What Simple Exclusion and its friends can teach us

Inaugural lecture

Márton Balázs

University of Bristol

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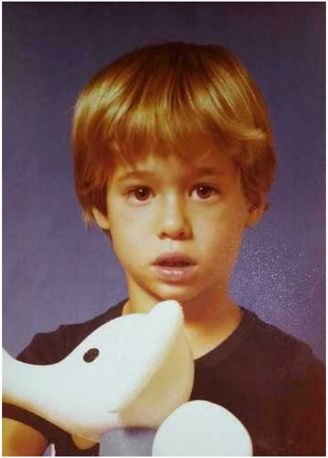
This presentation has 610 pages. ;-)

Thanks



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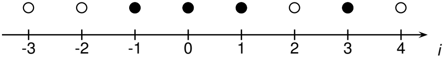

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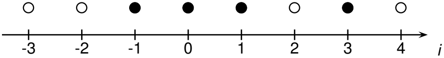

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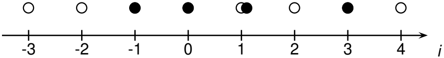

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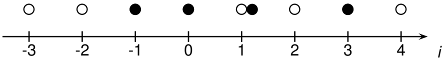

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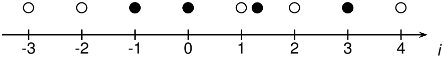

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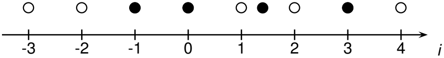

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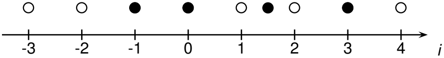

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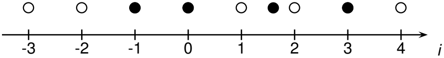

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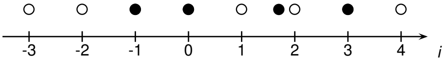

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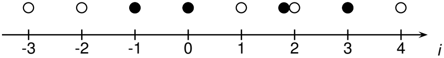

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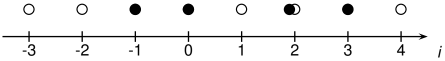

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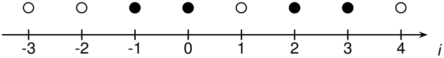

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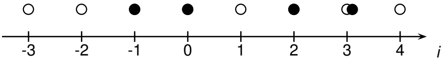

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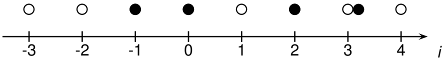

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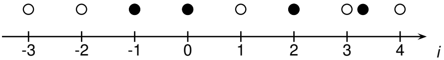

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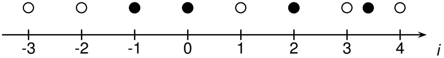

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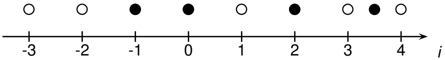

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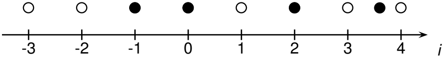

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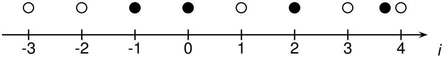

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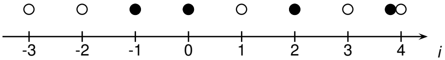

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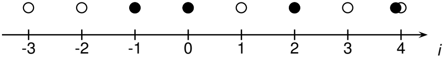

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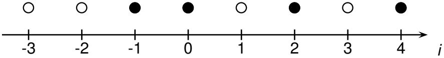

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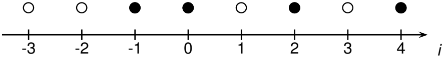

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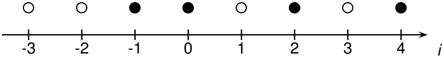

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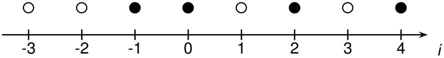

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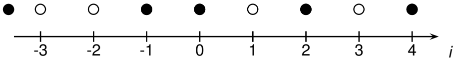

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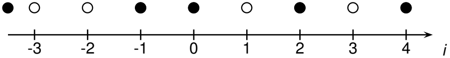

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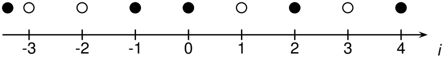

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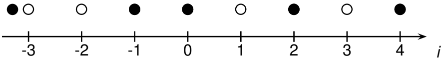

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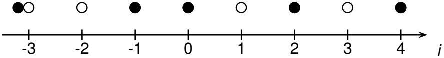

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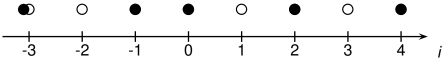

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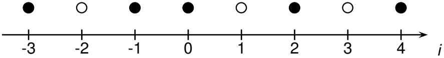

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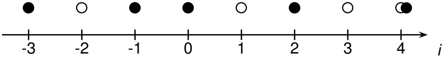

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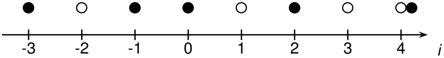

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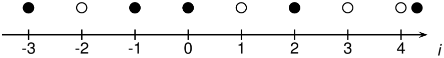

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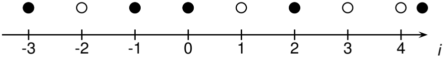

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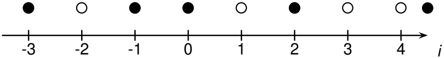

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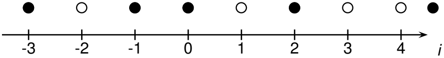

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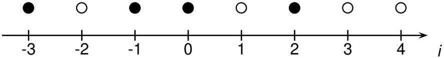

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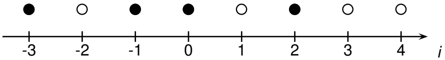

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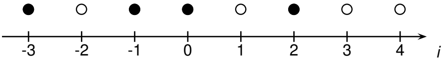

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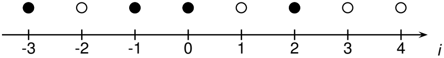

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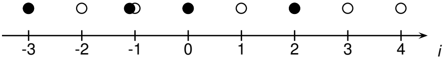

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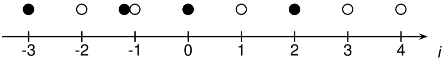

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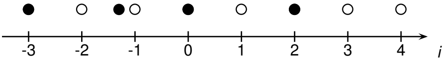

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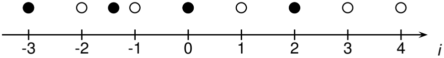

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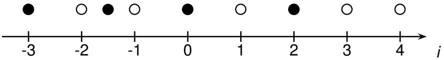

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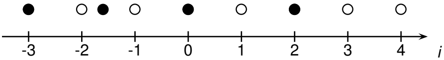

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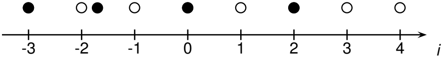

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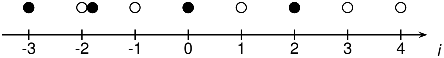

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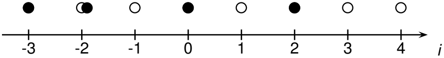

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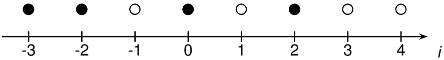

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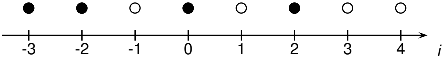

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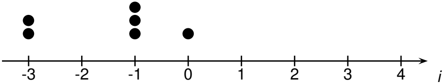

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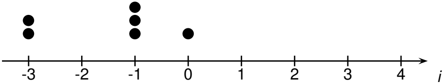

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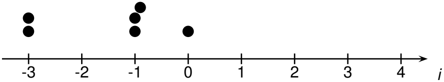

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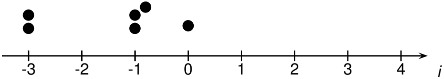

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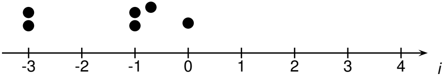

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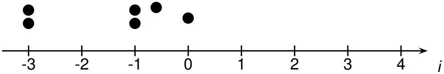

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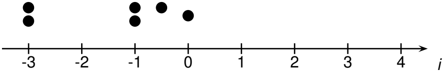

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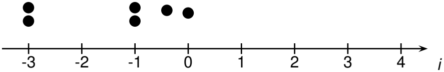

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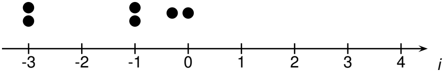

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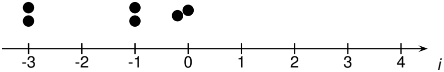

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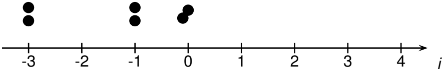

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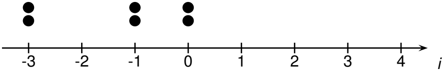

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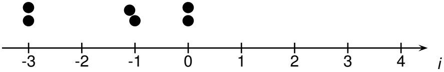

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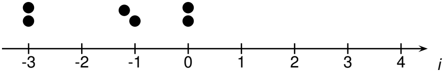

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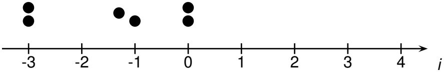

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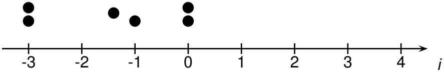

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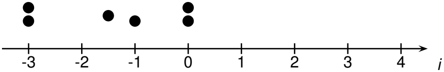

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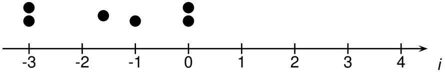

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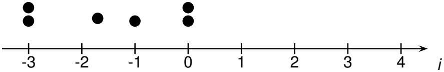

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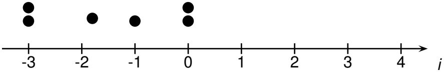

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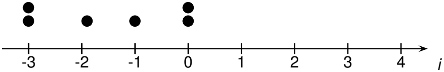

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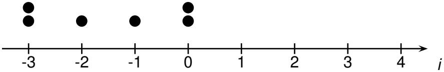

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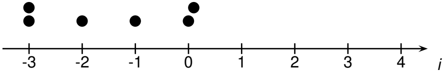

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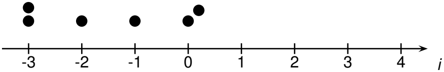

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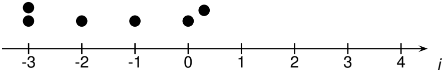

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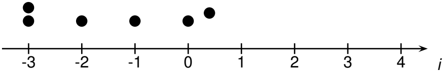

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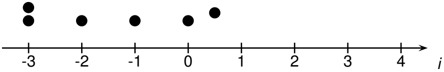

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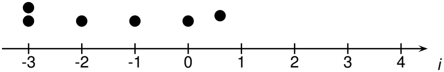

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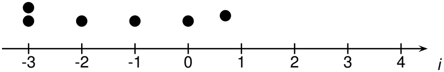

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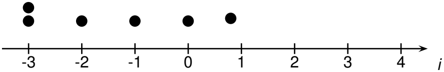

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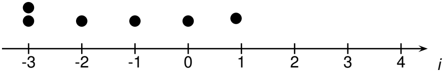

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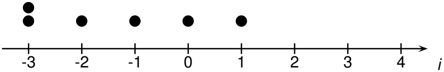

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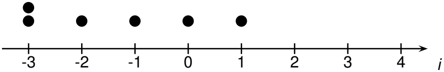

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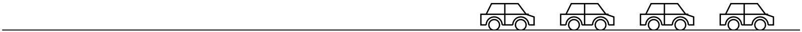

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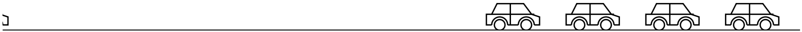

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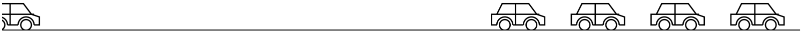

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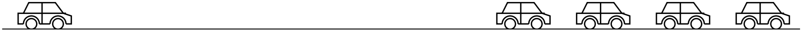

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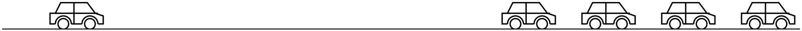

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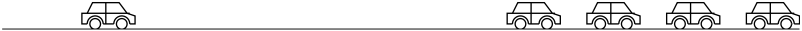

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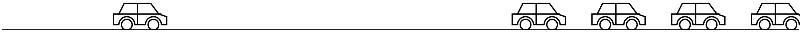

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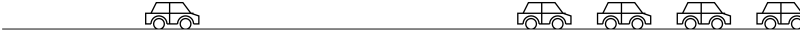

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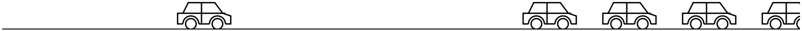

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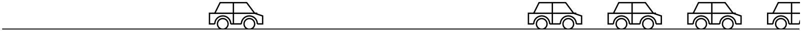

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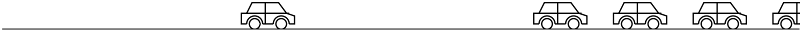

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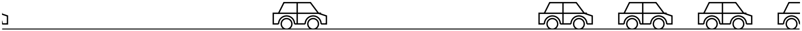

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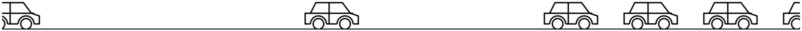

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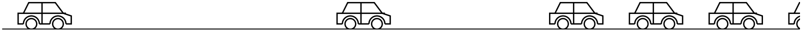

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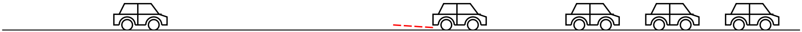

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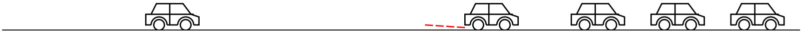

