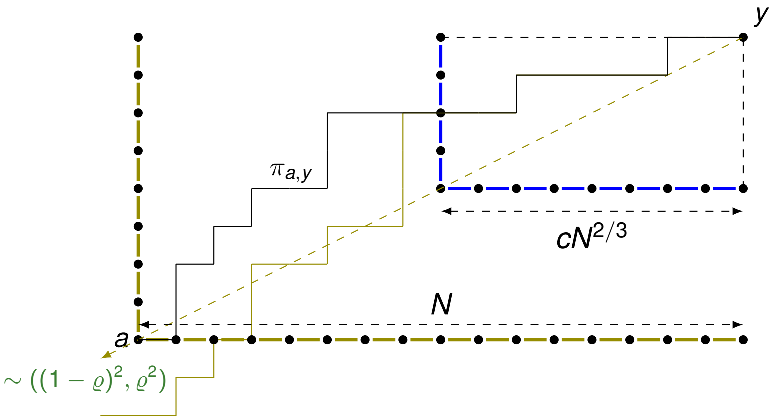
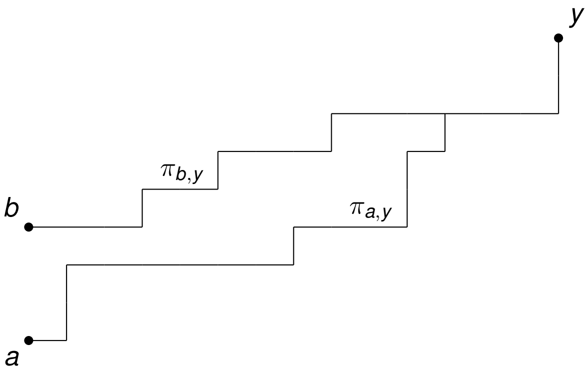


With probability at least $1 - Cc^{\frac{3}{8}}$, stationary and point-to-point paths already coalesce in the small box. (Busani, Ferrari '20)

Result 2)



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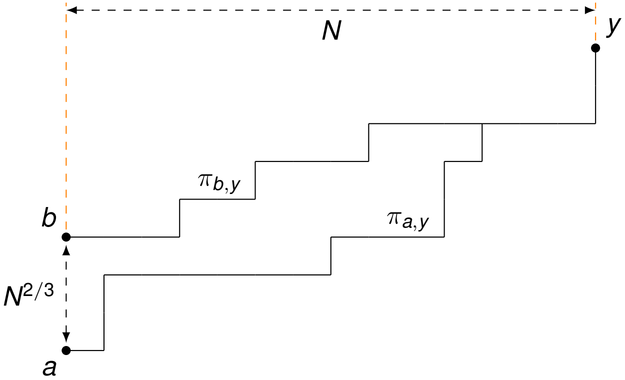

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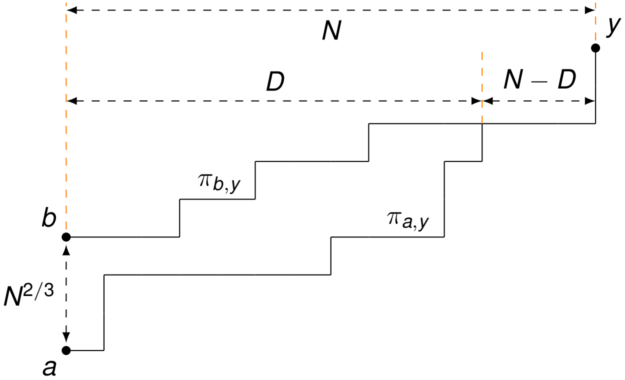


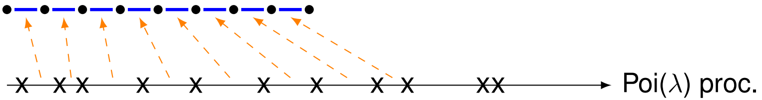
$$\begin{Bmatrix} \mathbf{P}\{D \le \alpha N\} \le C\alpha^2, \\ \mathbf{P}\{N - D \le \alpha N\} \le C\alpha^{\frac{2}{9}}. \end{Bmatrix}$$ (Basu, Sarkar, Sly '19; Zhang '20)

Result 3)

The Airy_2 process minus a parabola is locally well approximated in total variation by Brownian motion.

