Blocking measures is a combinatorial goldmine

Joint with
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Seminário de Probabilidade e Mecânica Estatística
IMPA
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The OMG slides

Definition (q-Pochhammer symbol)

$$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i)$$

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$$

Definition (q-binomial coefficient)

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m \cdot (q;q)_{n-m}}$$

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Theorem (Euler's identity)

$$\sum_{i=0}^{\infty} \frac{q^{\frac{i(i-1)}{2}} z^i}{(q;q)_i} = (-z;q)_\infty$$

(Breaking up the generating function of integer partitions w.r.t. the number of parts.)

$(a;q)_n = \prod_{i=0}^{n} (1 - aq^i)$

The OMG slides

Theorem (q-binomial theorem)

$$\sum_{i=0}^{m} q^{\frac{i(i-1)}{2}} z^i \begin{bmatrix} m \\ i \end{bmatrix}_q = (-z;q)_m$$

(Breaking up the generating function of integer partitions w.r.t. the number and maximal size of parts.)

$$(a;q)_n = \prod_{i=0}^{n} (1 - aq^i)$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m \cdot (q;q)_{n-m}}$$

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Theorem (Durfee rectangles identity)

$$\sum_{i=n^-}^{\infty} \frac{q^{i(n+i)}}{(q;q)_{n+i} \cdot (q;q)_i} = \frac{1}{(q;q)_\infty}$$

(Breaking up integer partitions w.r.t. the largest rectangles in their diagrams.)

$(a;q)_n = \prod_{i=0}^{n} (1 - aq^i)$

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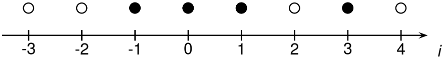
Theorem (Jacobi triple product)

$$\sum_{i=-\infty}^{\infty} q^{\frac{i(i+1)}{2}} z^i = (q;q)_\infty \cdot (-qz;q)_\infty \cdot \left(-\frac{1}{z};q\right)_\infty$$

(Connecting partitions and Frobenius partitions.)

$$(a;q)_n = \prod_{i=0}^{n} (1 - aq^i)$$

Asymmetric simple exclusion



$$\eta_i \in \{0, 1\}.$$

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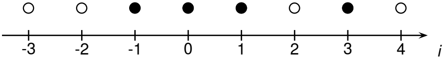

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to the right with rate p,
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The jump is suppressed if the destination site is occupied by another particle.

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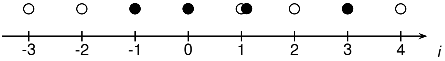

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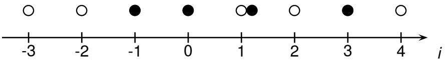

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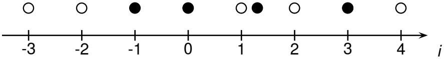

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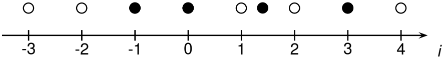

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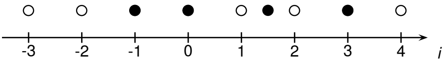

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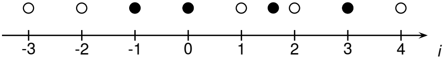

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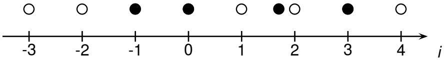

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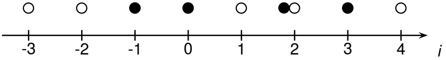

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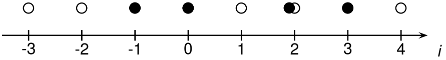

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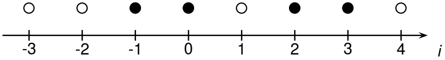

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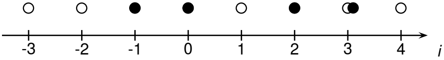

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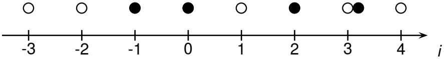

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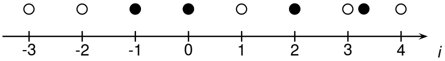

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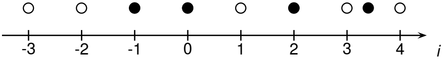

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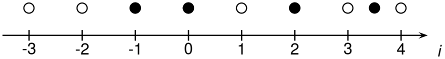

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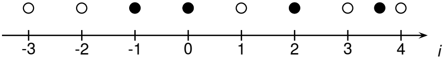

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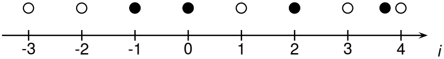

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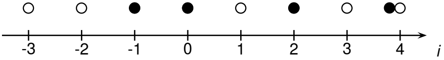

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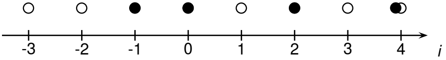

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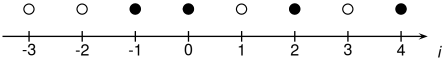

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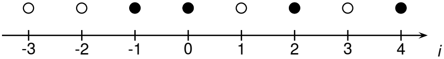

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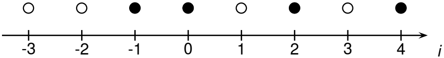

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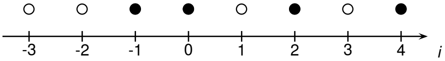

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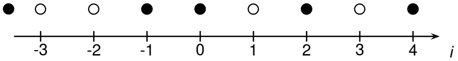

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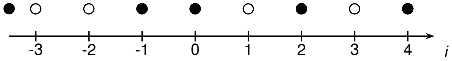

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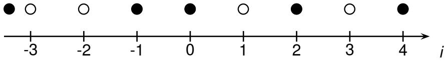

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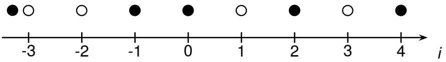

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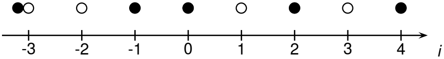

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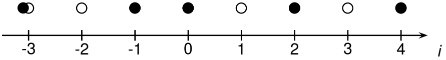

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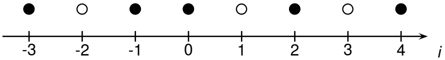

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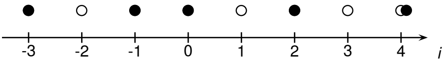

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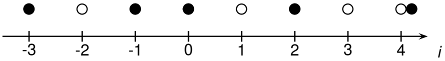

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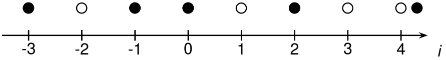

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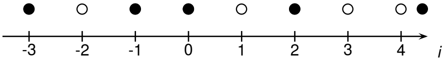

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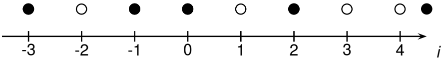

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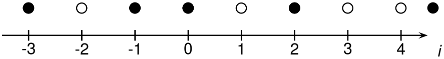

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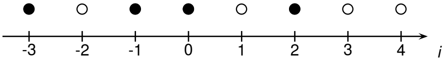

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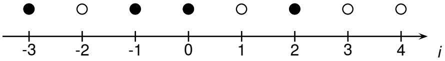

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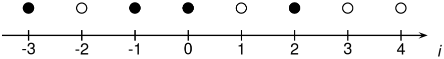

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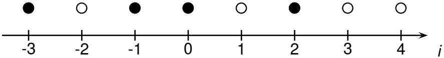

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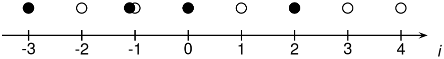

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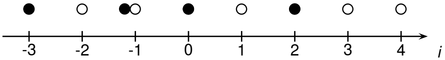

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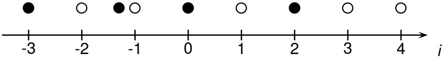

$$\eta_i \in \{0, 1\}.$$

Particles try to jump

to the right with rate p,
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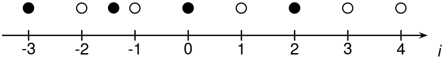

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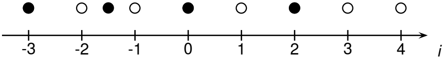

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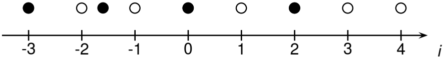

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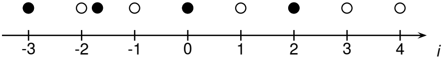

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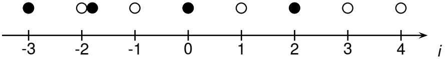

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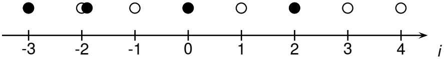

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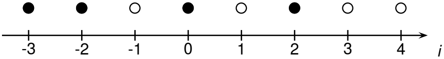

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Product blocking measures

Can we have a reversible stationary distribution in product form:

$$\underline{\mu}(\underline{\omega}) = \bigotimes_i \mu_i(\omega_i);$$

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}\big(\underline{\omega} \to \underline{\omega}^{i\curvearrowright i+1}\big) = \underline{\mu}\big(\underline{\omega}^{i\curvearrowright i+1}\big) \cdot \text{rate}\big(\underline{\omega}^{i\curvearrowright i+1} \to \underline{\omega}\big) \quad ?$$

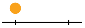
Here

$$\underline{\omega}^{i\curvearrowright i+1} = \underline{\omega} - \underline{\delta}_i + \underline{\delta}_{i+1}.$$

Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}\big(\underline{\eta} \to \underline{\eta}^{i\curvearrowright i+1}\big) = \underline{\mu}(\underline{\eta}^{i\curvearrowright i+1}) \cdot \text{rate}\big(\underline{\eta}^{i\curvearrowright i+1} \to \underline{\eta}\big)$$

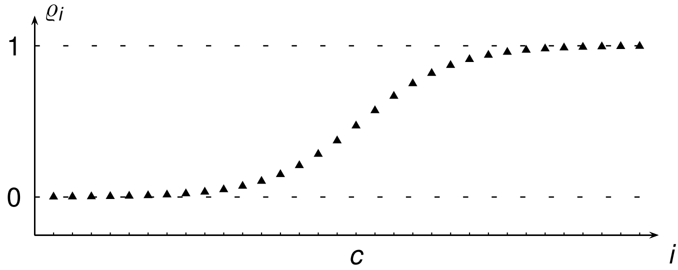
ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$;