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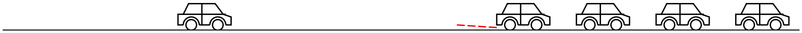

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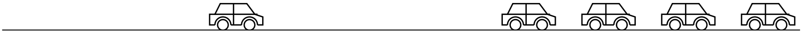

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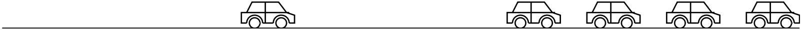

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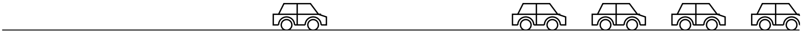

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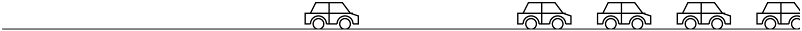

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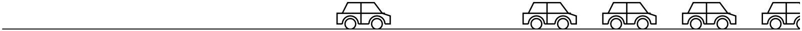

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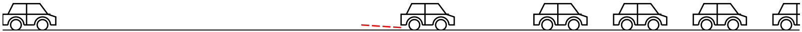

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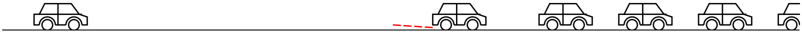

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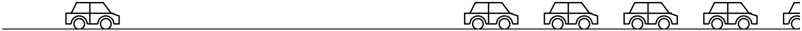

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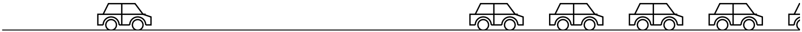

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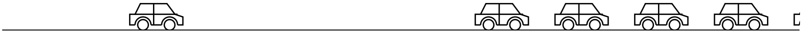

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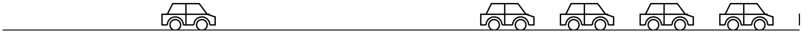

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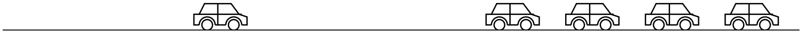

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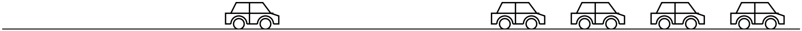

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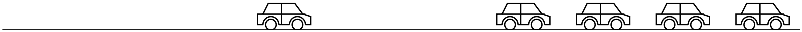

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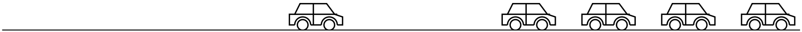

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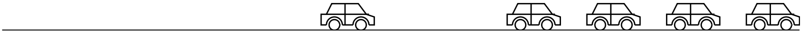

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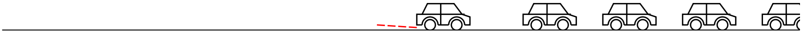

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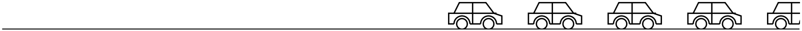

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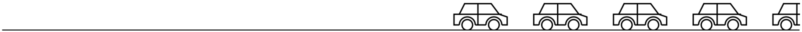

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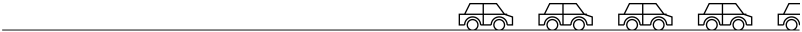

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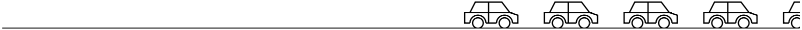

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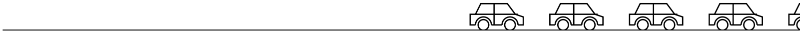

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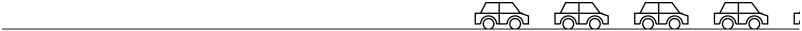

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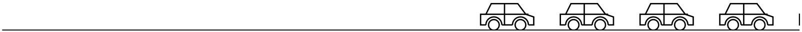

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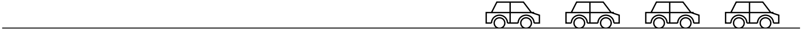

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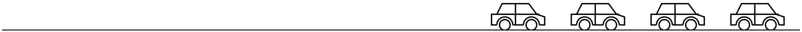

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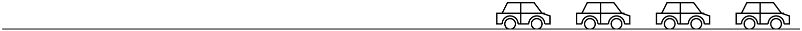

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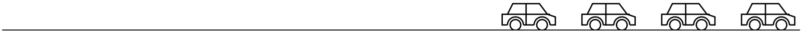

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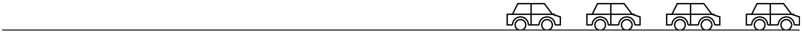

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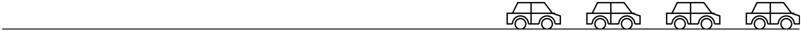
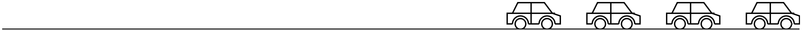


We notice the slow cars ⇝ strong braking immediately.

Arriving to a traffic jam is always sharp.

Arriving to a traffic jam

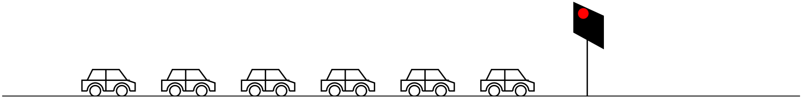


We notice the slow cars $\rightsquigarrow$ strong braking immediately.

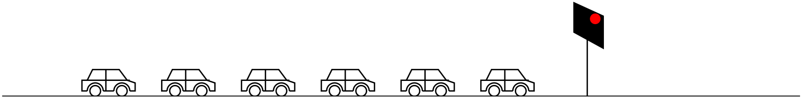
Arriving to a traffic jam is always sharp.

This is one aspect that makes motorways dangerous places.

Leaving a traffic jam



Leaving a traffic jam



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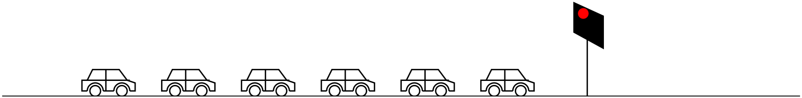

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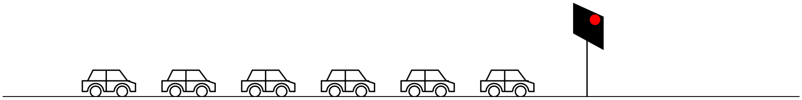

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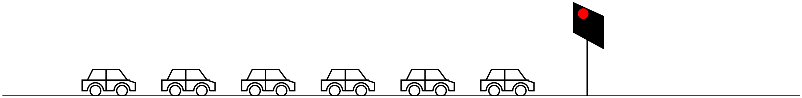

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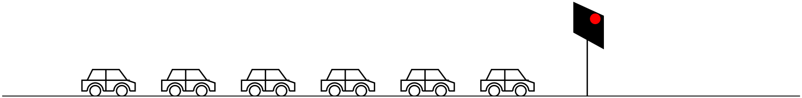

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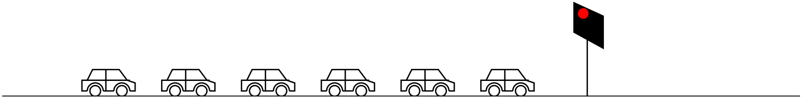

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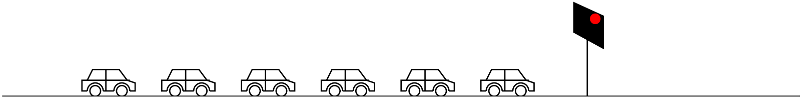

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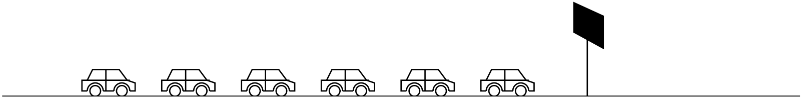

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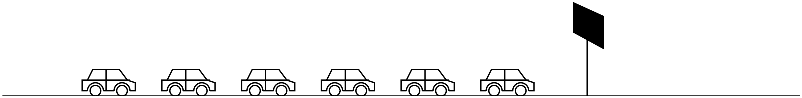

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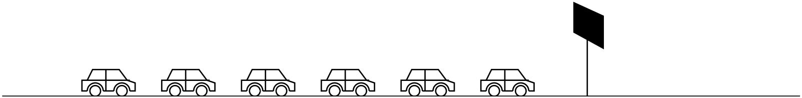

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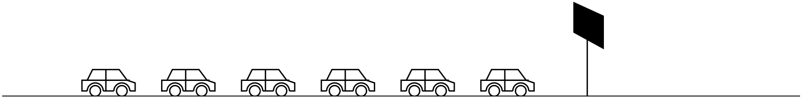

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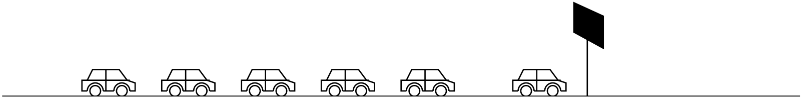

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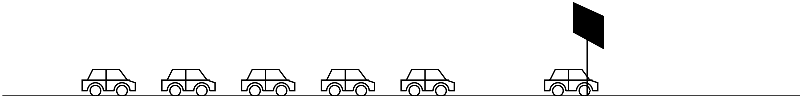

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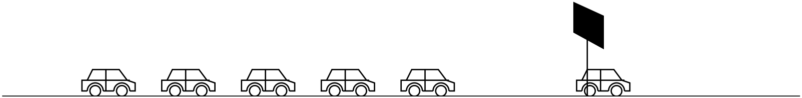

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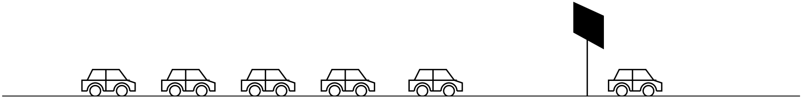

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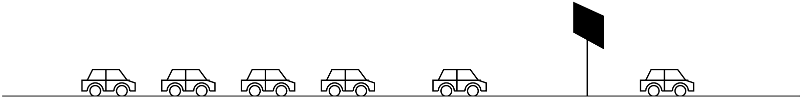

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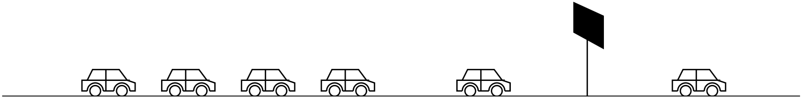

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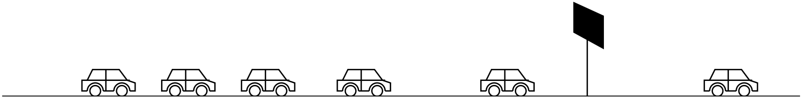

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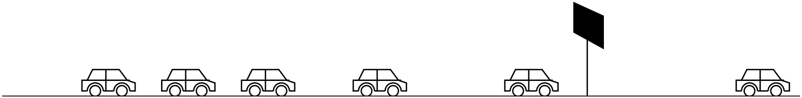

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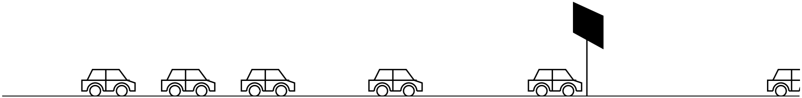

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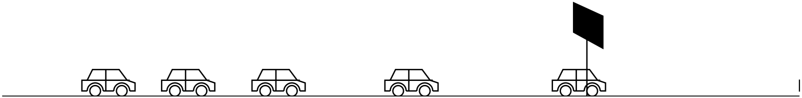

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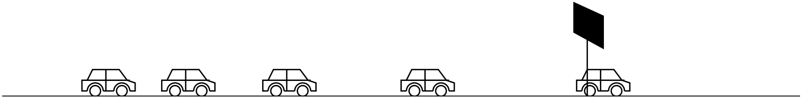

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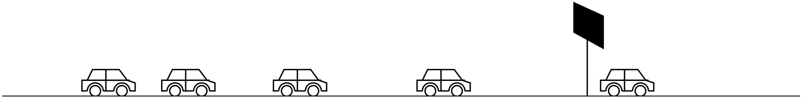

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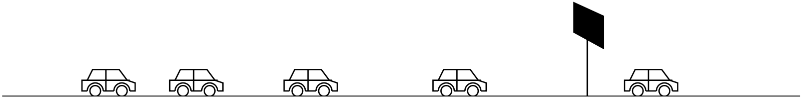

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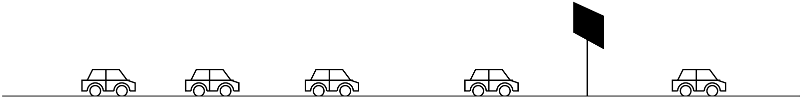

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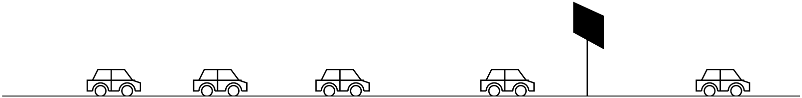

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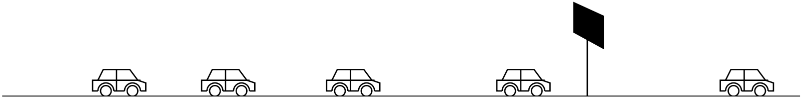

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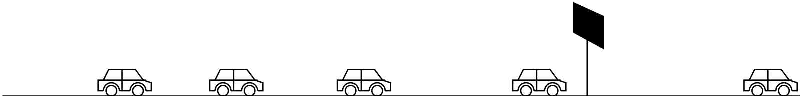

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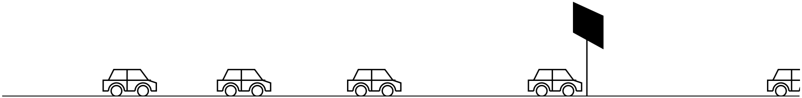