Queues

What is an i.i.d. Exp(λ) boundary?



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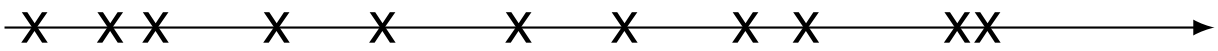

Poi(λ) pr.

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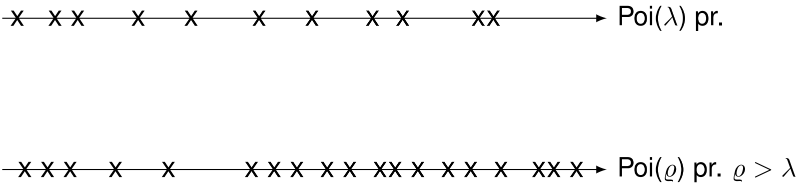

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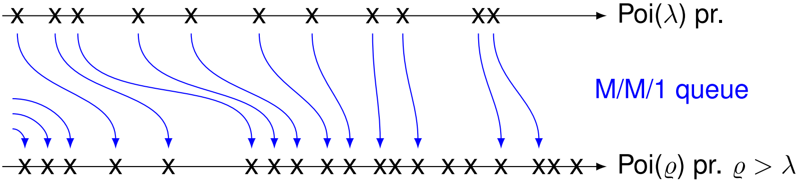

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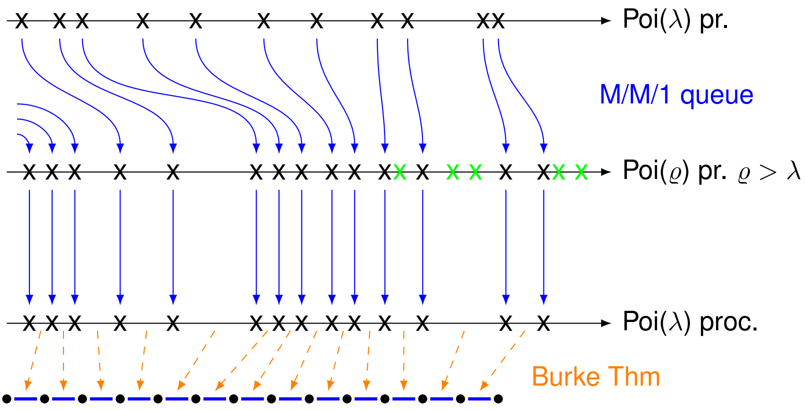

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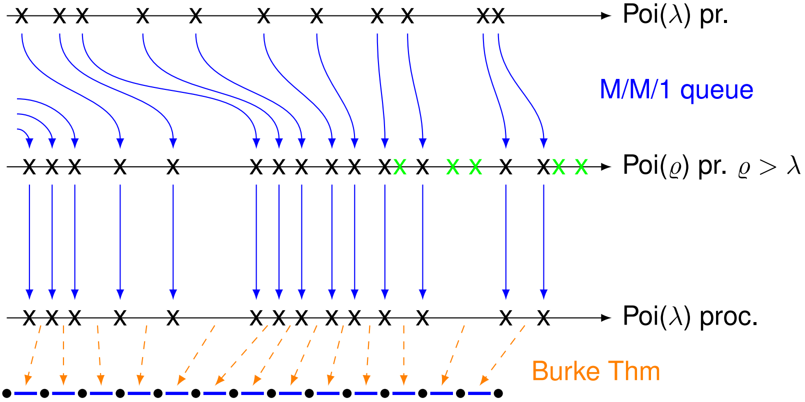