$\underline{\eta}$

$$\varrho_i(1-\varrho_{i+1}) \cdot p = (1-\varrho_i)\varrho_{i+1} \cdot q$$

Solution: $$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1+(\frac{p}{q})^{i-c}} = \frac{1}{(\frac{q}{p})^{i-c}+1}$$



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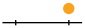

$\underline{\eta}^{i \curvearrowright i+1}$

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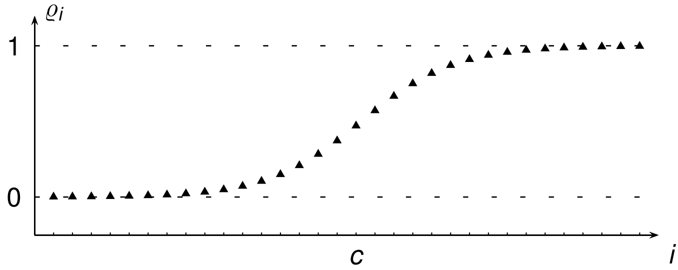


Asymmetric Simple Exclusion

Notice:

$$\mathbf{P}\{\eta_i = 0\} = 1 - \varrho_i = \frac{1}{1 + (\frac{p}{q})^{i-c}} \qquad \text{as } i \to \infty$$

$$\mathbf{P}\{\eta_i = 1\} = \varrho_i = \frac{1}{(\frac{q}{p})^{i-c} + 1} \qquad \text{as } i \to -\infty$$

are both summable. Hence by Borel-Cantelli there is $\underline{\mu}$-a.s. a rightmost hole and a leftmost particle,

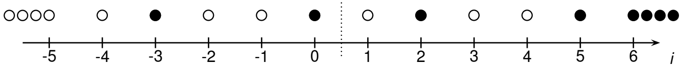
$$N := \sum_{i=1}^{\infty}(1 - \eta_i) - \sum_{i=-\infty}^{0} \eta_i$$

is $\underline{\mu}$-a.s. finite.

Asymmetric Simple Exclusion

$$N(\underline{\eta}) := \sum_{i=1}^{\infty}(1-\eta_i) - \sum_{i=-\infty}^{0}\eta_i$$

$$1 = 3 - 2$$



Also: it is conserved. Define

$$\Omega^n := \{0,\ 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

the irreducible components of the state space.

$$\underline{\mu}\Big(\bigcup_{n=-\infty}^{\infty}\Omega^n\Big) = 1.$$

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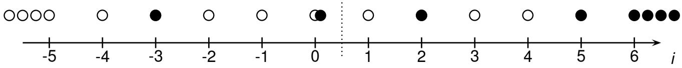

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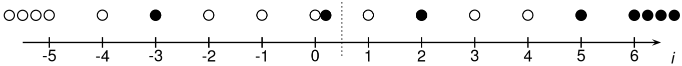

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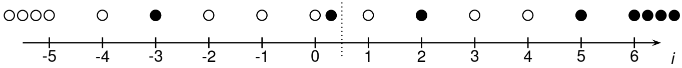

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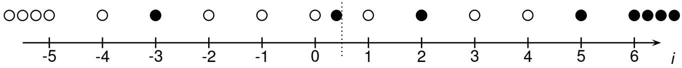

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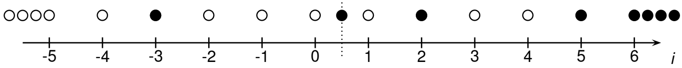

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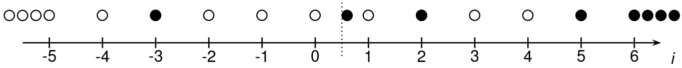

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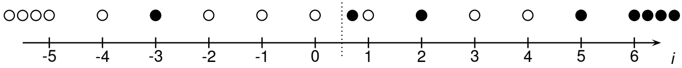

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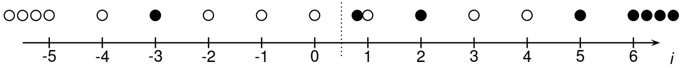

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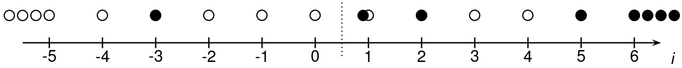

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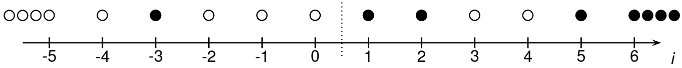

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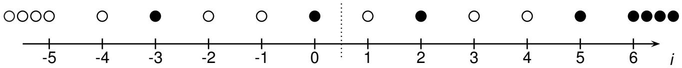

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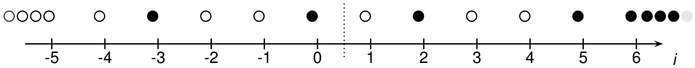

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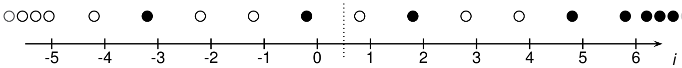

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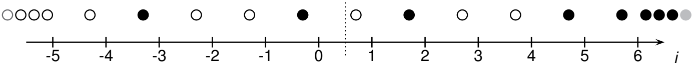

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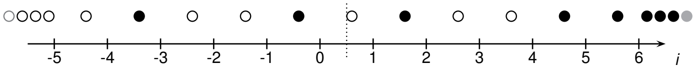

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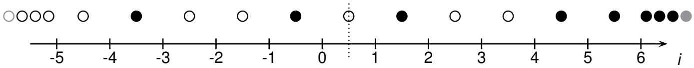

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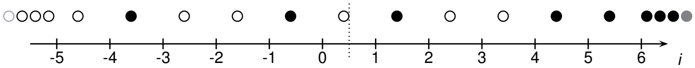

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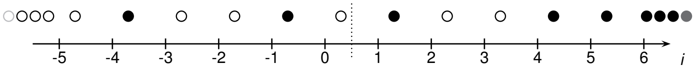

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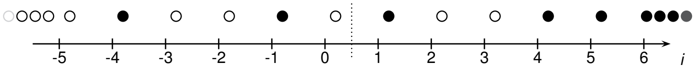

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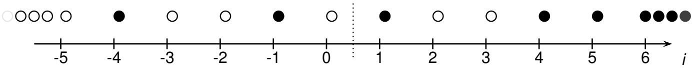

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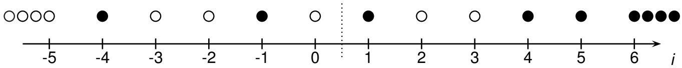

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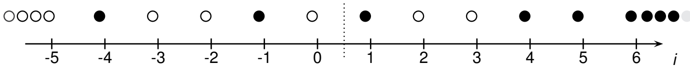

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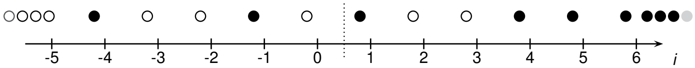

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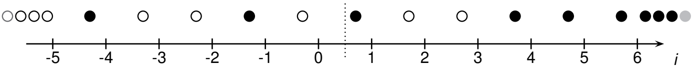

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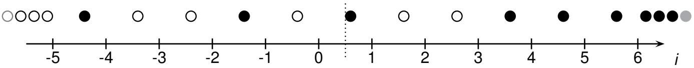

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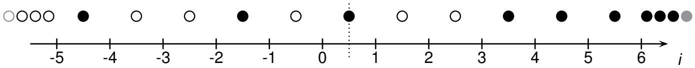

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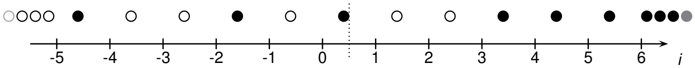

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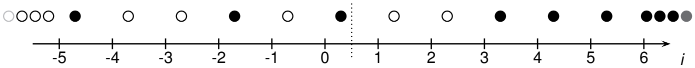

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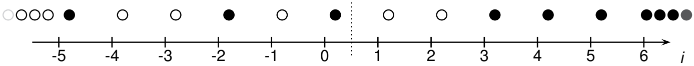

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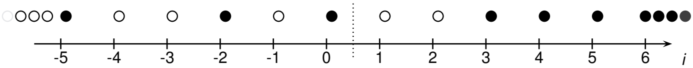

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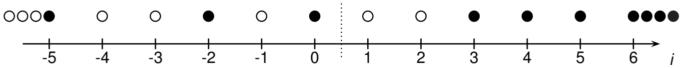

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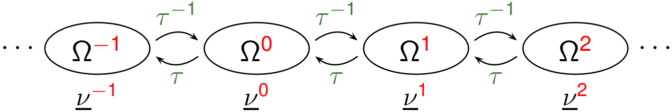

$$\underline{\mu}(\cdot) = \sum_{n=-\infty}^{\infty} \underline{\mu}(\cdot \,|\, N(\cdot) = n)\underline{\mu}(N(\cdot) = n) = \sum_{n=-\infty}^{\infty} \underline{\nu}^n(\cdot)\underline{\mu}(N(\cdot) = n).$$

Ergodic decomposition of $\underline{\mu}$ with $\underline{\nu}^n(\cdot) := \underline{\mu}(\cdot \,|\, N(\cdot) = n)$.

Let's find the coefficients $\underline{\mu}(N(\cdot) = n)$!

Asymmetric Simple Exclusion

<u>Recall:</u>

$$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1 + (\frac{p}{q})^{i-c}} = \frac{1}{(\frac{q}{p})^{i-c} + 1}$$

Asymmetric Simple Exclusion

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$$\underline{\mu}(\underline{\eta}) = \prod_{i \le 0} \frac{\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{1 + \left(\frac{p}{q}\right)^{i-c}} \cdot \prod_{i > 0} \frac{\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

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$$\underline{\mu}(\tau\underline{\eta}) = \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}$$

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$$= \cdots = \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}.$$

Asymmetric Simple Exclusion

So,

$$\underline{\mu}(N = n - 1) = \sum_{\underline{\eta}:N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta})$$

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So,

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Solution:

$$\underline{\mu}(N = n) = \frac{\big(\frac{q}{p}\big)^{\frac{n^2+n}{2}-cn}}{\sum\big(\frac{q}{p}\big)^{\frac{m^2+m}{2}-cm}}$$

discrete Gaussian.