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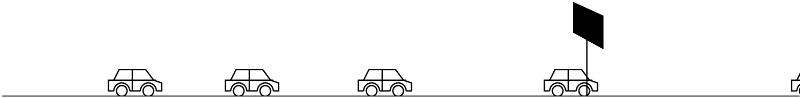

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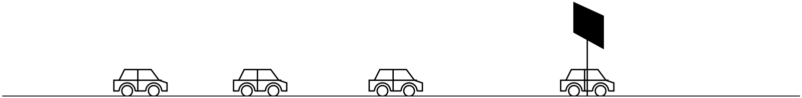

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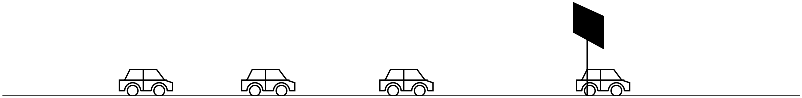

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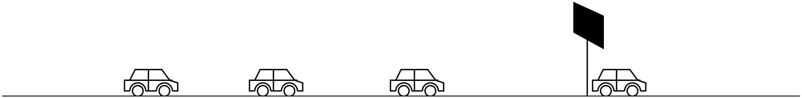

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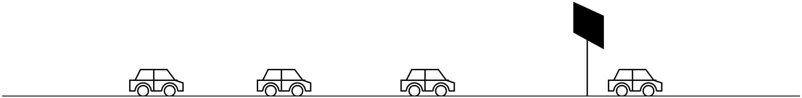

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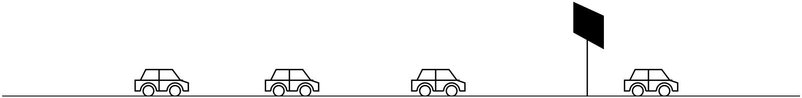

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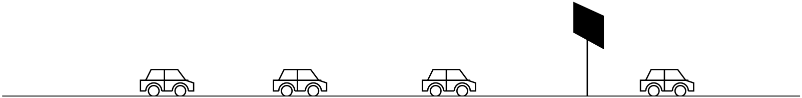

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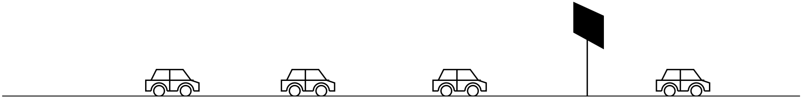

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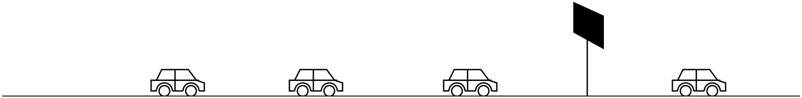

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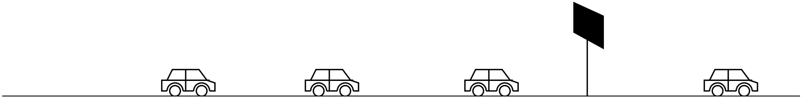

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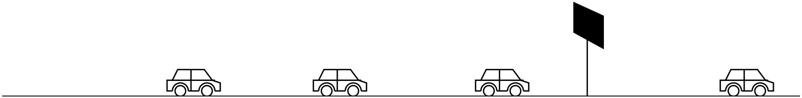

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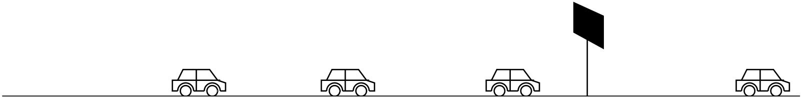

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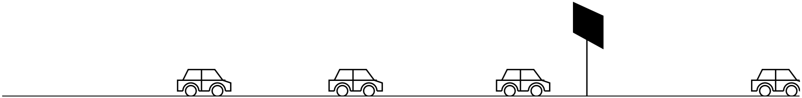

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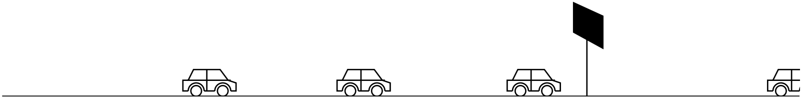

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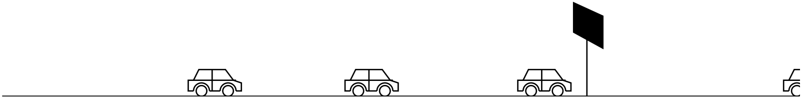

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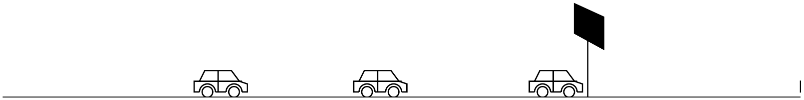

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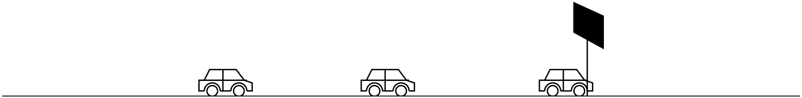

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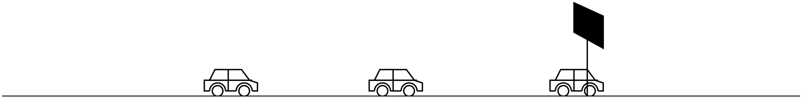

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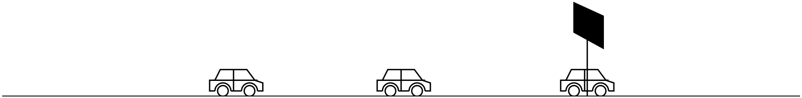

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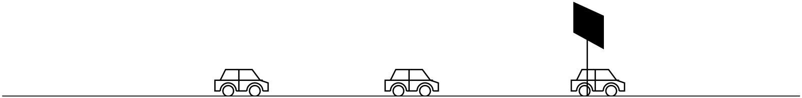

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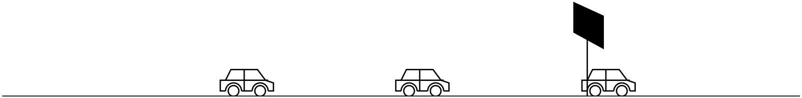

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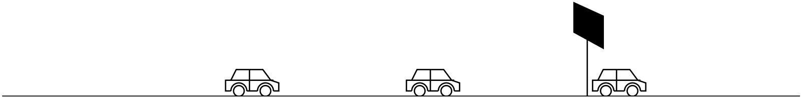

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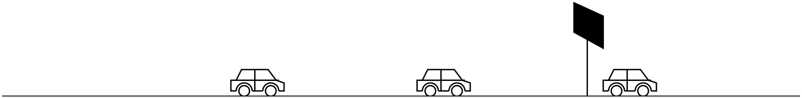

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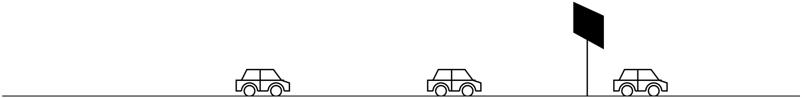

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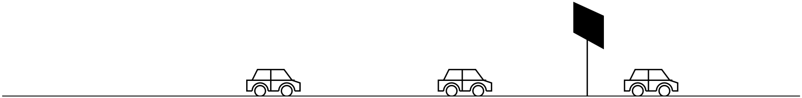

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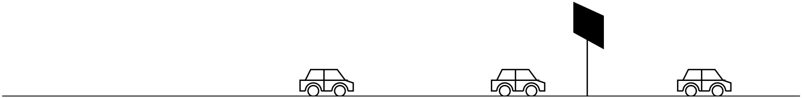

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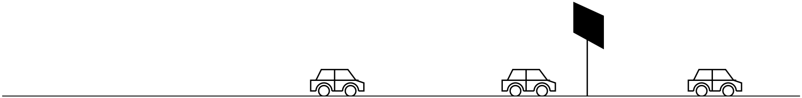

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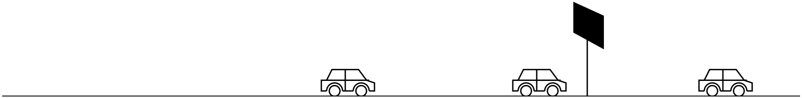

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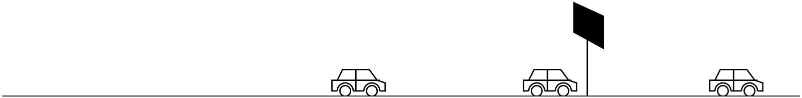

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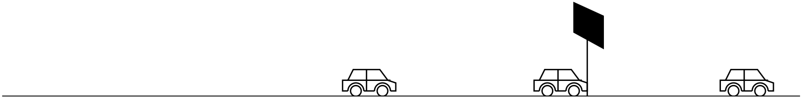

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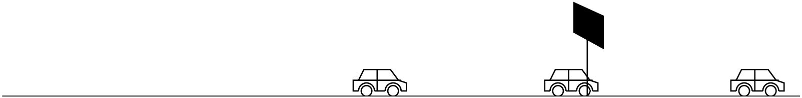

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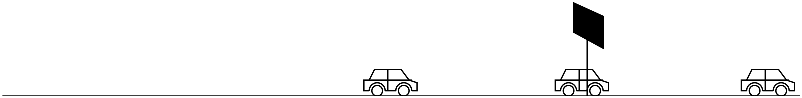

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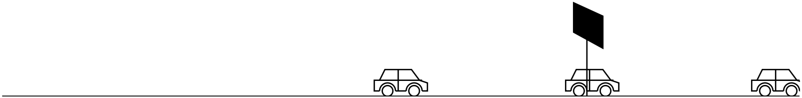

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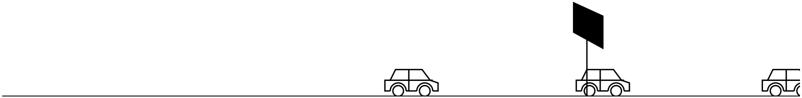

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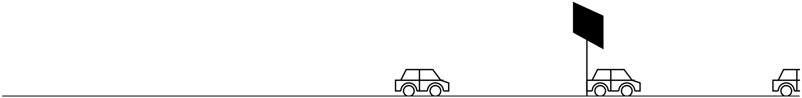

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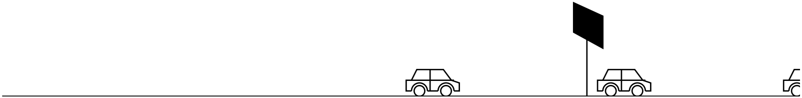

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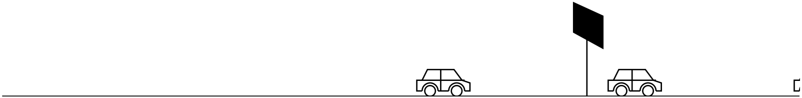

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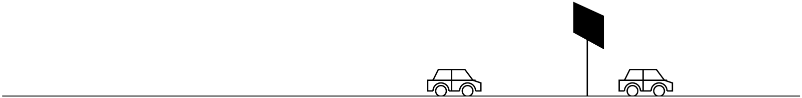

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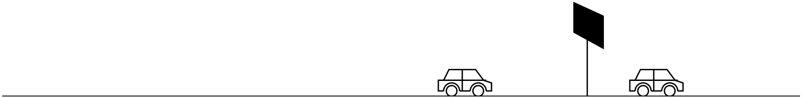

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Continuous, long acceleration for those starting from the rear

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Leaving a traffic jam is always soft, “blurry”.

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Why is there such a difference between the two ends of a traffic jam?

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Leaving a traffic jam is always soft, “blurry”.

Why is there such a difference between the two ends of a traffic jam?

TASEP: let’s go large scale!

On large scales

Rent a helicopter and view particles (cars) from high above.

That is, rescale space (X) and time (T) of TASEP.