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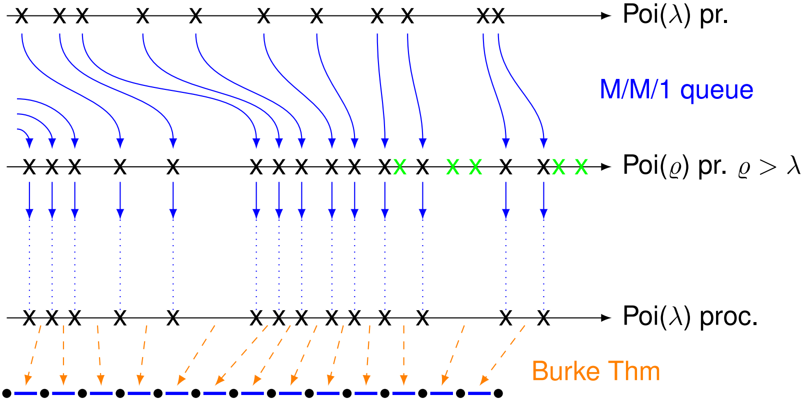

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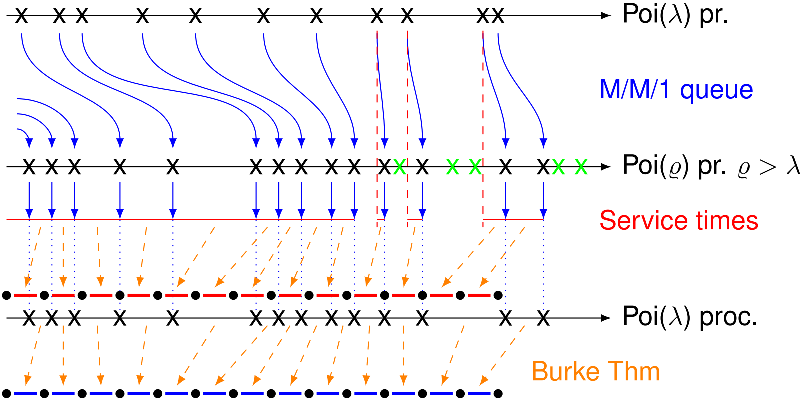

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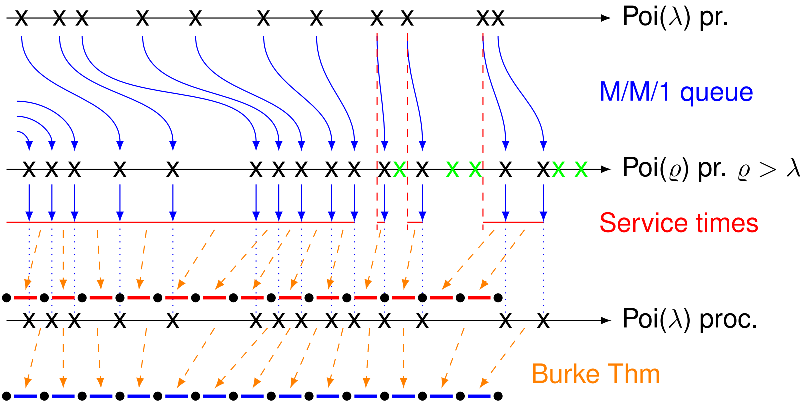

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(Ferrari, Martin '06; Fan, Seppäläinen '20)

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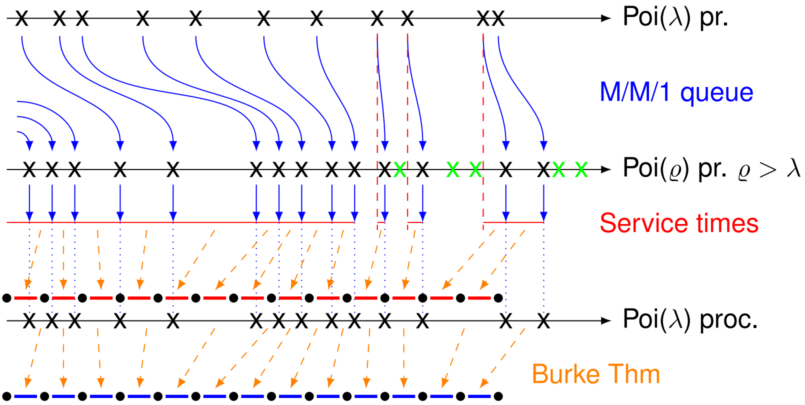