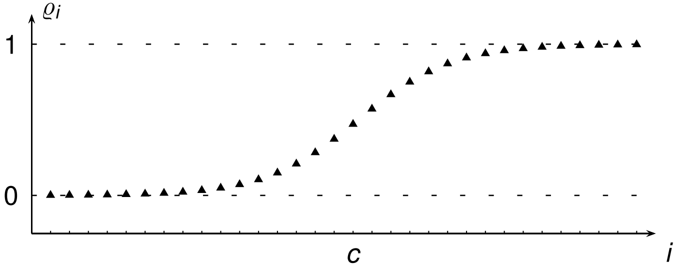
Asymmetric Simple Exclusion

Also:

$$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1+(\frac{p}{q})^{i-c}} = \frac{1}{(\frac{q}{p})^{i-c}+1}$$

so it's not surprising that

$$\underline{\mu}^c(\tau\underline{\eta}) = \underline{\mu}^{c+1}(\underline{\eta}).$$



Euler's identity; q-binomial theorem

Let $N^{\overleftarrow{\mathrm{p}}}_{1/2}$ be the number of particles on the left of $\frac{1}{2}$.

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Let $N^{\overleftarrow{\mathrm{p}}}_{1/2}$ be the number of particles on the left of $\frac{1}{2}$.

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- But this is just a shift, so $\underline{\mu}^{c}(N^{\overleftarrow{\mathrm{p}}}_{-1/2} = k) = \underline{\mu}^{c+1}(N^{\overleftarrow{\mathrm{p}}}_{1/2} = k)$.

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- Solve it, get the $\underline{\mu}^{c}$-distribution of $N^{\overleftarrow{p}}_{1/2}$.

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- That this sums to one is Euler's identity.

$\underline{\mu}^c(\tau\underline{\eta}) = \underline{\mu}^{c+1}(\underline{\eta})$

$$\sum_{i=0}^{\infty} \frac{q^{\frac{i(i-1)}{2}} z^i}{(q;q)_i} = (-z;q)_\infty$$

Euler's identity; q-binomial theorem

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- That this sums to one is Euler's identity.
- Repeat the same with the number of particles in a finite interval; get the q-binomial theorem.

$\underline{\mu}^c(\tau\underline{\eta}) = \underline{\mu}^{c+1}(\underline{\eta})$

$$\sum_{i=0}^{m} q^{\frac{i(i-1)}{2}} z^i \begin{bmatrix} m \\ i \end{bmatrix}_q = (-z;q)_m$$

Durfee rectangles identity

- Recolor particles into holes, holes into particles. Get an ASEP drifting left with finite number of particles on the right and holes on the left.

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- Follow through what happens to $N^{\overleftarrow{\mathrm{p}}}_{1/2}$:

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- Follow through what happens to $\overleftarrow{N}^{\mathrm{p}}_{1/2}$:

  $\underline{\mu}^c(\overleftarrow{N}^{\mathrm{p}}_{1/2}) = \underline{\mu}^{1-c}(\overrightarrow{N}^{\mathrm{h}}_{1/2})$.
- Add this to the convolution $N = \overrightarrow{N}^{\mathrm{h}}_{1/2} - \overleftarrow{N}^{\mathrm{p}}_{1/2}$ to get the Durfee rectangles identity.

$$\sum_{i=n^-}^{\infty} \frac{q^{i(n+i)}}{(q;q)_{n+i}\cdot(q;q)_i} = \frac{1}{(q;q)_\infty}$$

Second class particles

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- We get an explicit formula for the distribution of locations of a fixed number of second class particles under $\underline{\mu}^c$.
- No new identities there, although some we have can be reproved using the second class particles. They provide a wormhole between $\dots, \Omega^{-1}, \Omega^0, \Omega^1, \dots$

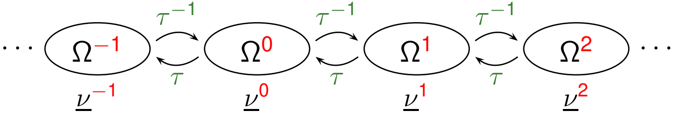
The ergodic measure

And, if $N(\underline{\eta}) = n$,

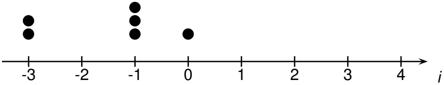
$$\underline{\nu}^n(\underline{\eta}) = \underline{\mu}(\underline{\eta} \,|\, N(\underline{\eta}) = n) = \frac{\underline{\mu}(\underline{\eta})}{\underline{\mu}(N(\underline{\eta}) = n)}$$

$$= \frac{\prod\limits_{i\le 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod\limits_{i\le 0}\left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod\limits_{i>0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod\limits_{i>0}\left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}.$$

This is the unique stationary distribution on Ω^n.



The asymmetric zero range process



$$\omega_i \in \mathbb{Z}^+.$$

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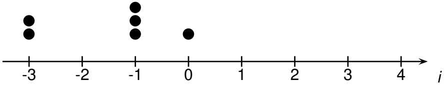

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Particles jump
to the right with rate p
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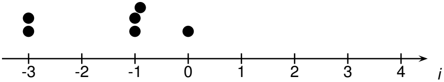

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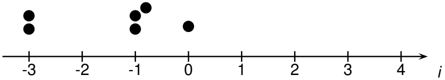

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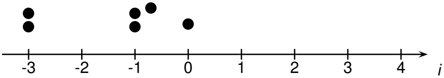

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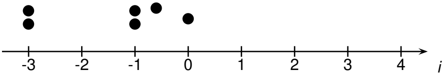

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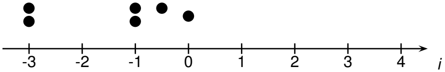

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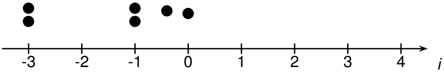

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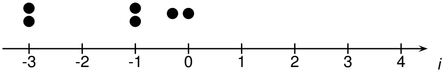

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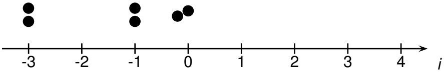

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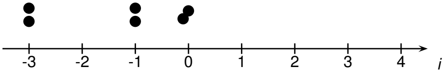

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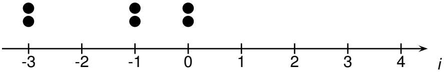

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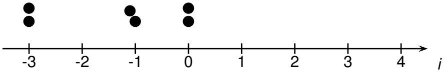

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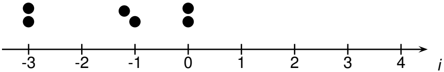

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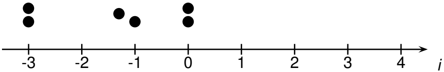

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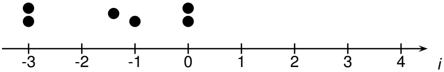

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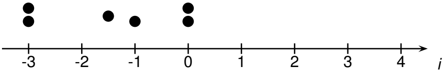

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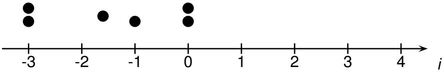

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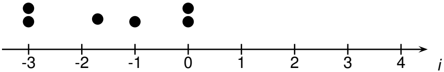

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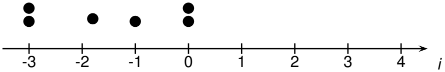

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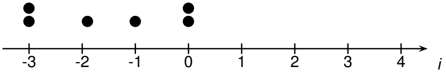

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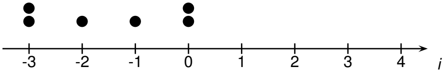

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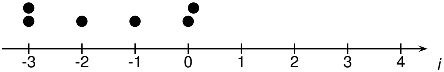

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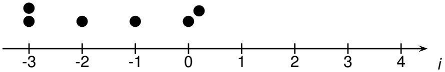

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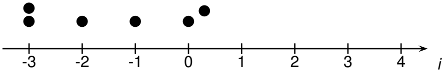

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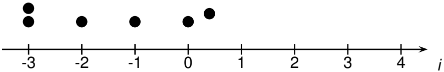

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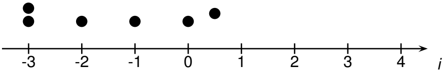

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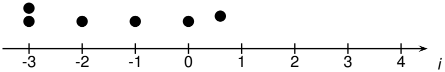

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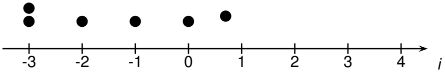

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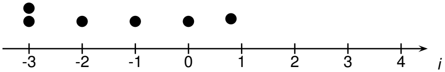

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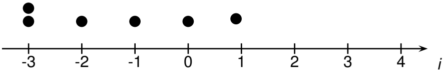

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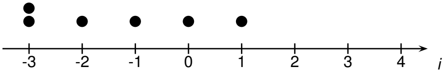

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Asymmetric zero range process

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}\big(\underline{\omega} \to \underline{\omega}^{i\curvearrowright i+1}\big) = \underline{\mu}(\underline{\omega}^{i\curvearrowright i+1}) \cdot \text{rate}\big(\underline{\omega}^{i\curvearrowright i+1} \to \underline{\omega}\big) \quad ?$$

AZRP:

$$\mu_i(\omega_i)\mu_{i+1}(\omega_{i+1}) \cdot p\mathbf{1}\{\omega_i > 0\} = \mu_i(\omega_i - 1)\mu_{i+1}(\omega_{i+1} + 1) \cdot q$$

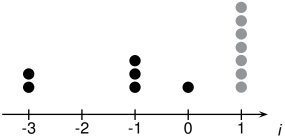
Solution: $\mu_i \sim \text{Geometric}\Big(1 - \Big(\frac{p}{q}\Big)^{i-\text{const}}\Big).$

State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

$\rightsquigarrow$ we have a problem: cannot do this for all i! We'll pick const $= 1$ and have a *right boundary* instead.