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The density $\varrho(T, X)$ of particles satisfies

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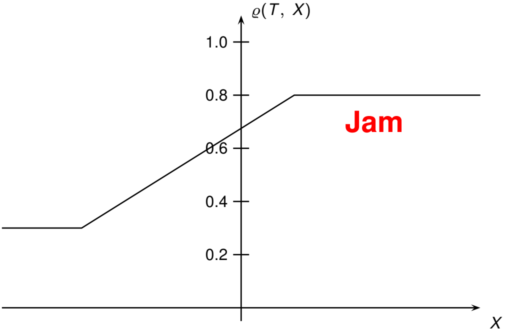
The following are solutions of this equation:

On large scales



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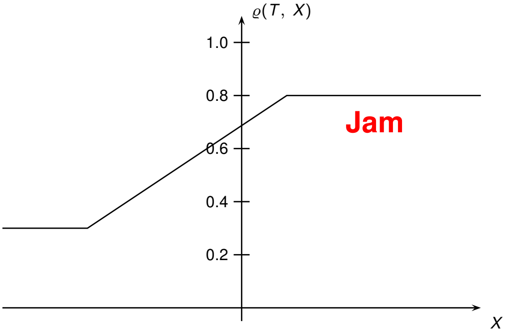

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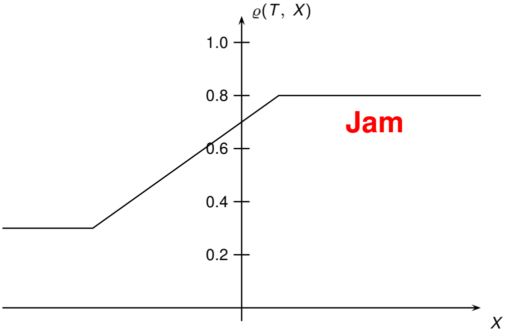

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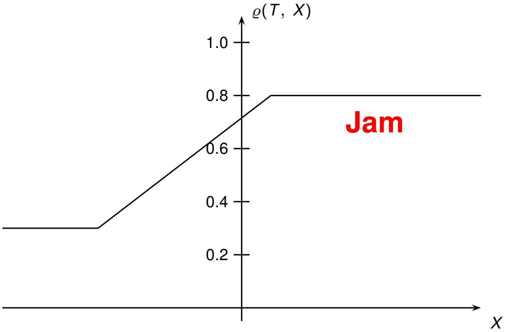

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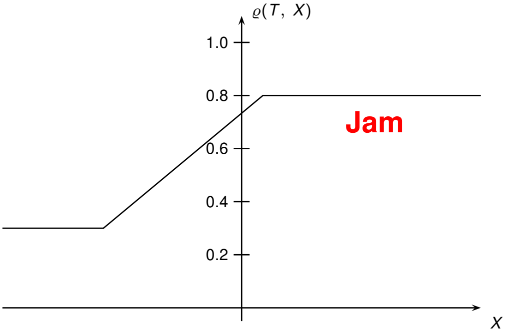

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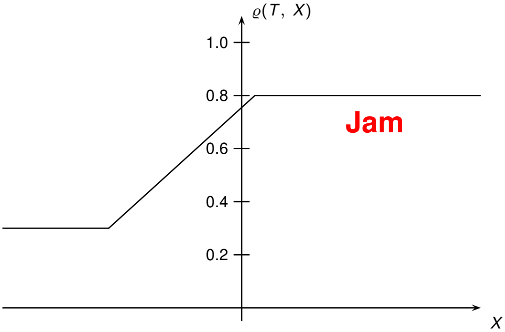

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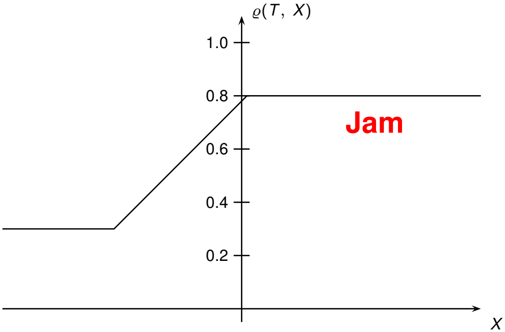

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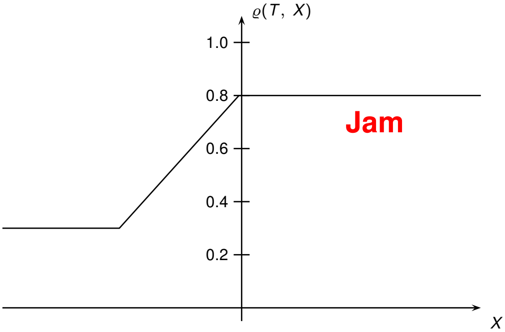

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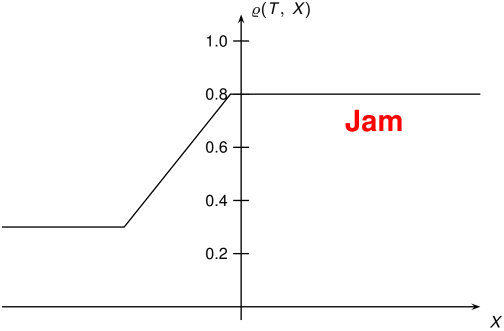

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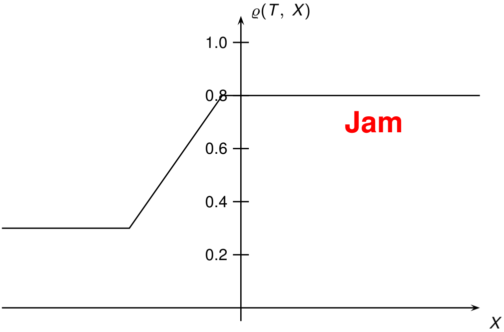

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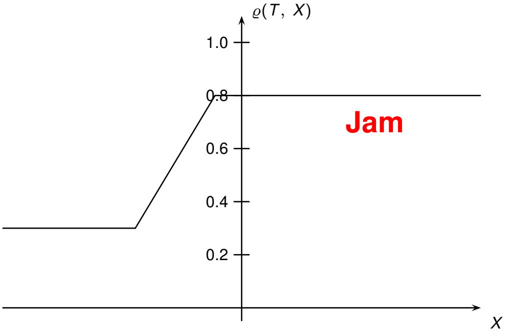

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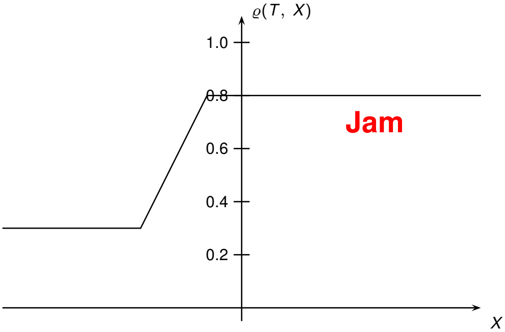

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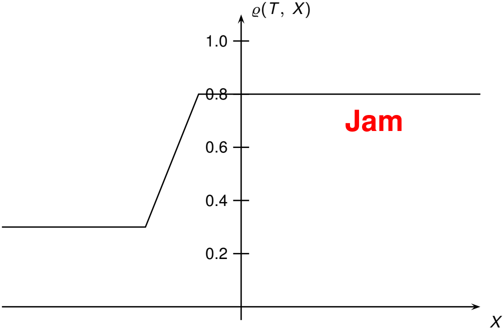

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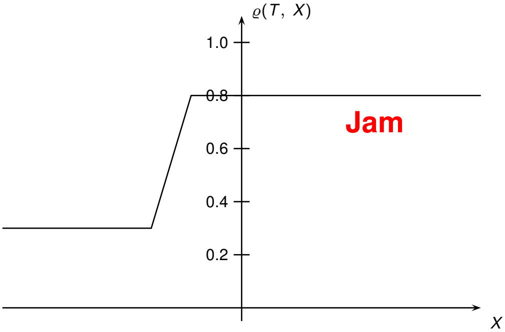

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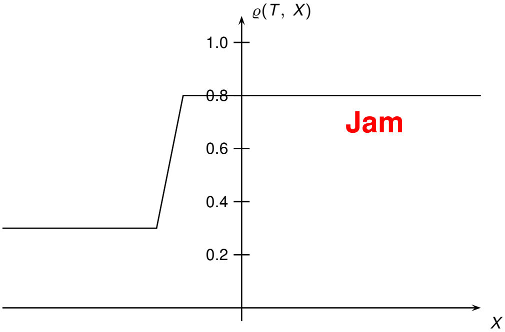

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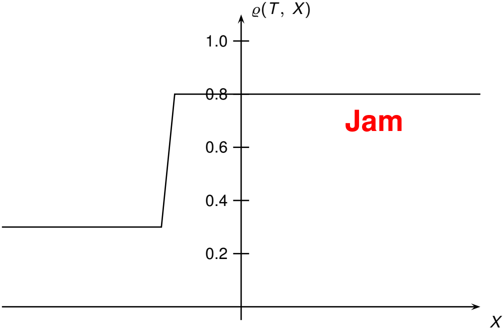

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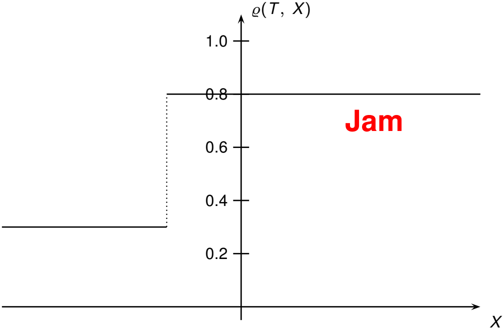

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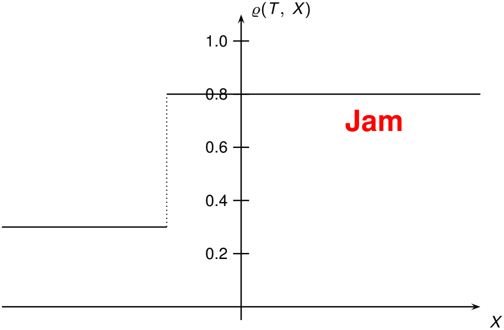

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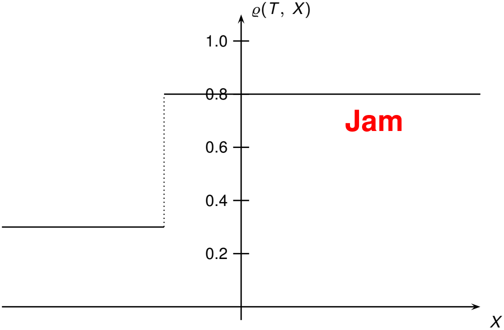

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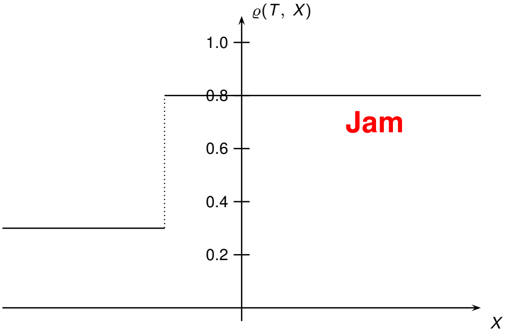

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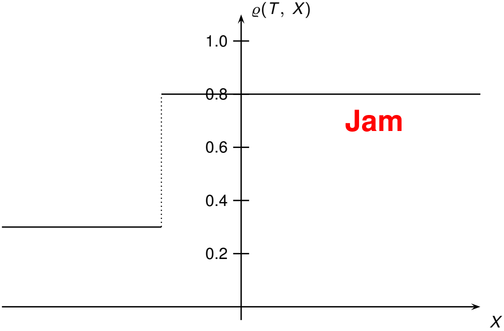

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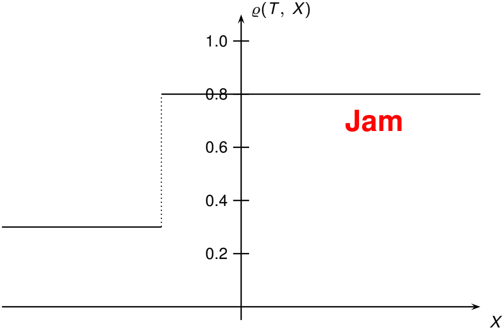

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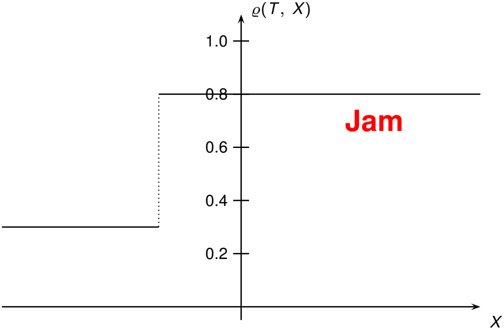
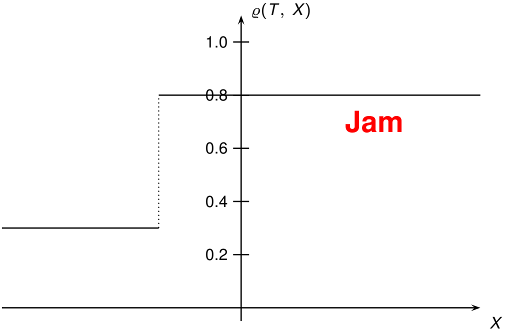


On large scales



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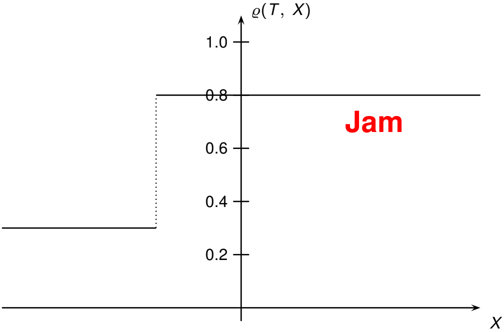

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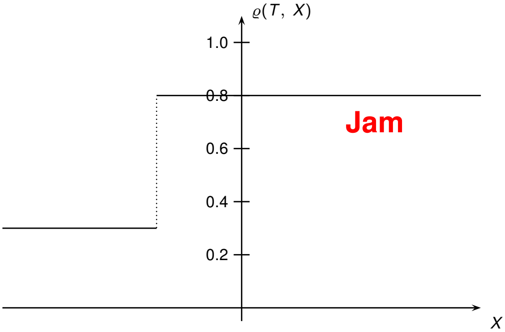

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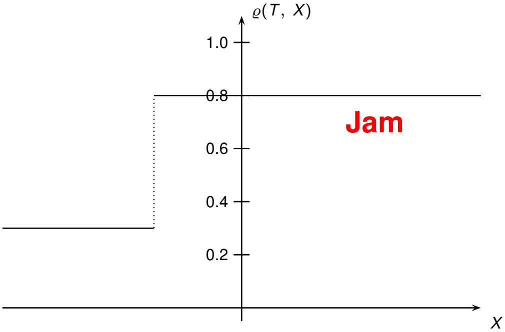

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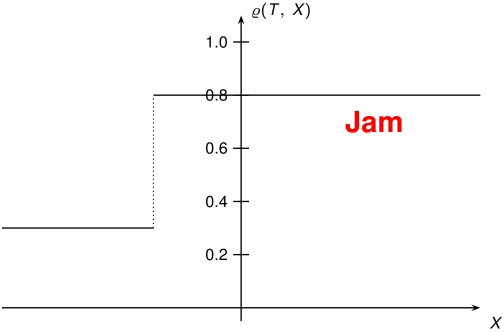

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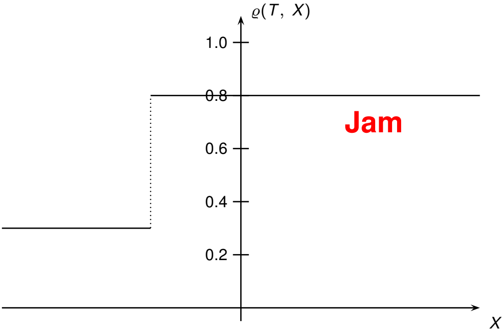