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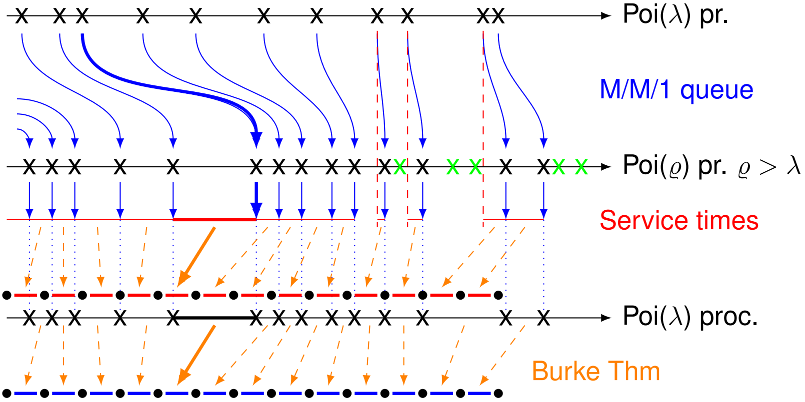

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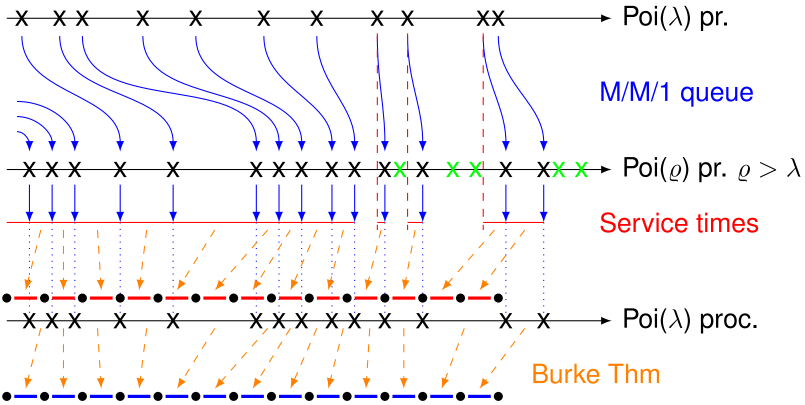

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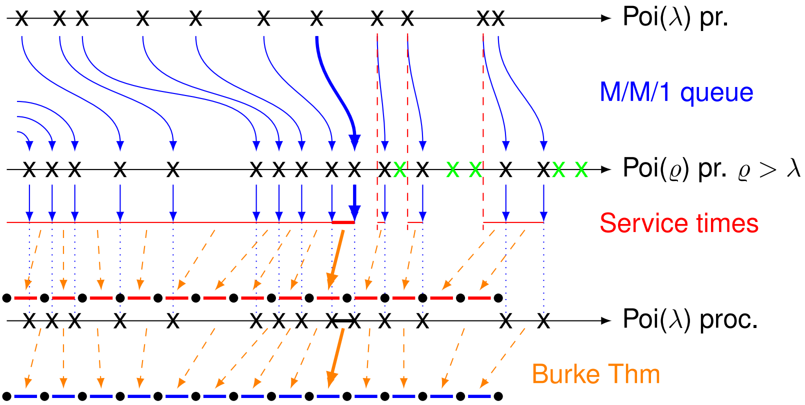

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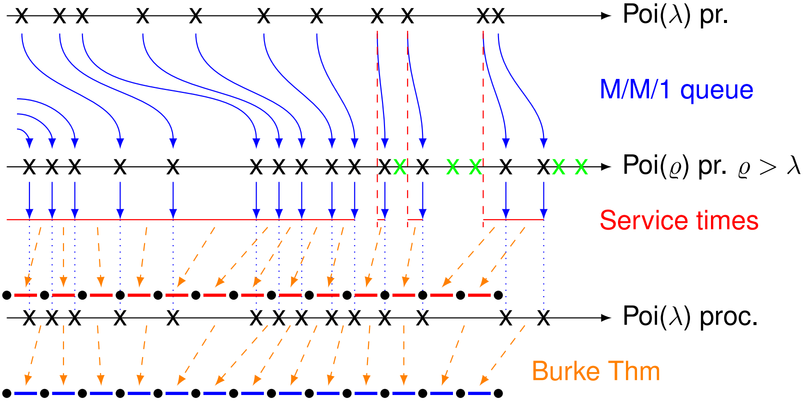