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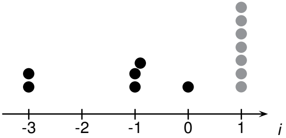

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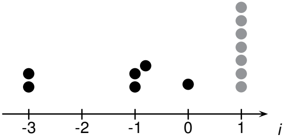

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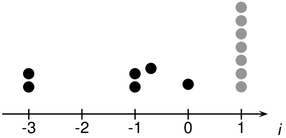

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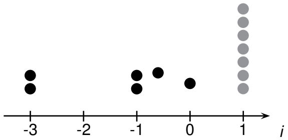

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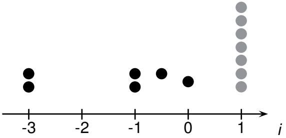

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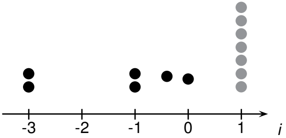

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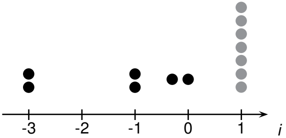

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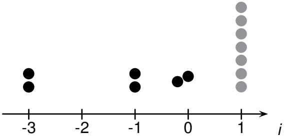

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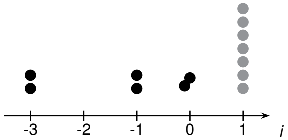

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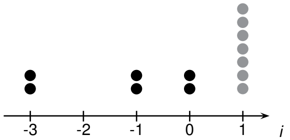

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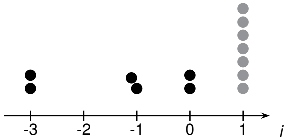

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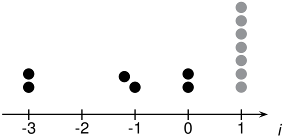

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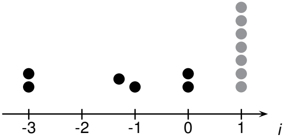

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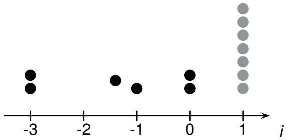

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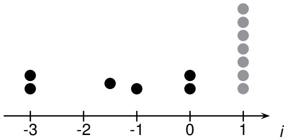

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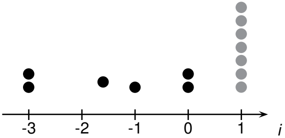

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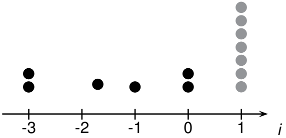

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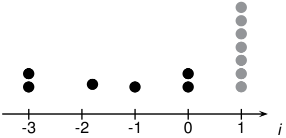

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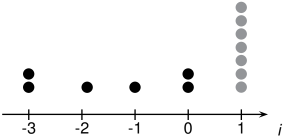

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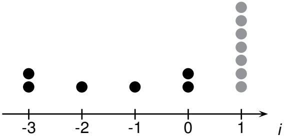

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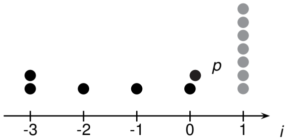

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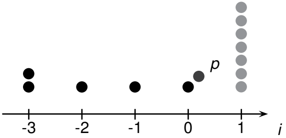

State space: AZRP

Recall: Stationary distribution with marginals

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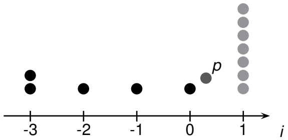

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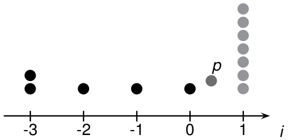

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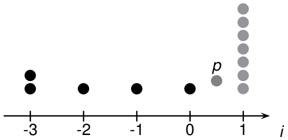

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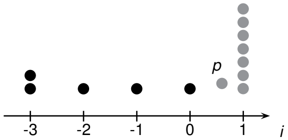

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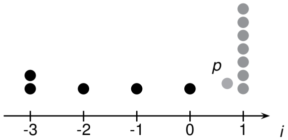

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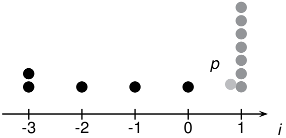

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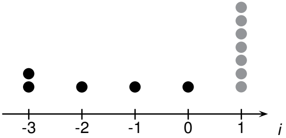

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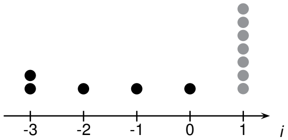

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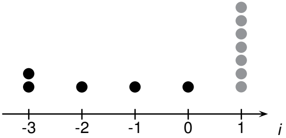

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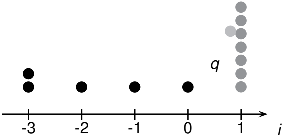

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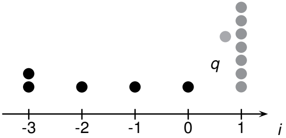

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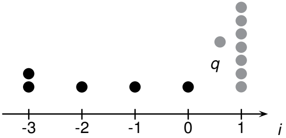

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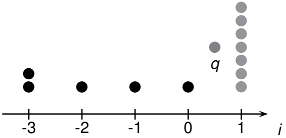

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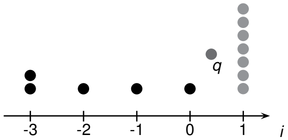

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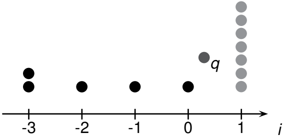

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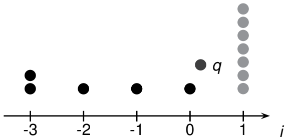

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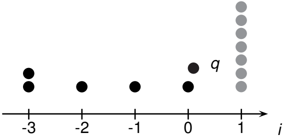

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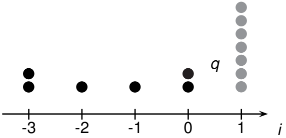
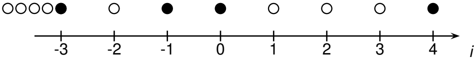


$\rightsquigarrow$ The product measure stays stationary on the half-line.

Lay down / stand up

ASEP



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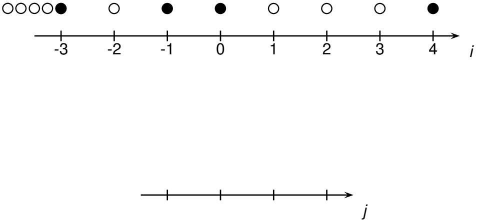

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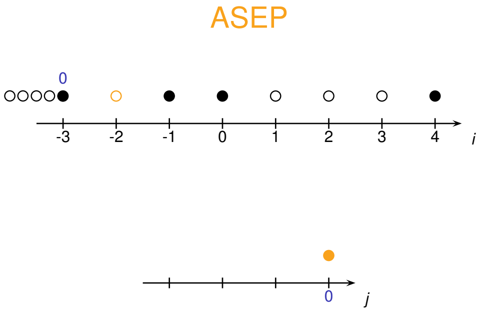

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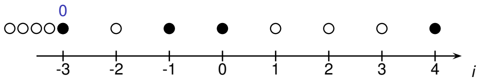

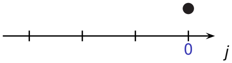

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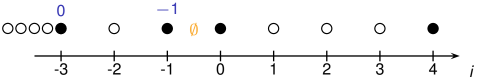

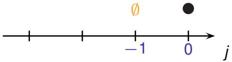

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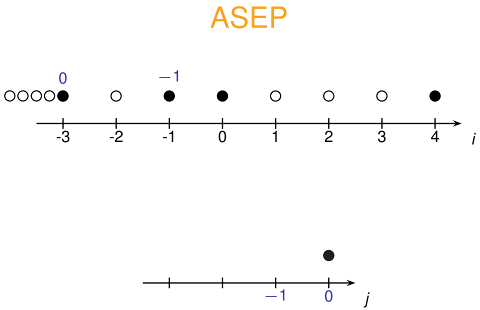

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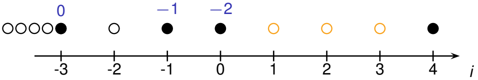

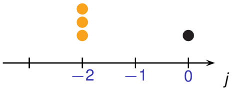

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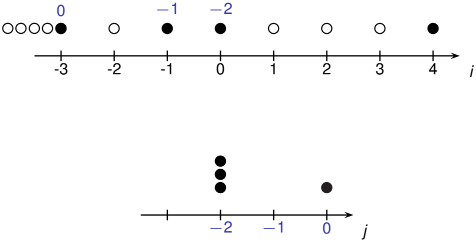

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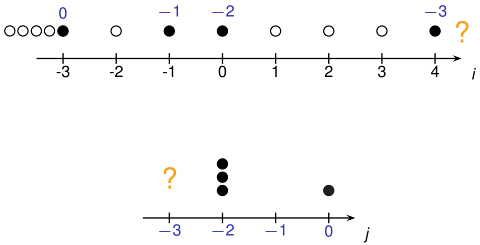

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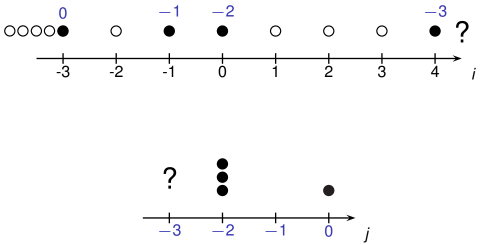

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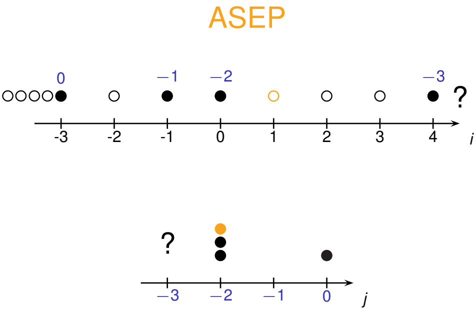

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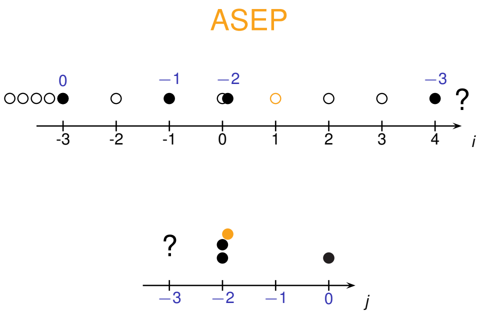

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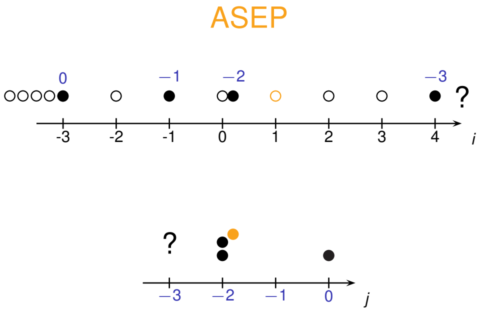

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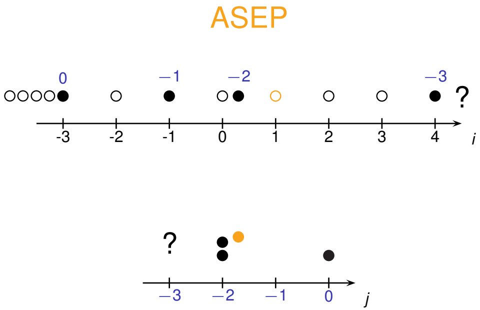