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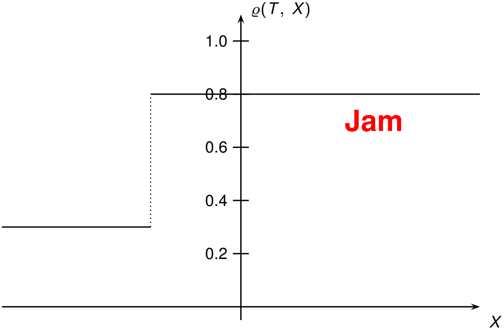

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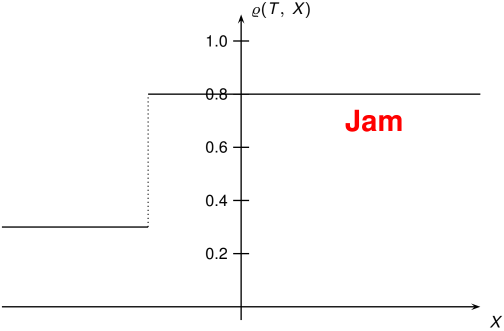

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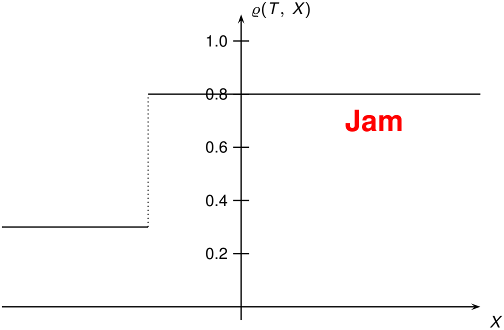

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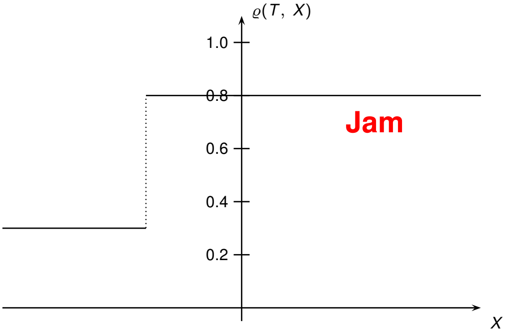

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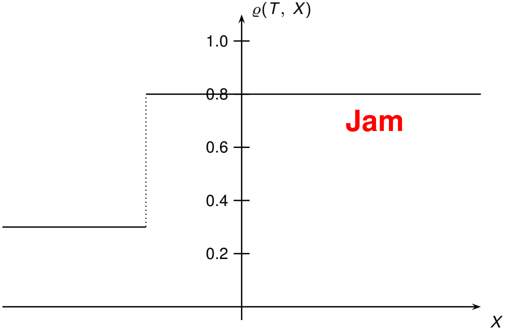

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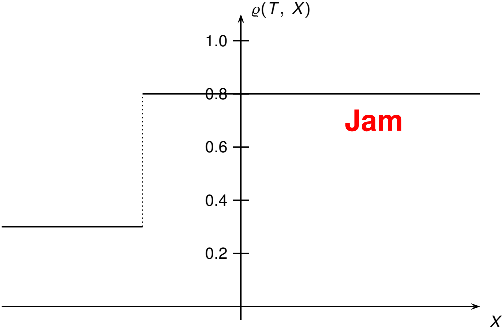

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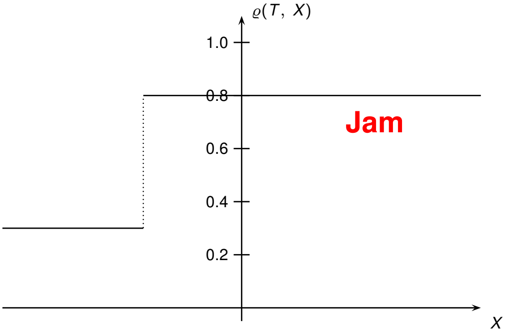

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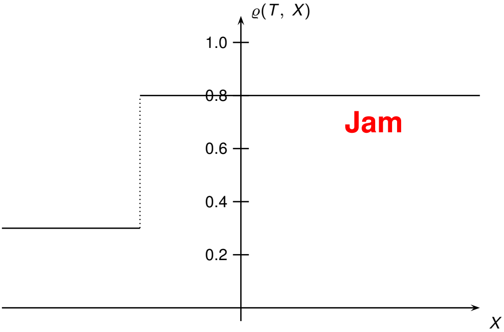

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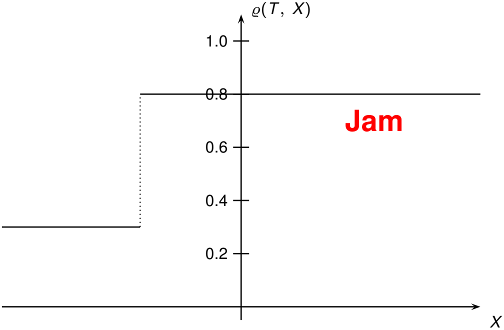

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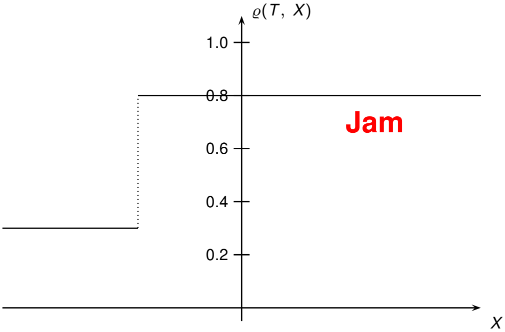

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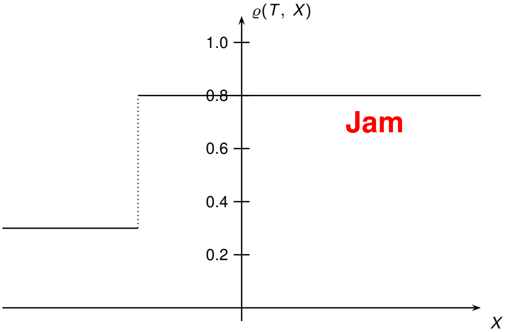

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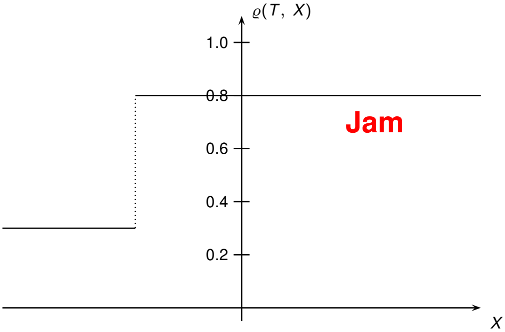

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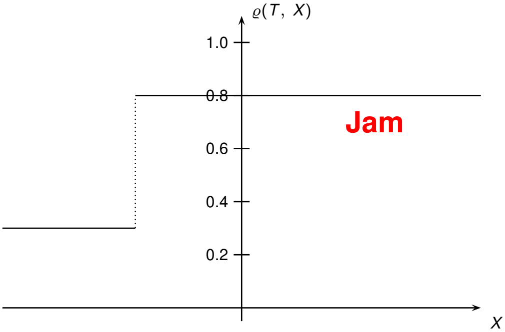

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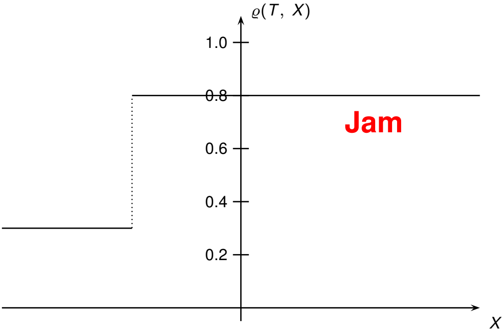

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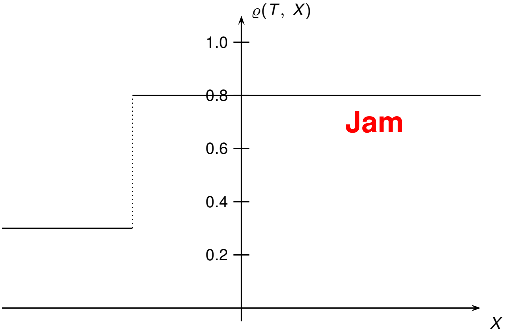

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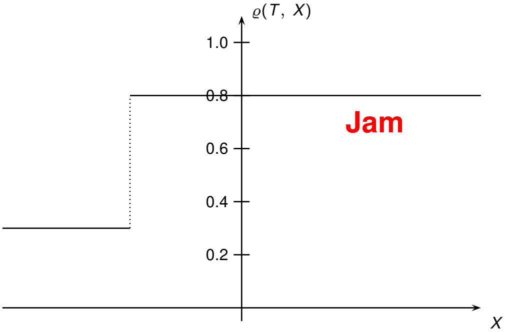

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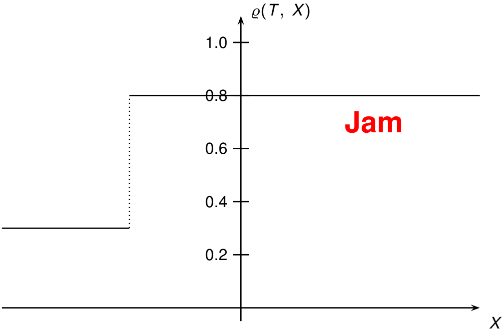

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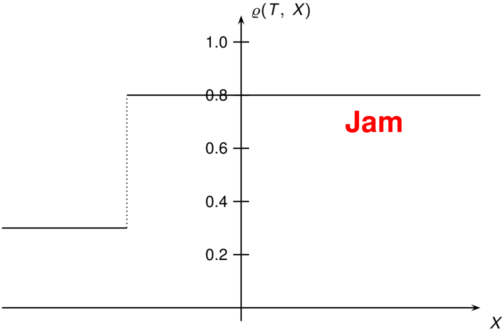

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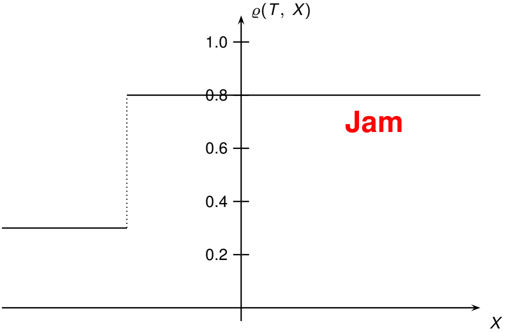

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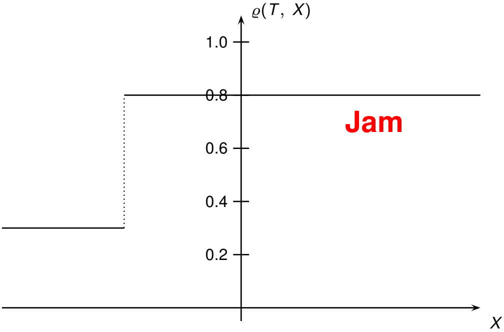

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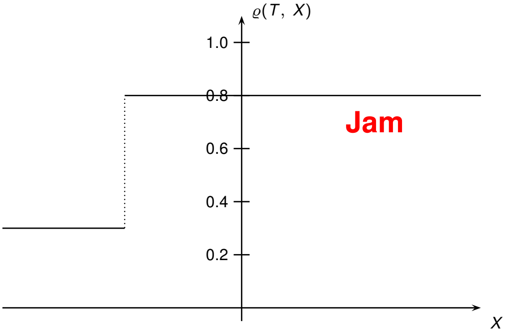

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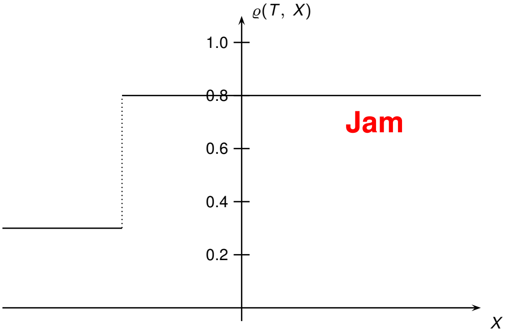

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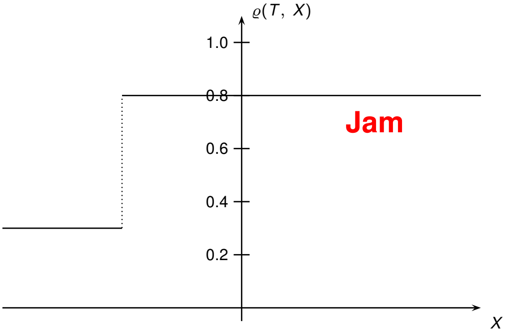

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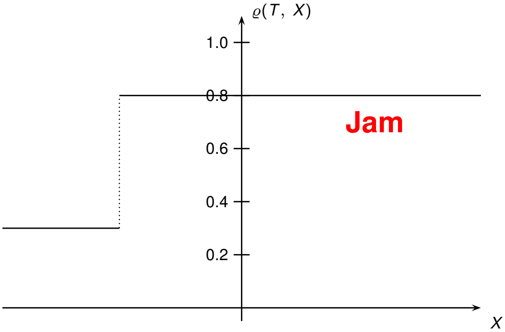

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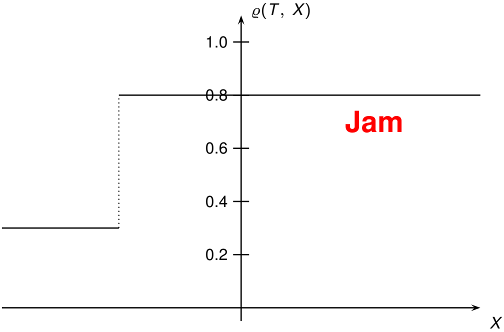