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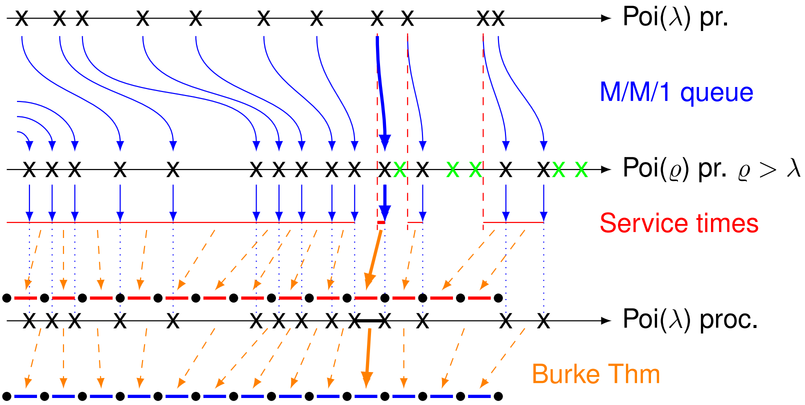

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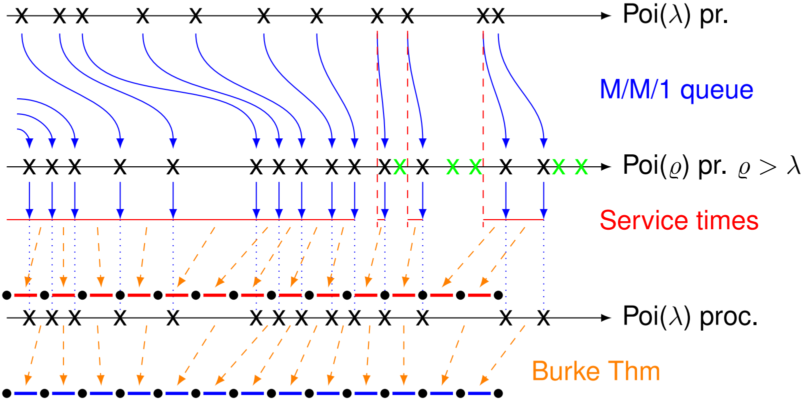

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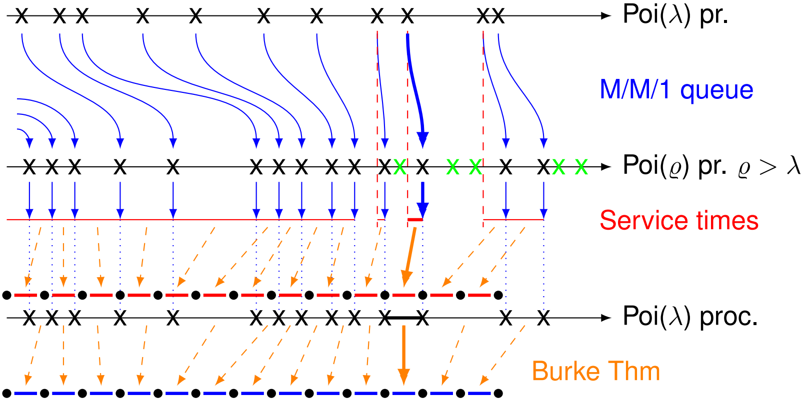

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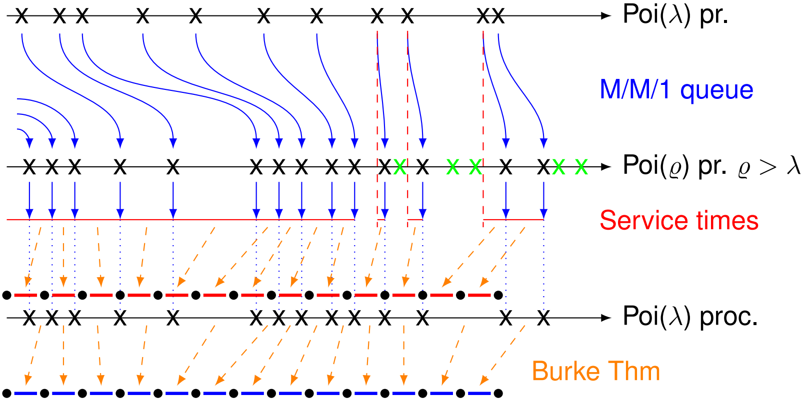