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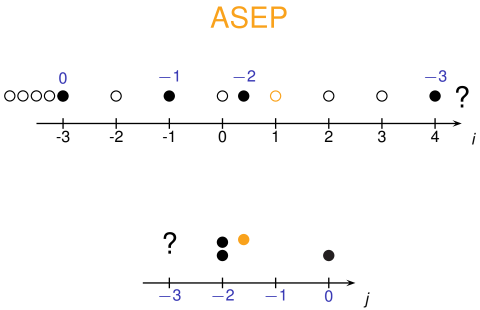

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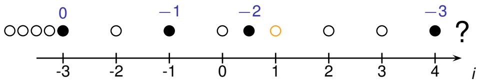

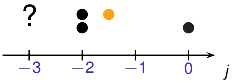

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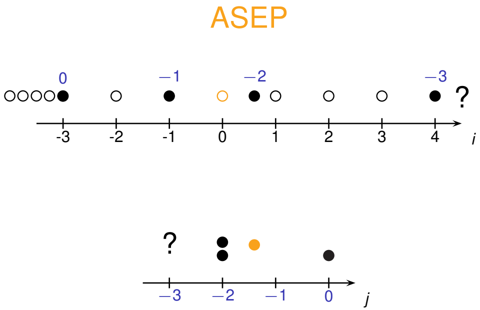

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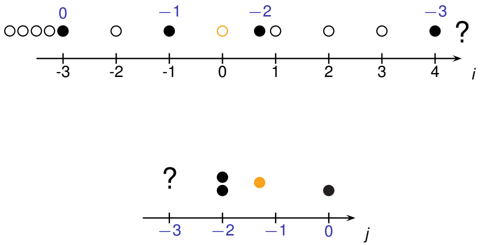

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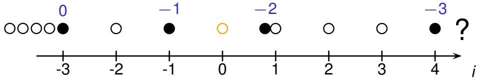

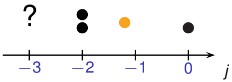

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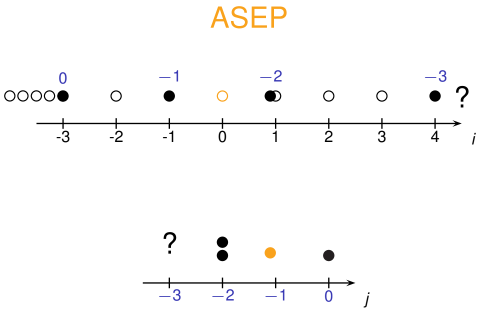

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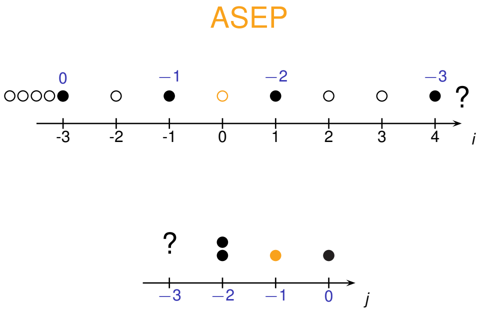

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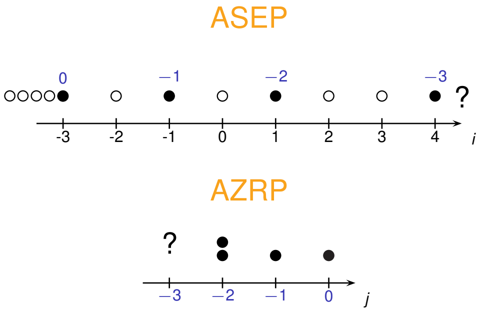

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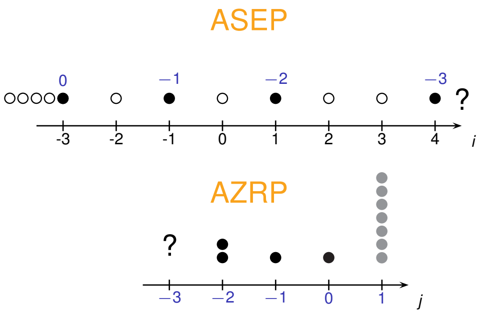

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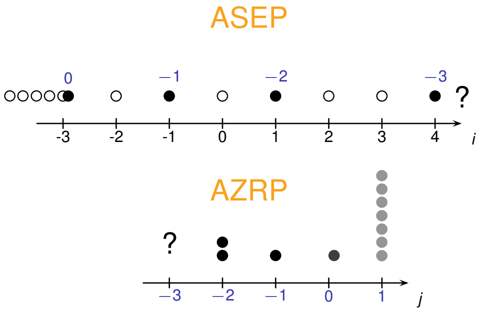

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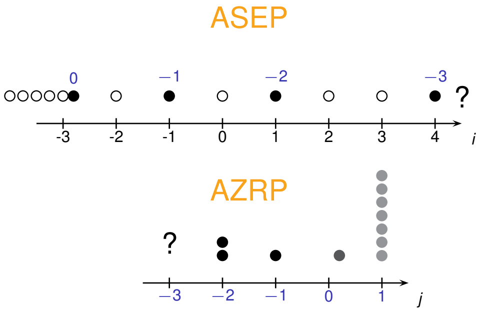

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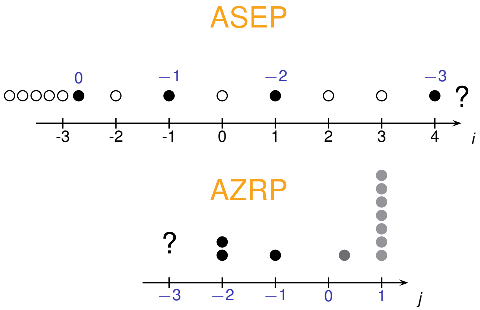

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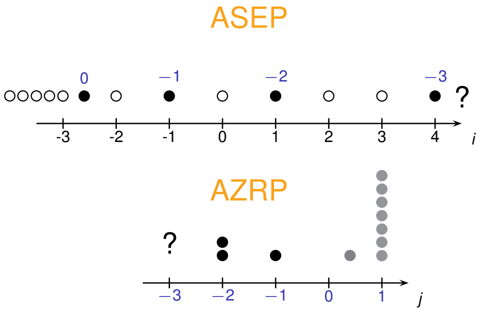

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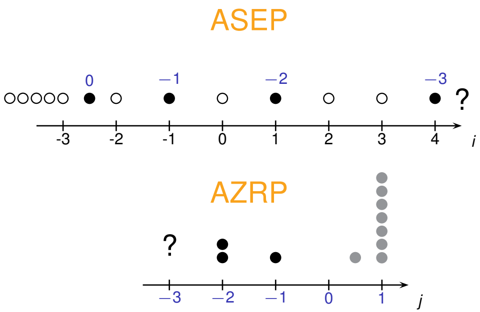

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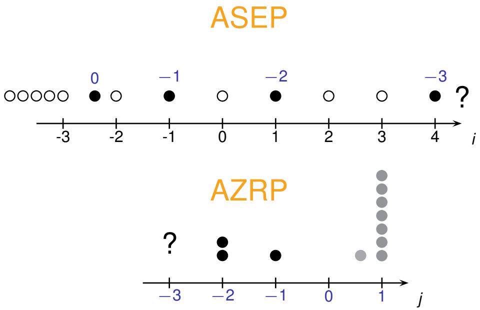

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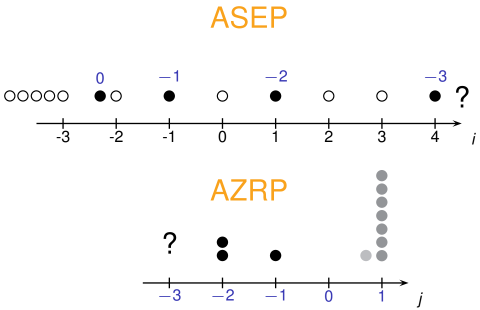

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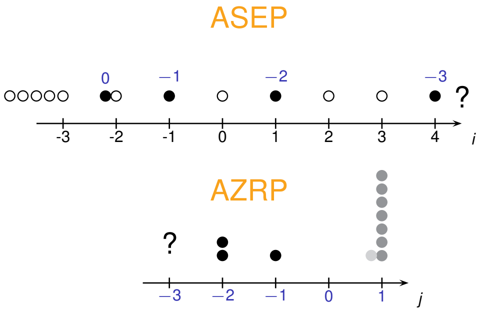

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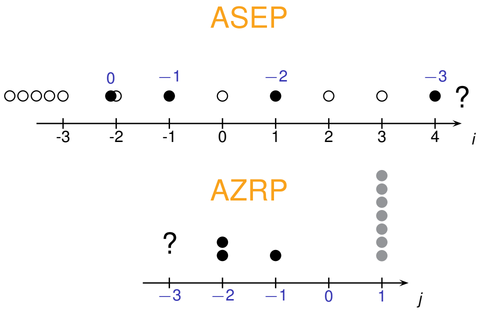

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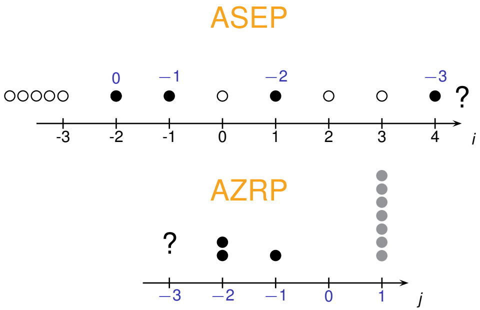
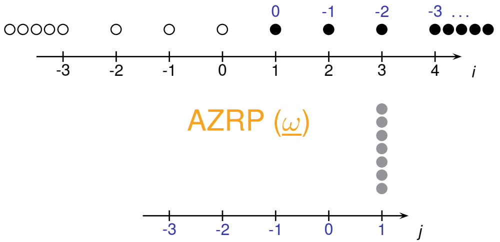


ASEP $\overset{T^n}{=}$ AZRP

$$\underline{\nu}^n \overset{T^n}{=} \prod_{i \le 0} \text{Geometric}\Big(1 - \Big(\frac{p}{q}\Big)^{i-1}\Big)$$

since stationary distributions of countable irreducible Markov chains are unique.