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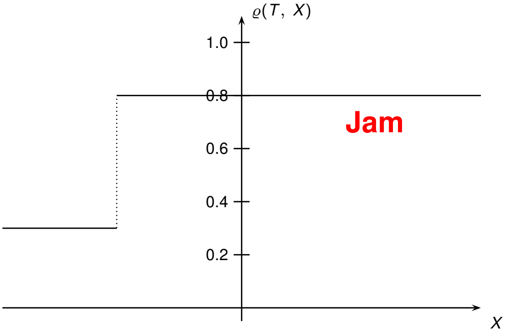

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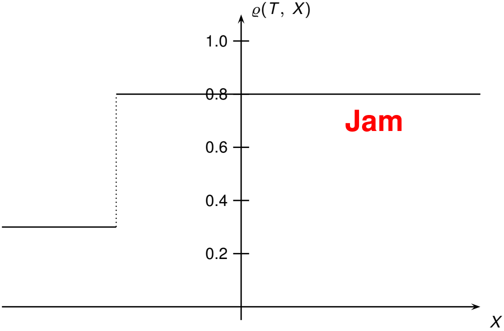

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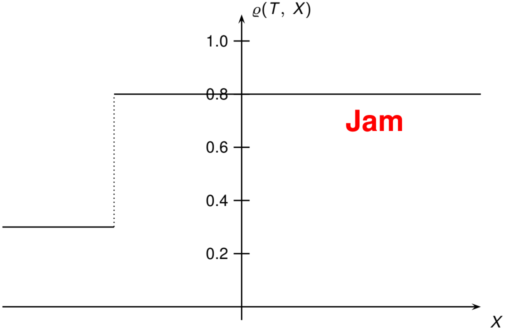

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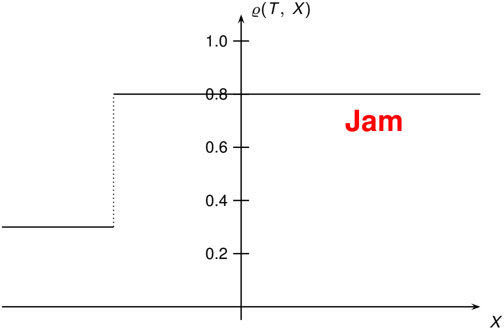

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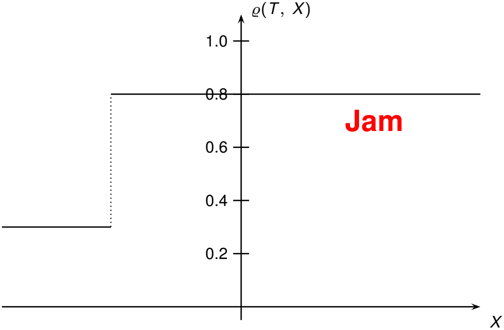

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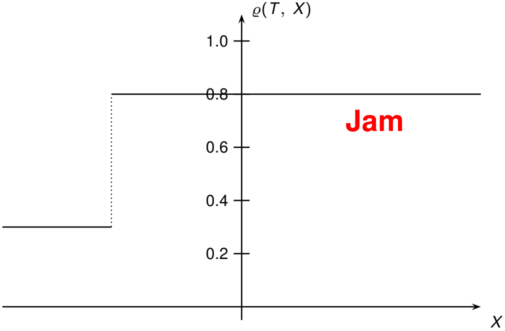

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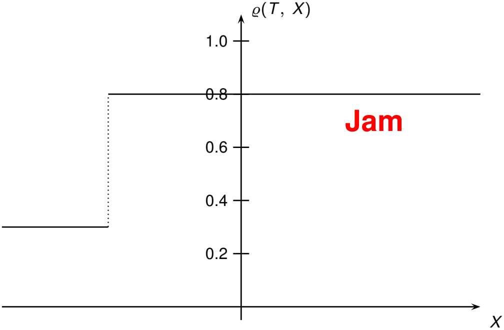

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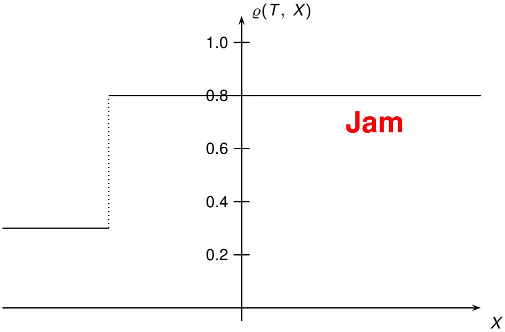
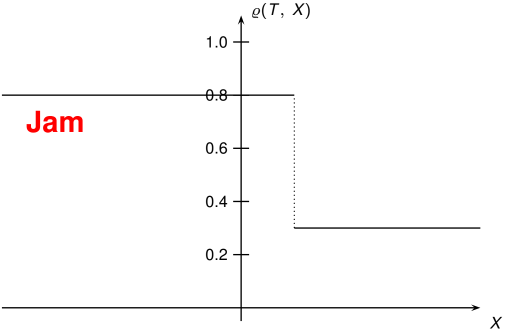
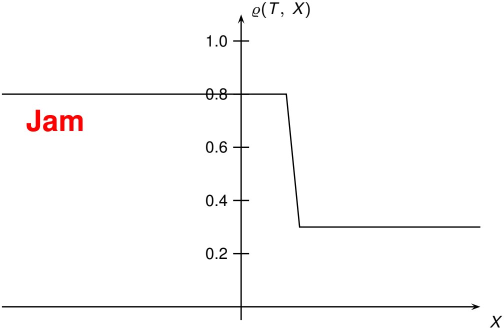


The start of the jam: sharpens. *Shock*

On large scales



On large scales



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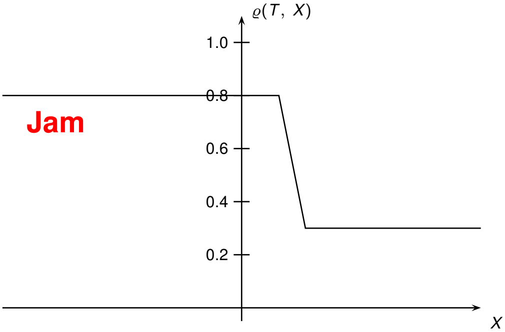

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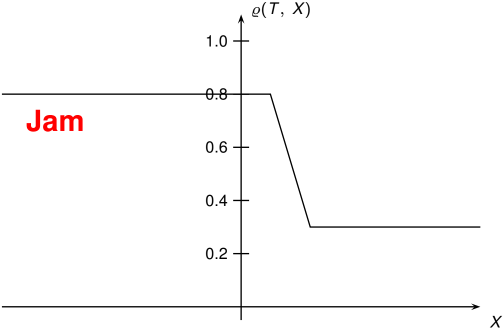

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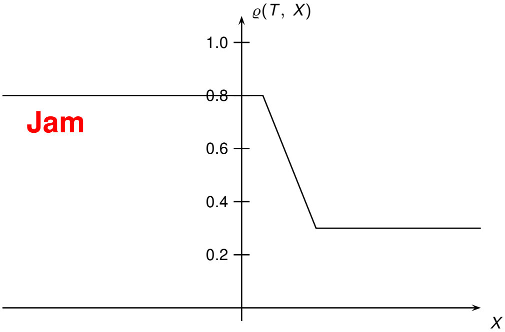
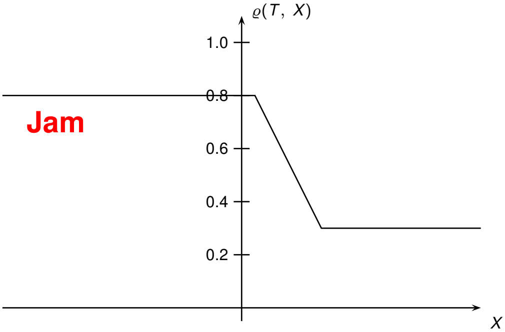
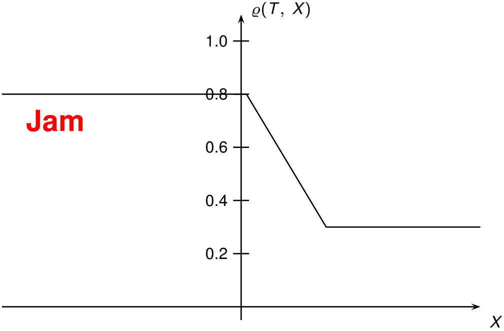


On large scales



On large scales



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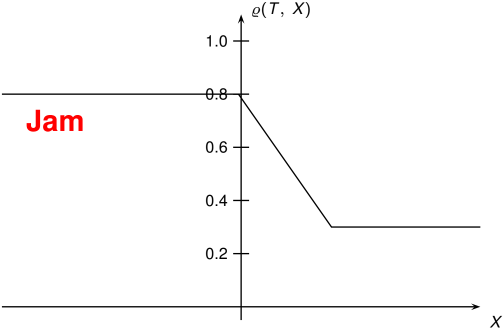
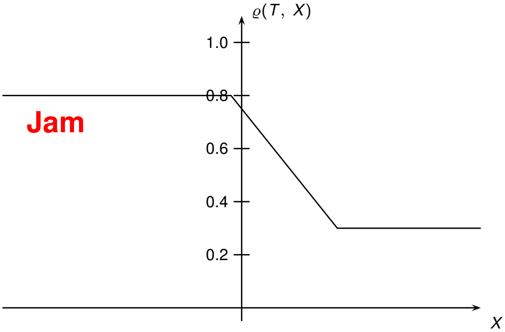


On large scales



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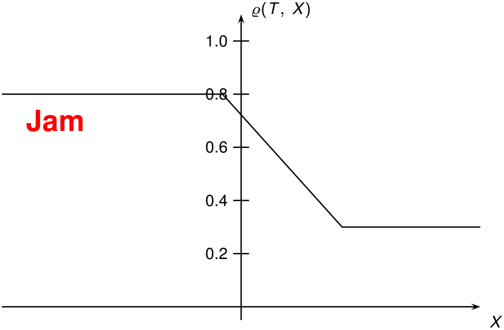

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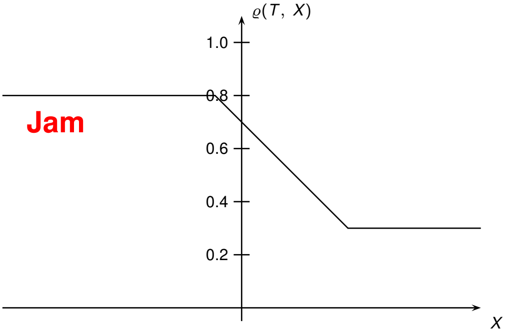
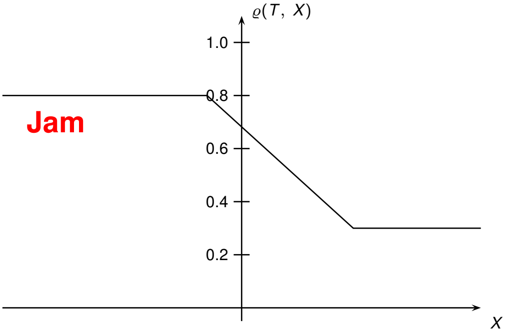
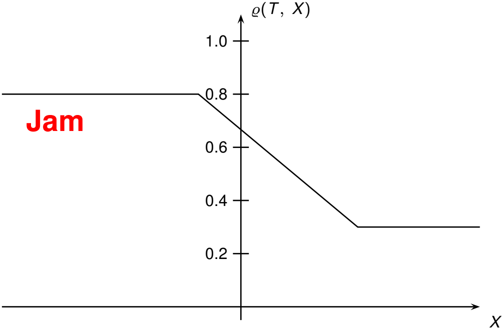


On large scales



On large scales



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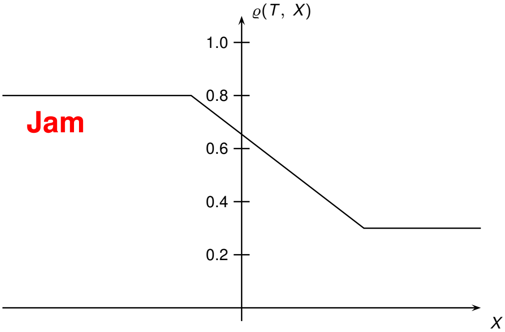

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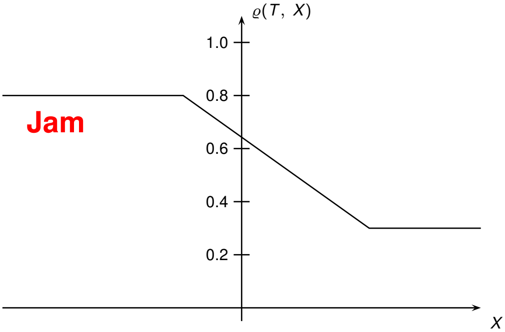

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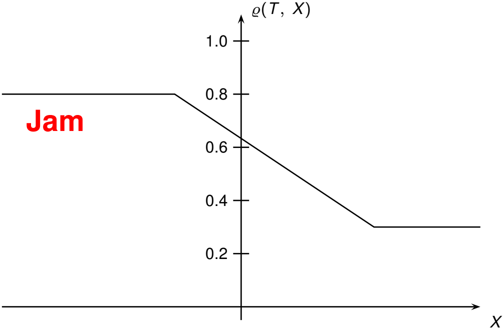

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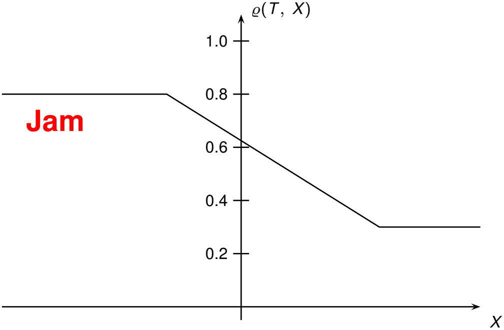