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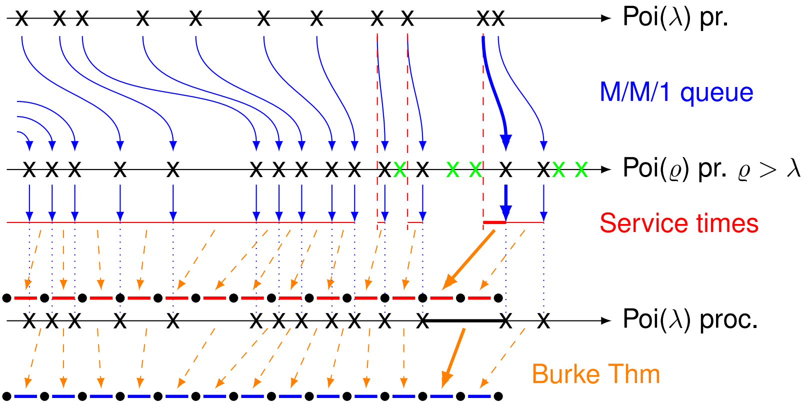

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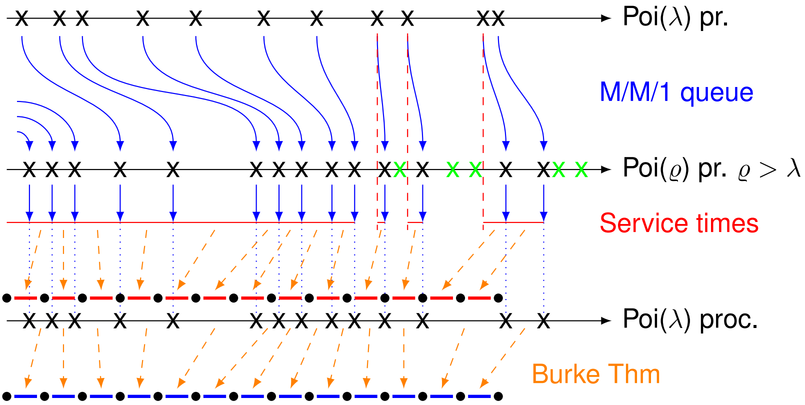

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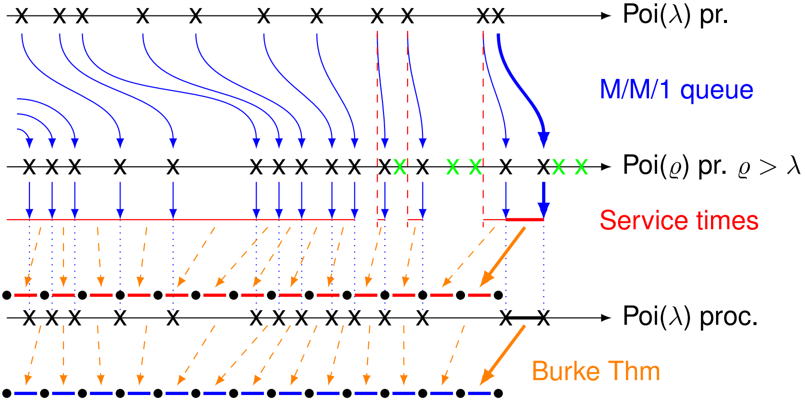

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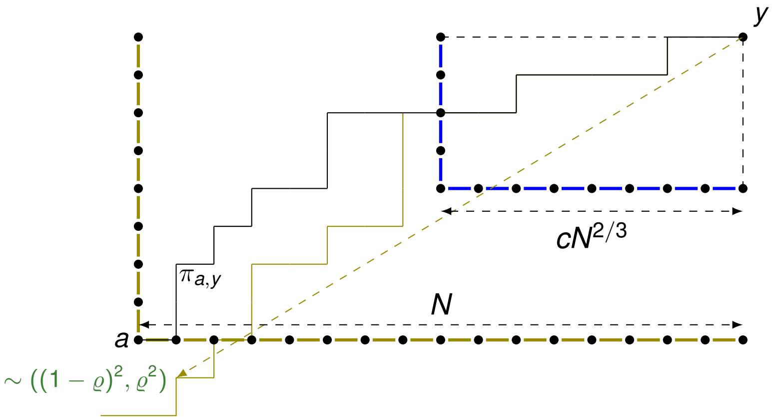
Result 1): P-2-P is like stati path