Jacobi triple product

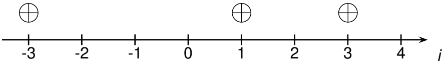


$$\eta_i = \mathbf{1}\{i \geq 1\}, \qquad N(\underline{\eta}) = 0, \qquad \omega_i \equiv 0.$$

$$\underline{\nu}^0(\underline{\eta}) = \frac{\prod_{i\leq 0}\left(\frac{p}{q}\right)^{(i-c)\cdot 0}}{\prod_{i\leq 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)\cdot(1-1)}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)} \cdot \frac{\sum\left(\frac{q}{p}\right)^{\frac{m^2+m}{2}-cm}}{\left(\frac{q}{p}\right)^{\frac{0^2+0}{2}-c\cdot 0}}$$

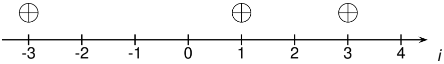
$$= \underline{\mu}(\underline{\omega}) = \prod_{i\leq 0}\left(1-\left(\frac{p}{q}\right)^{i-1}\right) \qquad \square$$

$\oplus$ $\ominus$ $\emptyset$ models



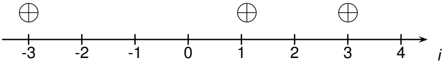
⊕ ⊖ ∅ models

⊕ to the right



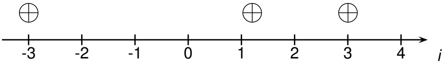
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



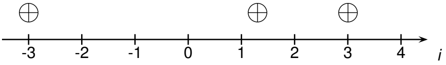
⊕ ⊖ ∅ models

⊕ to the right



$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



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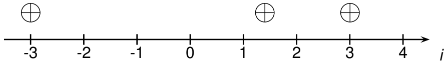

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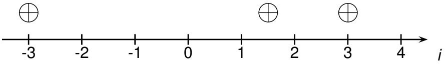
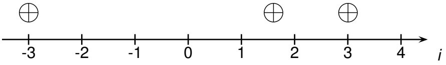
⊕ to the right



$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



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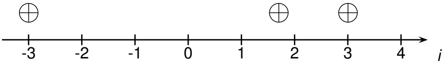

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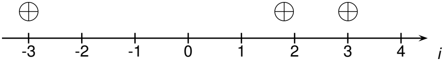

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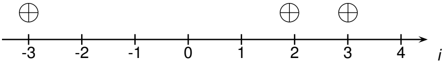

$\oplus$ $\ominus$ $\emptyset$ models

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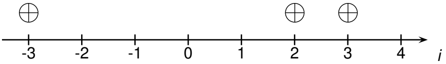
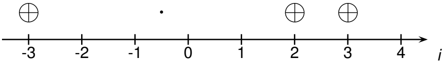


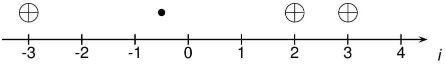
⊕ ⊖ ∅ models

pair creation from vacuum



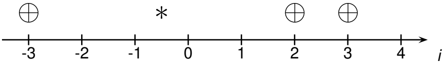
⊕ ⊖ ∅ models

pair creation from vacuum



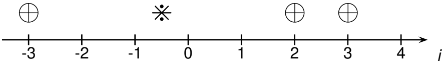
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



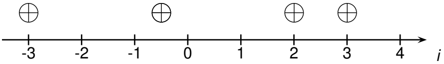
⊕ ⊖ ∅ models

pair creation from vacuum



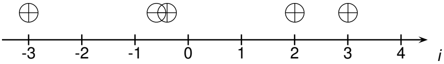
⊕ ⊖ ∅ models

pair creation from vacuum



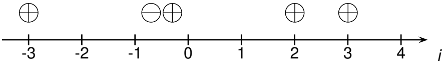
⊕ ⊖ ∅ models

pair creation from vacuum



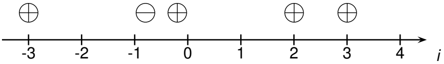
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



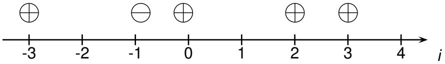
⊕ ⊖ ∅ models

pair creation from vacuum



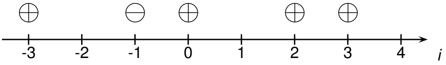
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



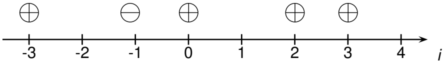
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



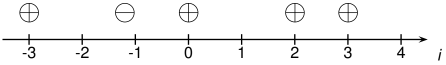
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



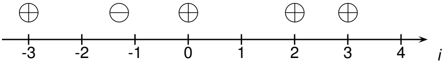
⊕ ⊖ ∅ models

⊖ to the left



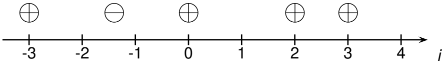
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



⊕ ⊖ ∅ models

⊖ to the left



⊕ ⊖ ∅ models

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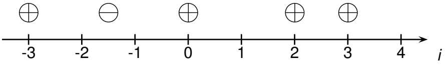

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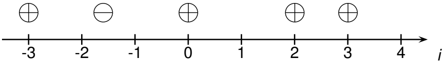

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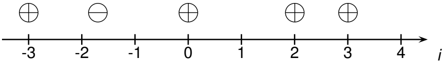

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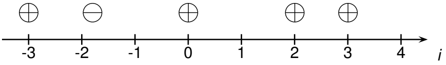

⊕ ⊖ ∅ models

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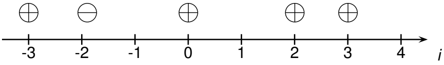
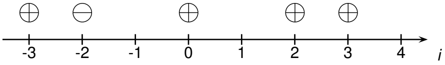


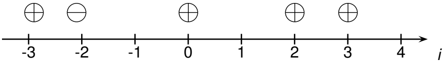
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



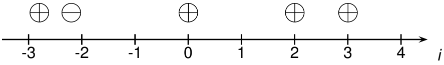
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



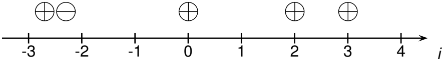
⊕ ⊖ ∅ models

annihilation



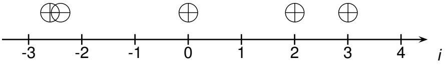
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



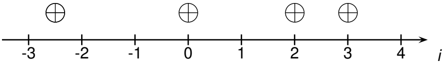
$\oplus \ominus \emptyset$ models

annihilation



$\oplus$ $\ominus$ $\emptyset$ models

annihilation



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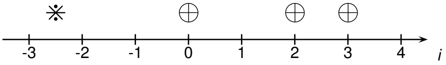
annihilation



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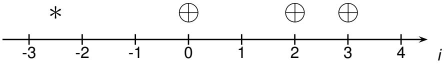
annihilation



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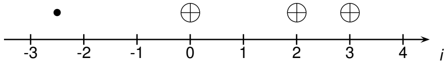
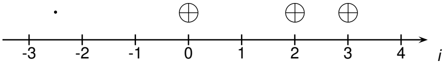


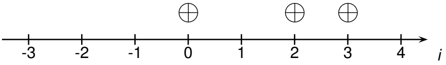
⊕ ⊖ ∅ models

annihilation

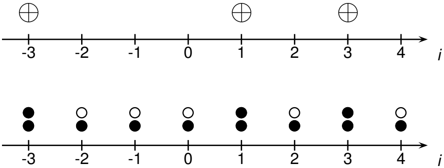


$\oplus$ $\ominus$ $\emptyset$ models

annihilation

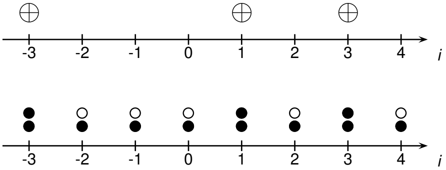


$\oplus$ $\ominus$ $\emptyset$ models



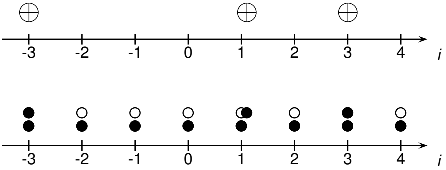
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



$\oplus$ $\ominus$ $\emptyset$ models

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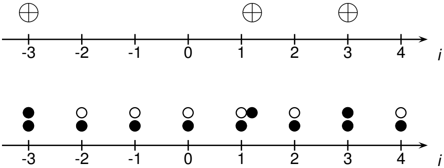

$\oplus$ $\ominus$ $\emptyset$ models

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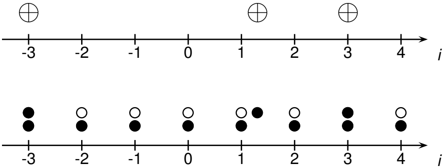

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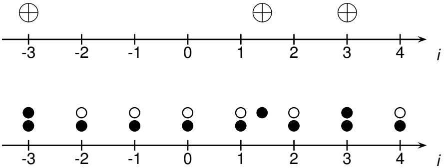
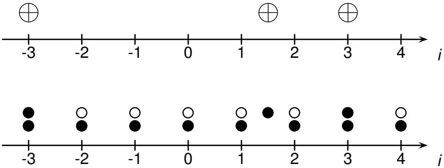


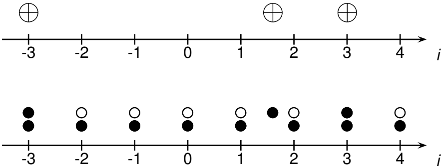
⊕ ⊖ ∅ models

⊕ to the right



$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



$\oplus$ $\ominus$ $\emptyset$ models

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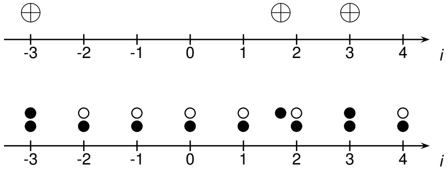

⊕ ⊖ ∅ models

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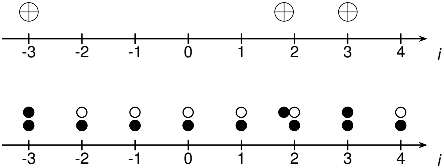

$\oplus$ $\ominus$ $\emptyset$ models

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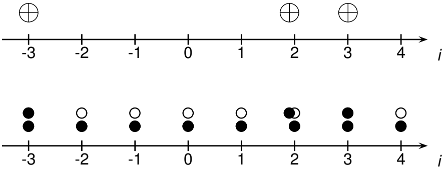
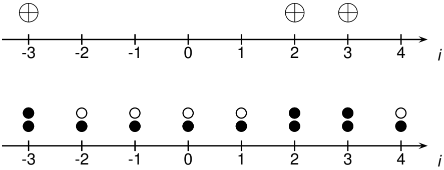
$\oplus$ to the right