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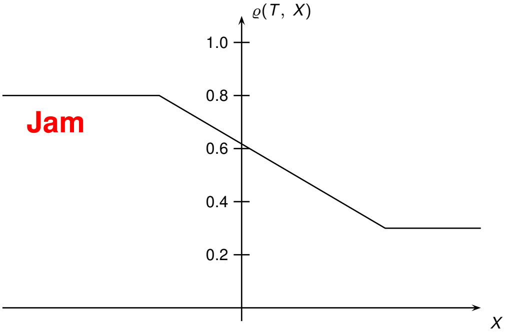

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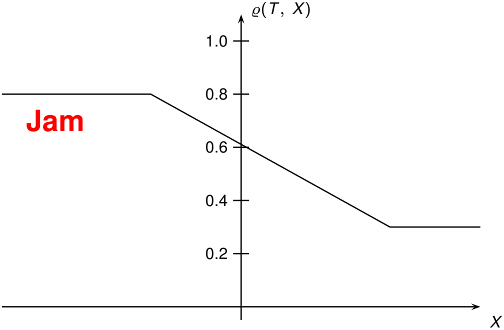

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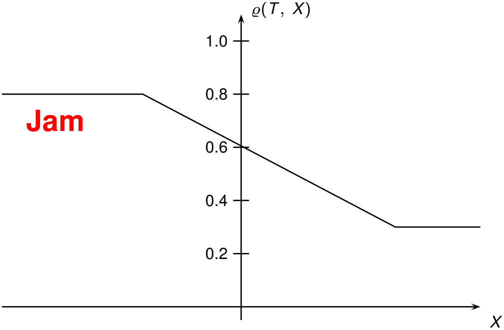

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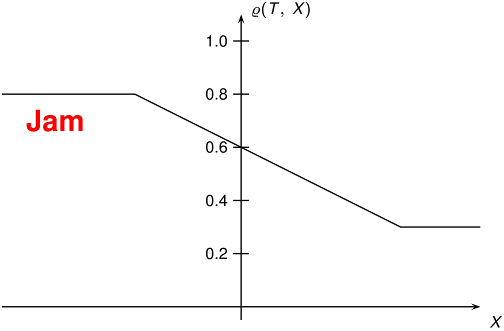

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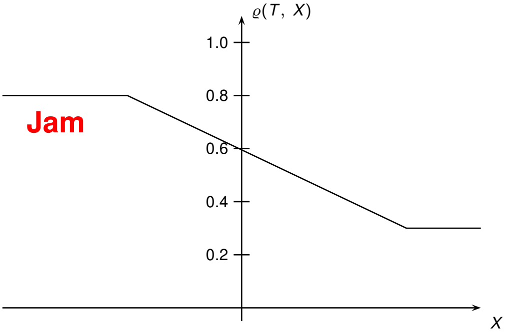

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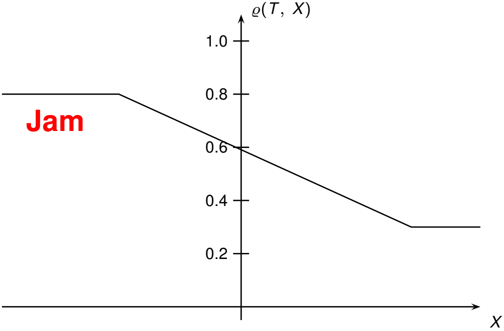

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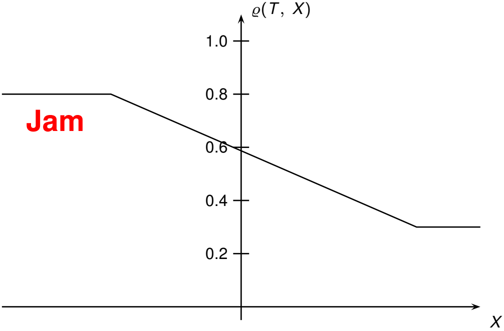

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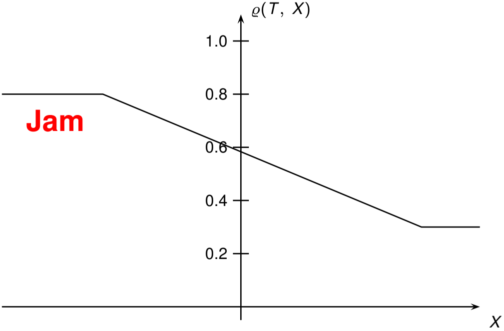

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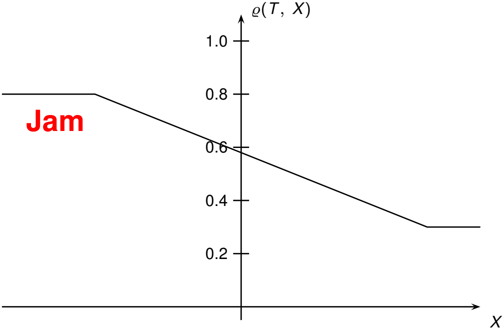

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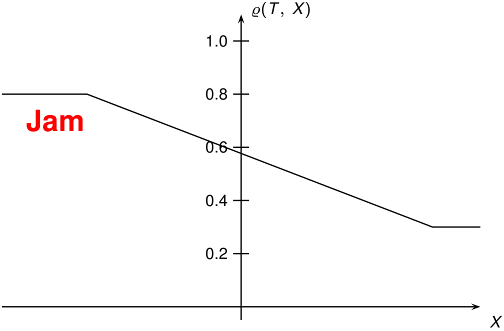

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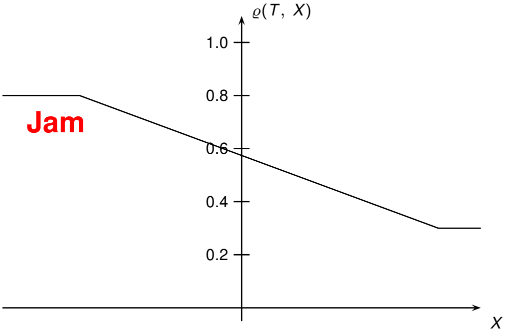

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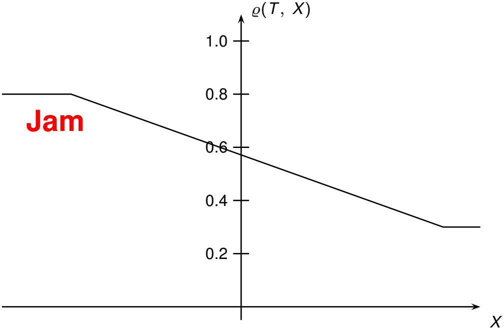

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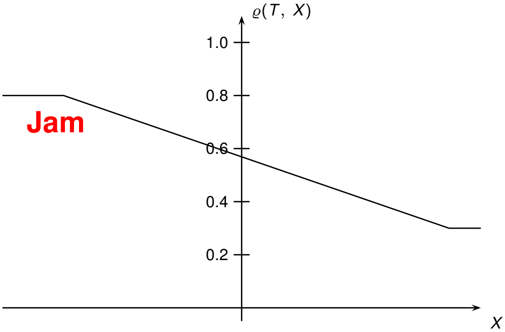

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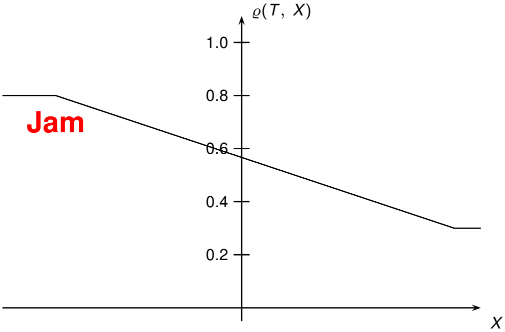

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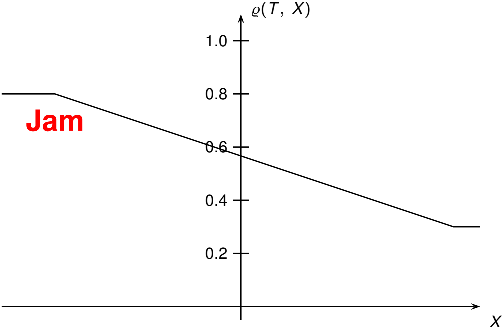


End of the jam: smoothens. *Rarefaction fan*

On large scales

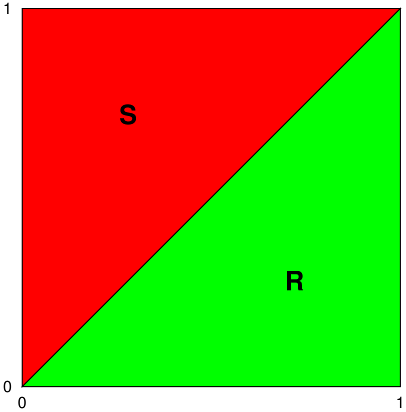
- Of course there are much more sophisticated models for traffic modelling.

On large scales

- Of course there are much more sophisticated models for traffic modelling.
- `https://youtu.be/7wm-pZp_mi0`

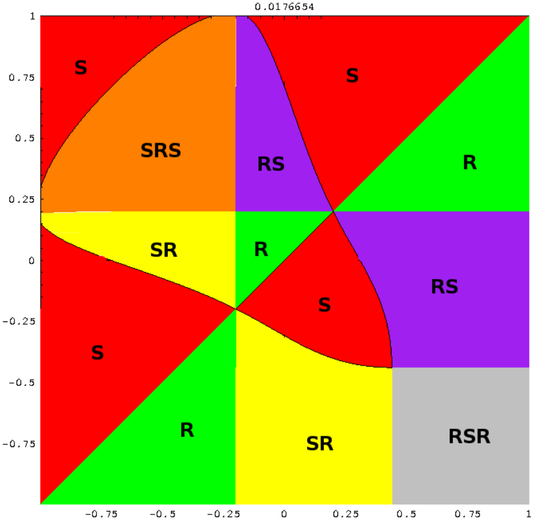
On large scales

TASEP (**R**: rarefaction fan, **S**: Shock):



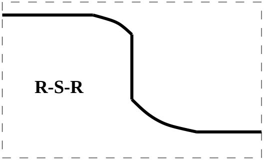
A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]

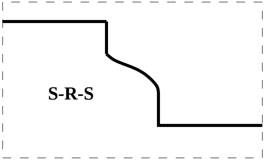
Here is what can also happen (**R**: rarefaction fan, **S**: Shock):



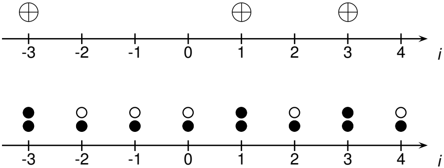
A fun model [B., A. L. Nagy, I. Tóth, B. Tóth '16]

Examples for $\varrho(T, X)$:





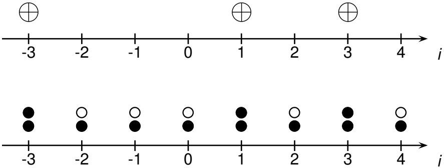
A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]



A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]

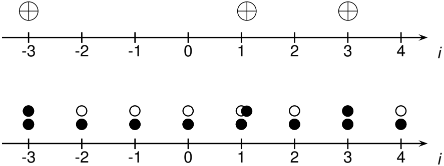
$\oplus \rightsquigarrow \substack{\bullet \\ \bullet}$ $\qquad$ $\emptyset \rightsquigarrow \substack{\circ \\ \bullet}$ $\qquad$ $\ominus \rightsquigarrow \substack{\circ \\ \circ}$

$\oplus$ to the right



A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]

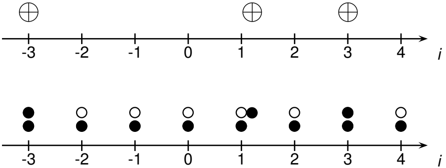
$\oplus$ to the right



A fun model [B., A. L. Nagy, I. Tóth, B. Tóth '16]

$$\oplus \rightsquigarrow \begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix} \qquad \emptyset \rightsquigarrow \begin{smallmatrix}\circ\\ \bullet\end{smallmatrix} \qquad \ominus \rightsquigarrow \begin{smallmatrix}\circ\\ \circ\end{smallmatrix}$$

$\oplus$ to the right



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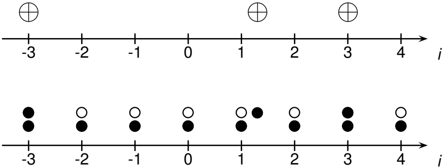
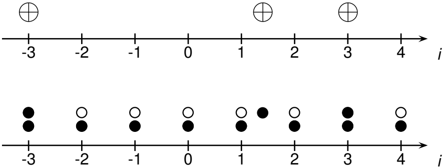


A fun model [B., A. L. Nagy, I. Tóth, B. Tóth '16]

$\oplus \rightsquigarrow \genfrac{}{}{0pt}{}{\bullet}{\bullet}$ $\quad$ $\emptyset \rightsquigarrow \genfrac{}{}{0pt}{}{\circ}{\bullet}$ $\quad$ $\ominus \rightsquigarrow \genfrac{}{}{0pt}{}{\circ}{\circ}$

$\oplus$ to the right



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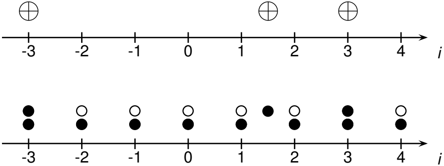

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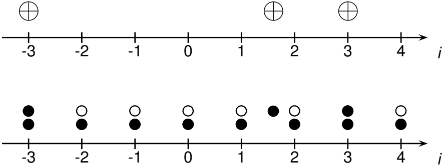

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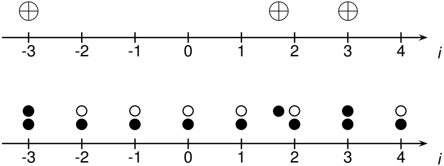