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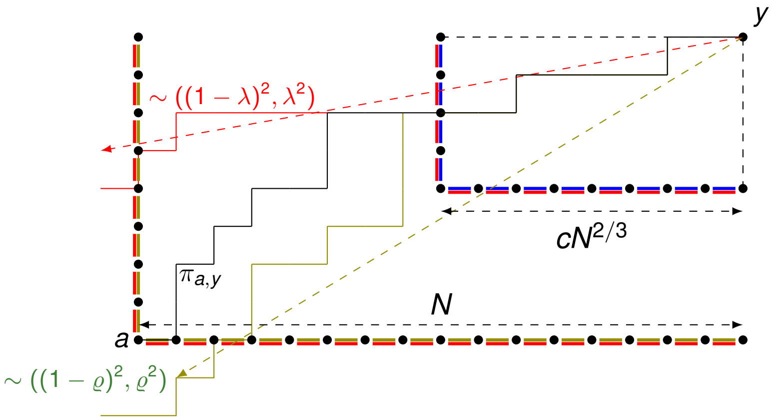
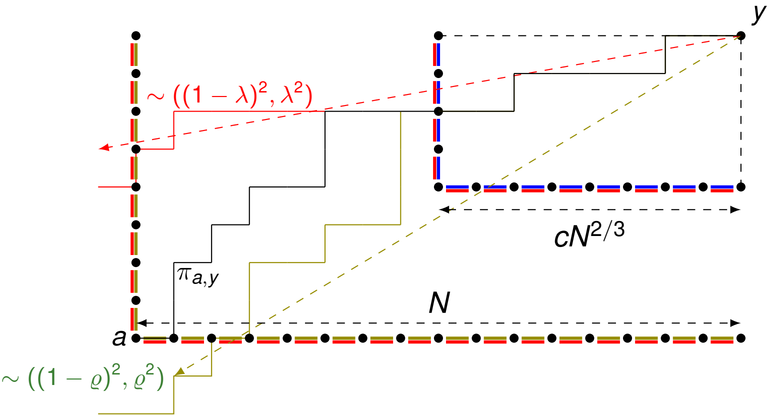


$$\lambda = \varrho - N^{-\frac{1}{3}}$$

Result 2): P-2-P paths coalesce soon



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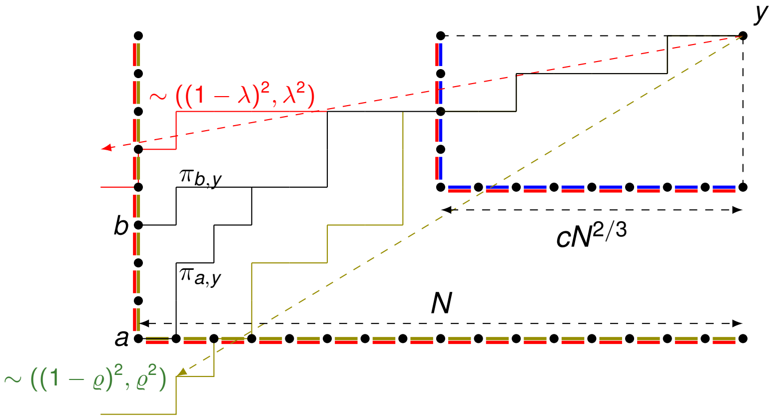