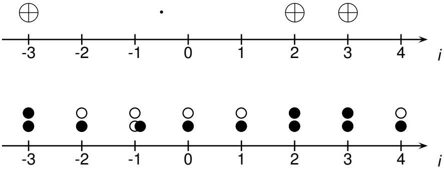
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



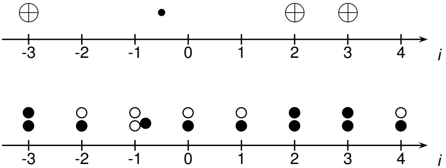
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



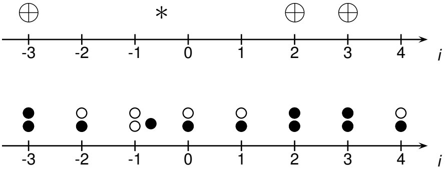
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



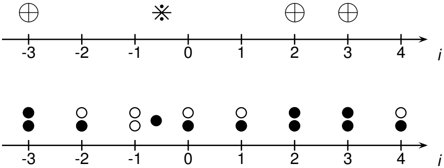
⊕ ⊖ ∅ models

pair creation from vacuum



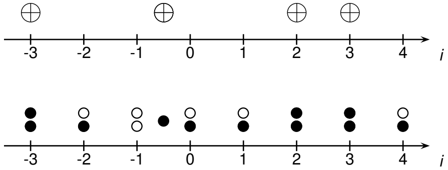
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



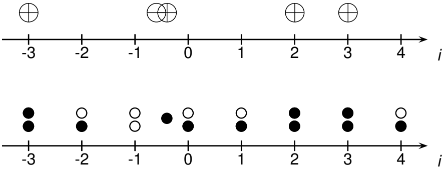
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



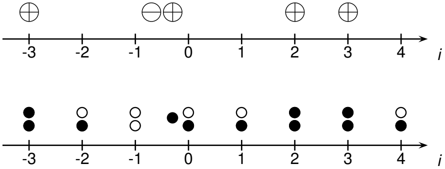
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



$\oplus$ $\ominus$ $\emptyset$ models

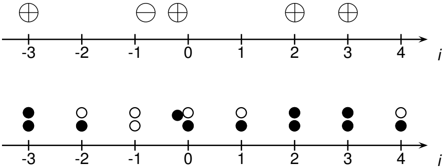
pair creation from vacuum



$\oplus$ $\ominus$ $\emptyset$ models

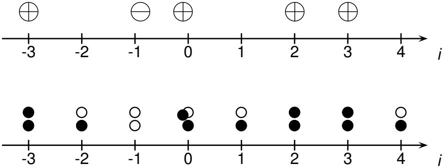
$\oplus \rightsquigarrow$ ●● $\qquad \emptyset \rightsquigarrow$ ○● $\qquad \ominus \rightsquigarrow$ ○○

pair creation from vacuum



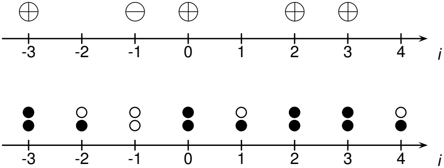
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



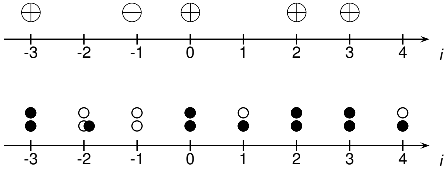
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



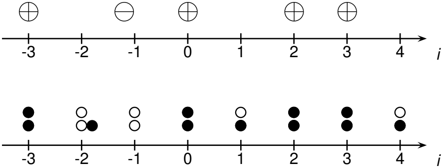
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



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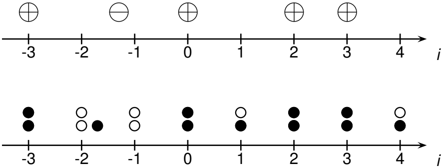

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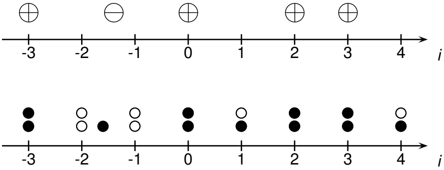

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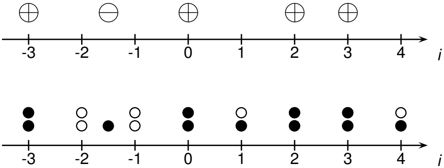

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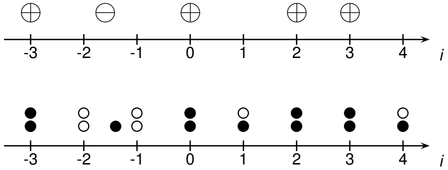

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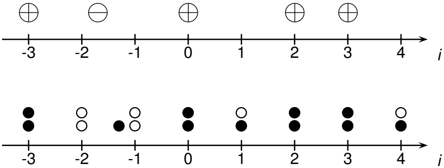

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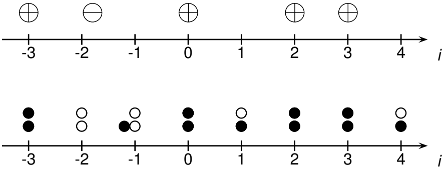
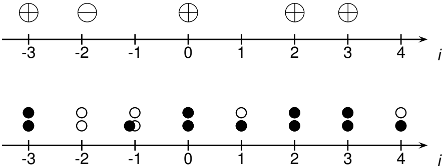
$\ominus$ to the left



$\oplus$ $\ominus$ $\emptyset$ models

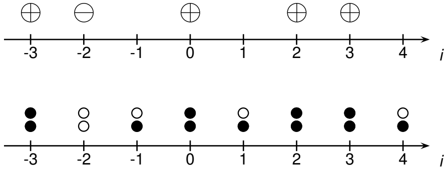
$\oplus \rightsquigarrow$ ●● $\qquad \emptyset \rightsquigarrow$ ○● $\qquad \ominus \rightsquigarrow$ ○○

$\ominus$ to the left



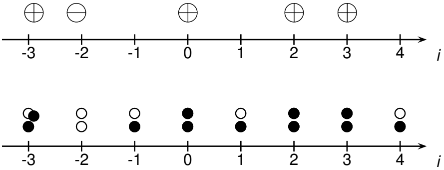
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



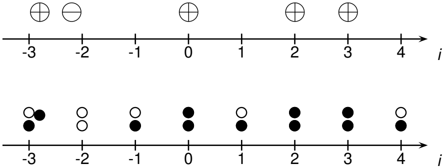
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



$\oplus$ $\ominus$ $\emptyset$ models

annihilation



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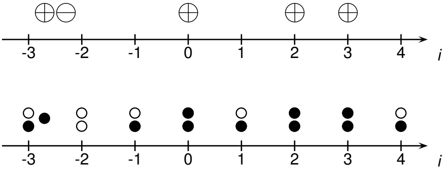
annihilation



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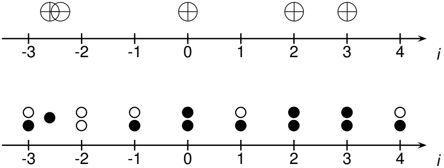
annihilation



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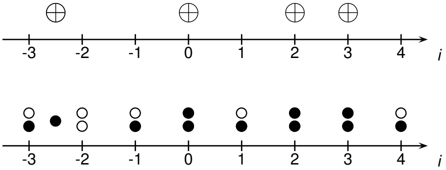
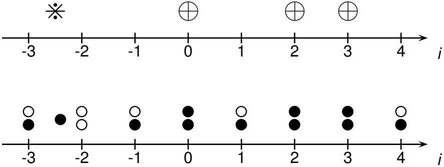
annihilation



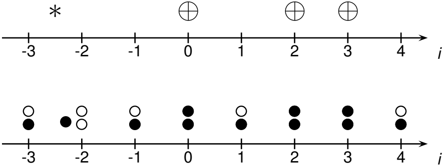
⊕ ⊖ ∅ models

annihilation



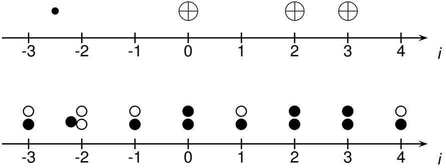
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



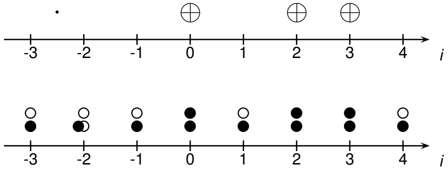
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



$\oplus$ $\ominus$ $\emptyset$ models

annihilation



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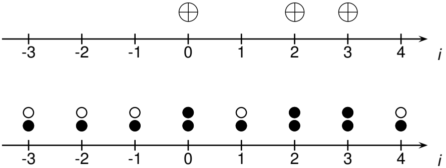
annihilation



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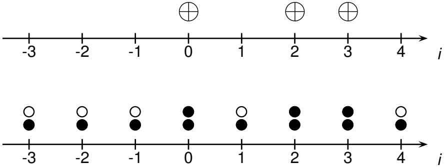
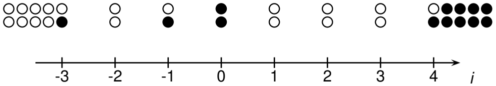


. . . and the same steps to the left.

Lay down / stand up a bit differently



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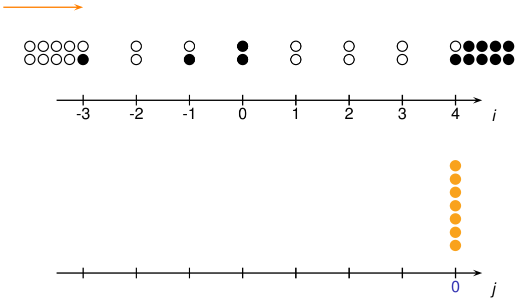

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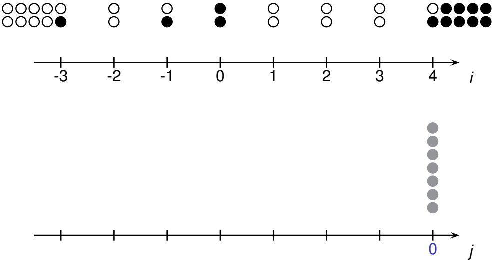

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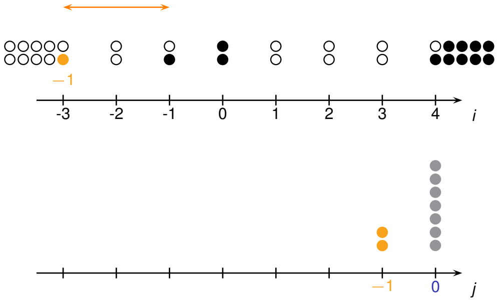

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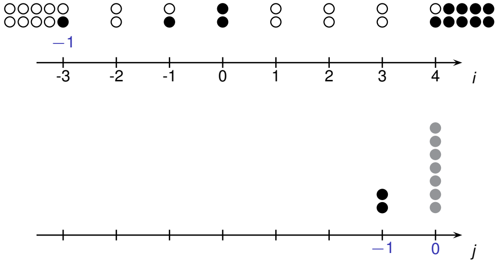

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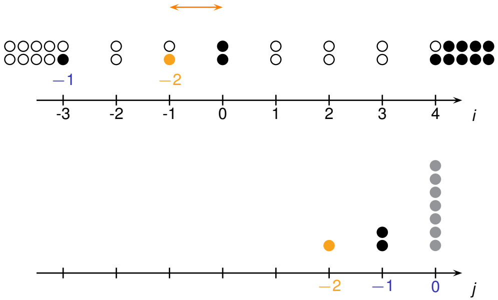

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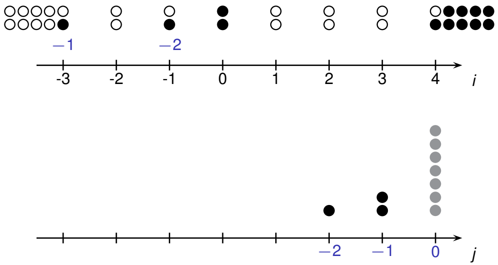

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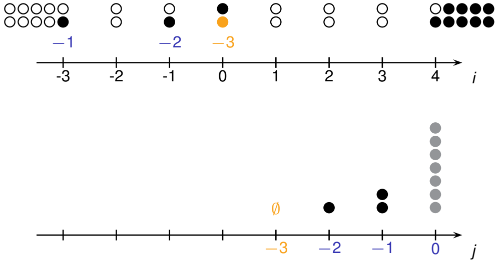

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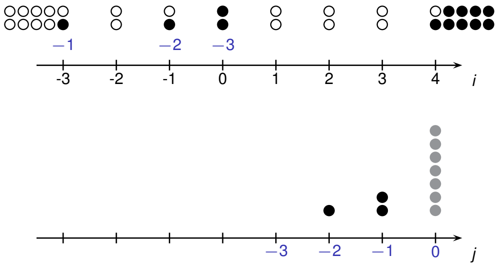

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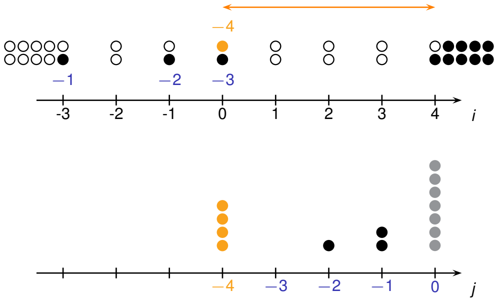

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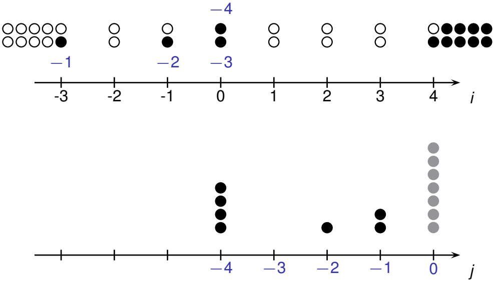

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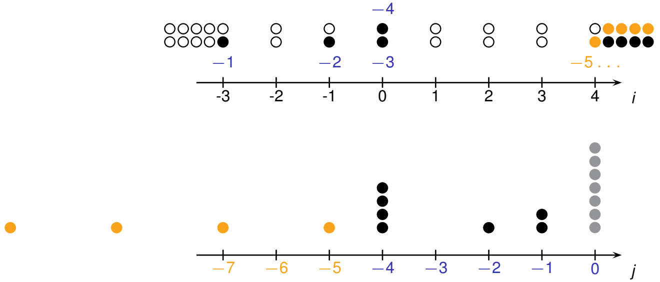

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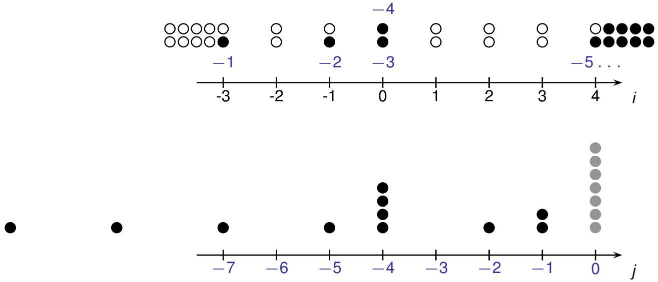

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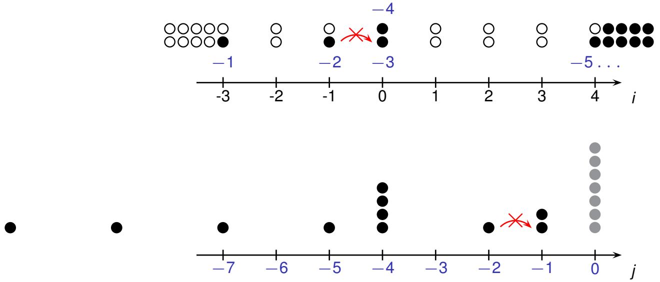
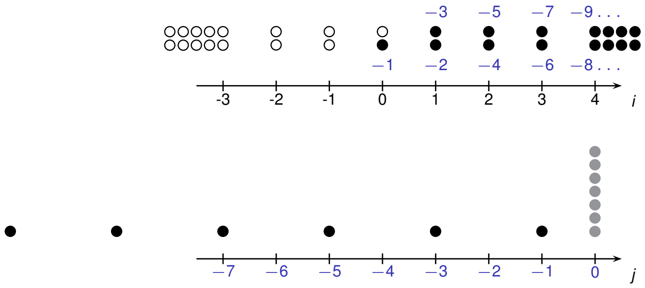


No two consecutive 0's!

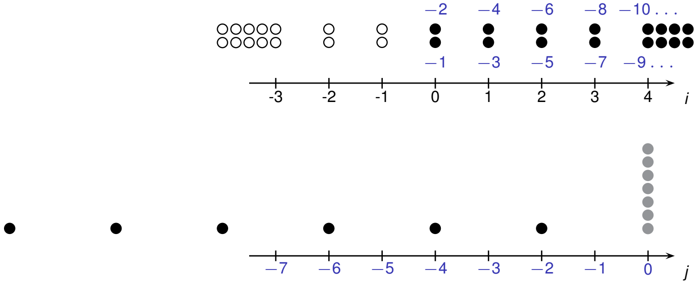
Lay down / stand up a bit differently

Odd ground state:



Lay down / stand up a bit differently

Even ground state:



Lay down / stand up a bit differently

- *No two consecutive 0's!* is a nonlocal constraint.

Lay down / stand up a bit differently

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- The stood up model is nice otherwise. It has reversible product blocking measures.

Lay down / stand up a bit differently

- *No two consecutive 0's!* is a nonlocal constraint.
- The stood up model is nice otherwise. It has reversible product blocking measures.
- *Reversible* measures survive forbidden jumps. Yay!

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

ASEP(q, 1) Carinci, Giardiná, Redig, Sasamoto:

Jump	Rate	Jump	Rate
$\oplus \rightarrow$	$q^{-1} + q^{-3}$	$\leftarrow \oplus$	$q + q^3$
$\leftarrow \ominus$	$q^{-1} + q^{-3}$	$\ominus \rightarrow$	$q + q^3$
$\emptyset \rightsquigarrow \ominus\oplus$	q^{-3}	$\emptyset \rightsquigarrow \oplus\ominus$	q^3
$\oplus\ominus \rightsquigarrow \emptyset$	$(1 + q^2)(q^{-1} + q^{-3})$	$\ominus\oplus \rightsquigarrow \emptyset$	$(1 + q^{-2})(q + q^3)$

Identity: *Has to do with the odd and even terms of Jacobi's triple product.*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

A nice three-state model:

Jump	**Rate**	**Jump**	**Rate**
$\oplus \rightarrow$	1	$\leftarrow \oplus$	q
$\leftarrow \ominus$	1	$\ominus \rightarrow$	q
$\emptyset \rightsquigarrow \ominus\oplus$	c	$\emptyset \rightsquigarrow \oplus\ominus$	qc
$\oplus\ominus \rightsquigarrow \emptyset$	2	$\ominus\oplus \rightsquigarrow \emptyset$	$2q$

Identity: *Has to do with the square of Jacobi's triple product.*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

<u>2-exclusion:</u>

Jump	**Rate**	**Jump**	**Rate**
$\oplus \rightarrow$	1	$\leftarrow \oplus$	q
$\leftarrow \ominus$	1	$\ominus \rightarrow$	q
$\emptyset \rightsquigarrow \ominus\oplus$	1	$\emptyset \rightsquigarrow \oplus\ominus$	q
$\oplus\ominus \rightsquigarrow \emptyset$	1	$\ominus\oplus \rightsquigarrow \emptyset$	q

Identity: *Looks new and interesting...*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

<u>K-exclusion:</u>

Identity: *Rather nice generalisation using the K^{th} roots of unity.*

New identities

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New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

K-exclusion:

Identity: *Rather nice generalisation using the* K^{th} *roots of unity.*

Zero range:

Identity: *The geometric sum formula. :-D*

Thank you.

Second class particles revealed

Theorem

The $\underline{\mu}^c$-probability of seeing d second class particles at locations $m_1 < m_2 < \cdots < m_d$ is

$$\frac{\prod_{i=1}^{d}(1-q^i)\cdot q^{dc-\sum_{k=1}^{d} m_k}}{\prod_{j=1}^{d}(1+q^{c+d-j-m_j})(1+q^{c+d+1-j-m_j})}$$

(ordered, otherwise independent Logistic distributions).