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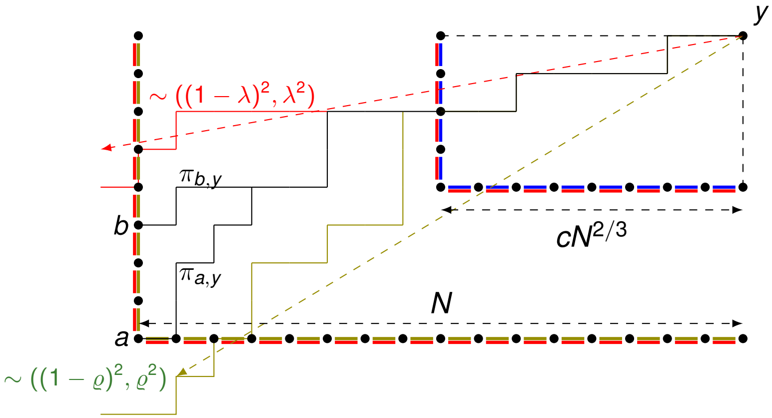

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This can be boosted by pulling the small box left by αN.

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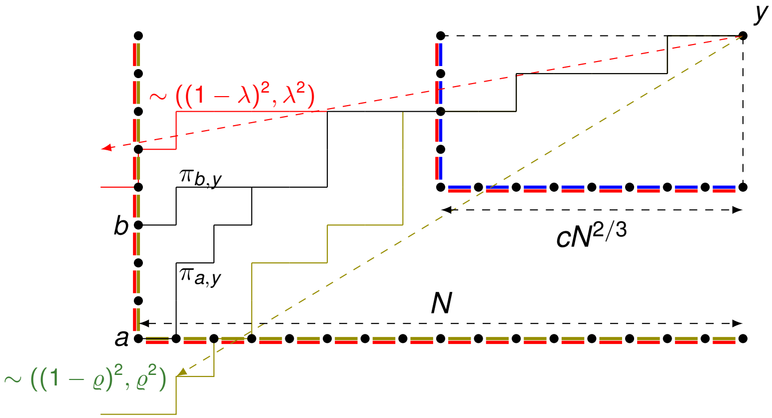

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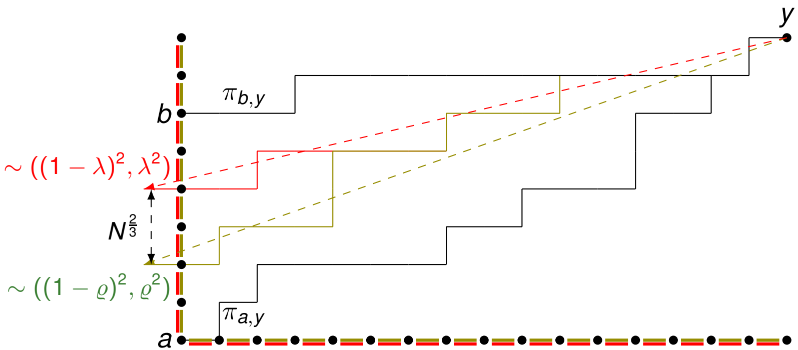
Rescale these boundaries: *Stationary Horizon* (Busani'21, +Seppäläinen, Sorensen'22).

Result 2): P-2-P paths don't coalesce soon



$$\lambda = \varrho - N^{-\frac{1}{3}}$$

Coalescing too soon would mean stationary paths getting squeezed to each other too soon so they bend.

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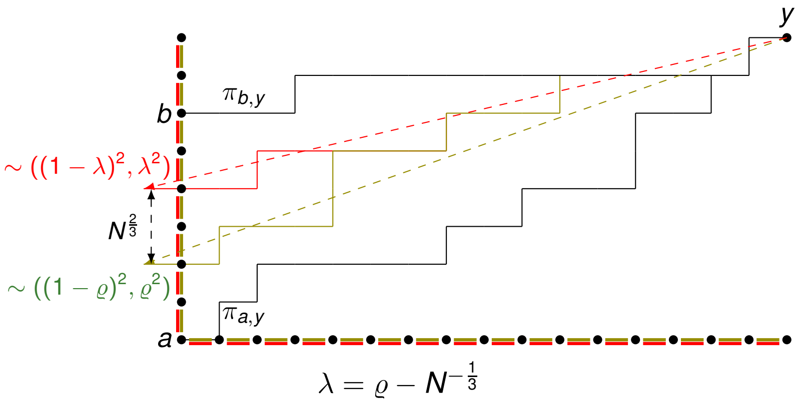

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Thank you.