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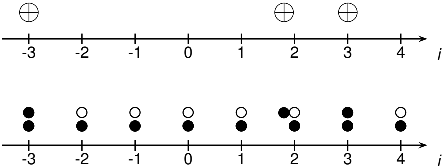

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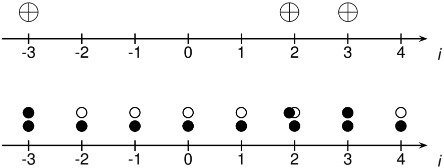

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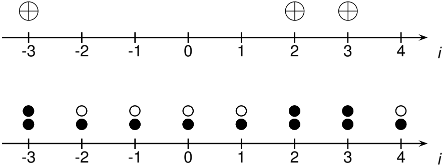
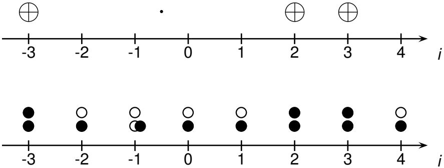


A fun model [B., A. L. Nagy, I. Tóth, B. Tóth '16]

pair creation from vacuum



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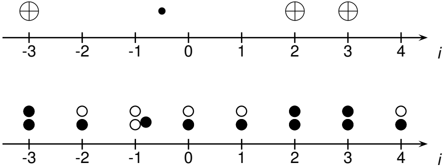
pair creation from vacuum



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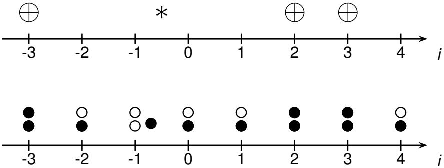
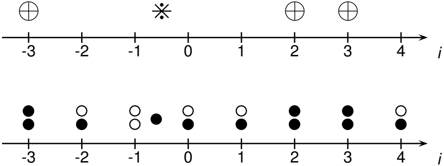
pair creation from vacuum



A fun model [B., A. L. Nagy, I. Tóth, B. Tóth '16]

pair creation from vacuum



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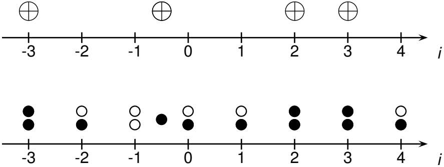

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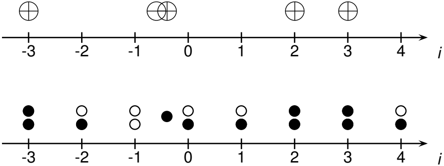
pair creation from vacuum



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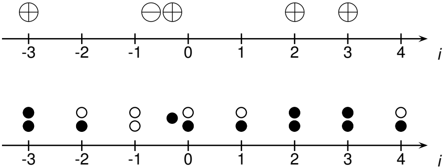
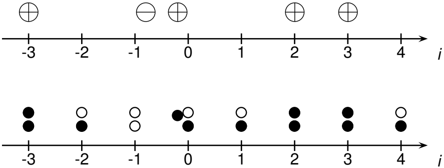


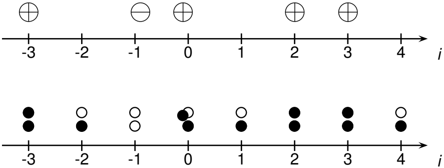
A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]

pair creation from vacuum



A fun model [B., A. L. Nagy, I. Tóth, B. Tóth '16]

pair creation from vacuum



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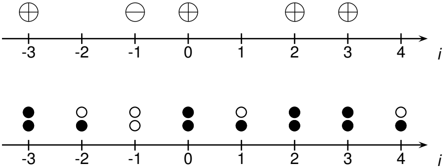
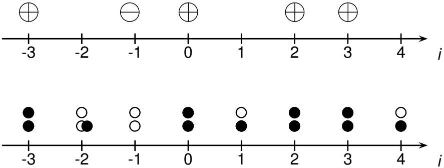


A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]

$\oplus \rightsquigarrow \begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}$ $\qquad \emptyset \rightsquigarrow \begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}$ $\qquad \ominus \rightsquigarrow \begin{smallmatrix}\circ\\ \circ\end{smallmatrix}$

$\ominus$ to the left



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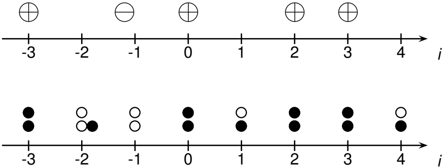

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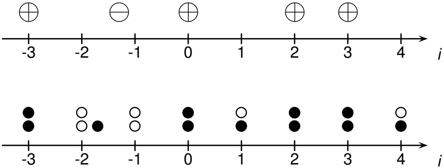

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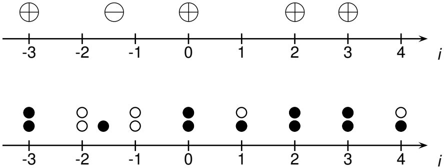

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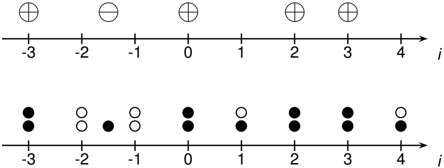

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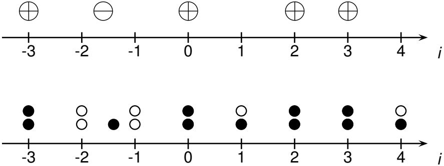
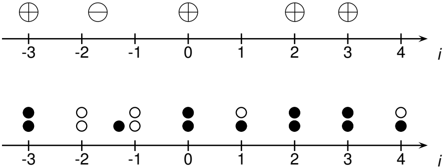


A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]

⊖ to the left



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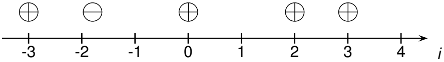

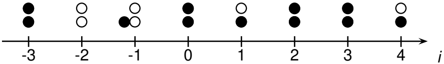

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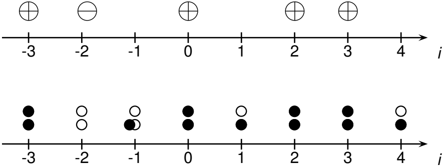
$\ominus$ to the left



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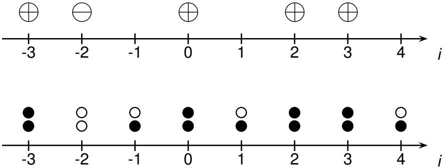
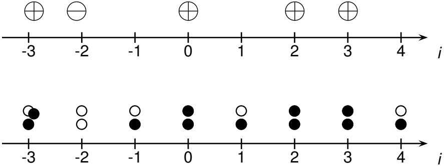


A fun model [B., A. L. Nagy, I. Tóth, B. Tóth '16]

annihilation



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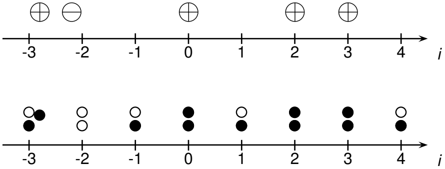

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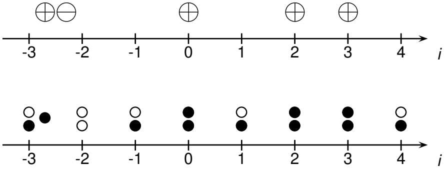

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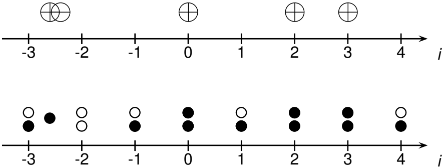

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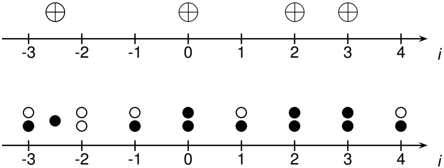

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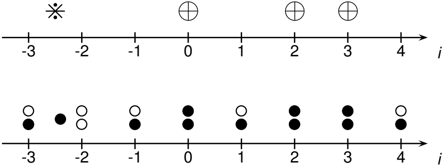

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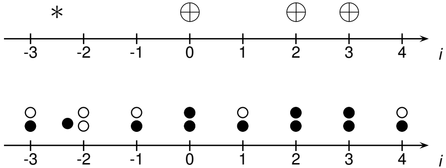
annihilation



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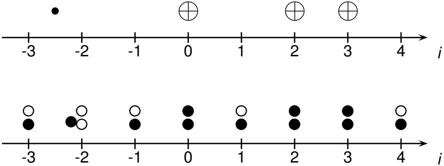

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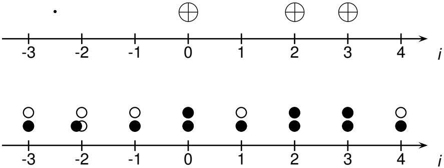

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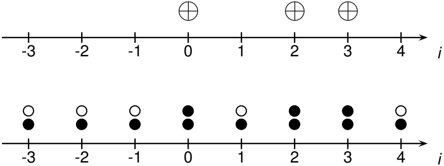
annihilation



A fun model [B., A. L. Nagy, I. Tóth, B. Tóth ’16]

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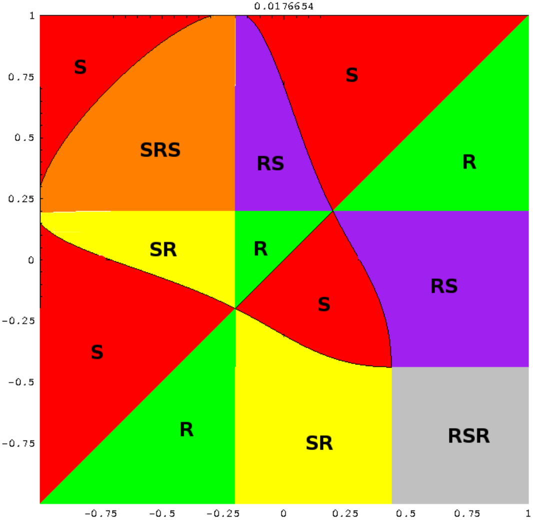
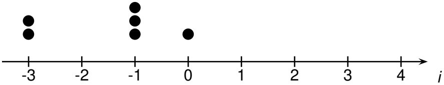
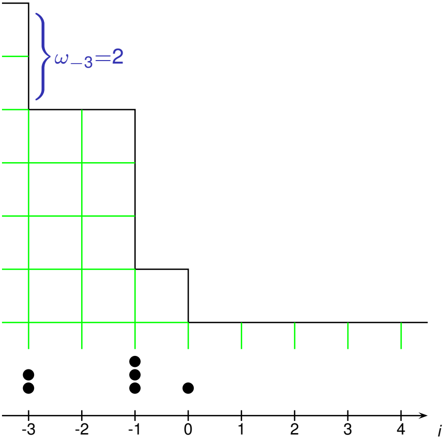


The asymmetric zero range process



$$\omega_i \geq 0.$$

The asymmetric zero range process



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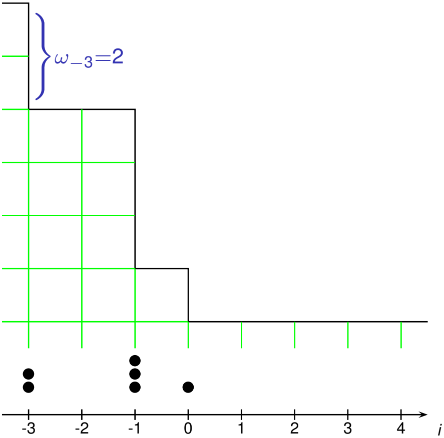

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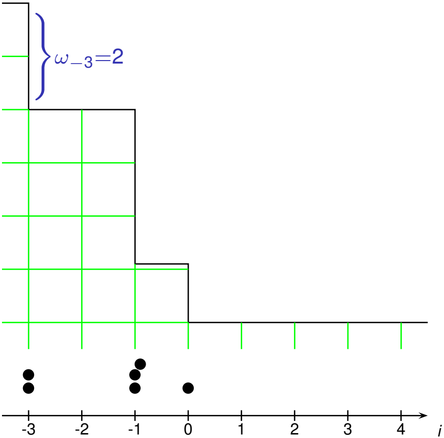
The asymmetric zero range process



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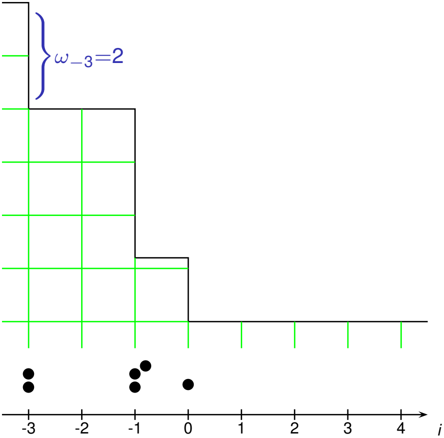

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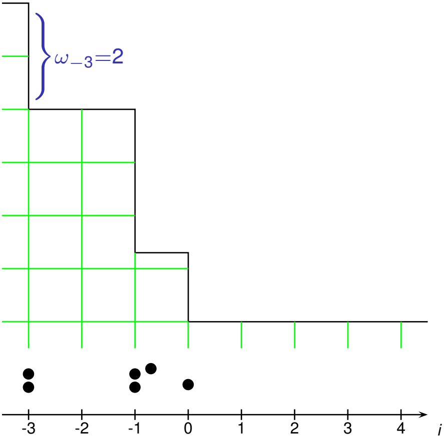

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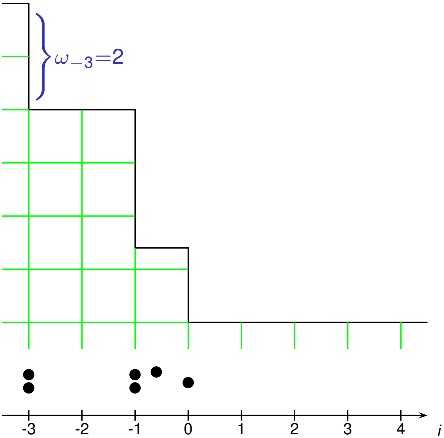

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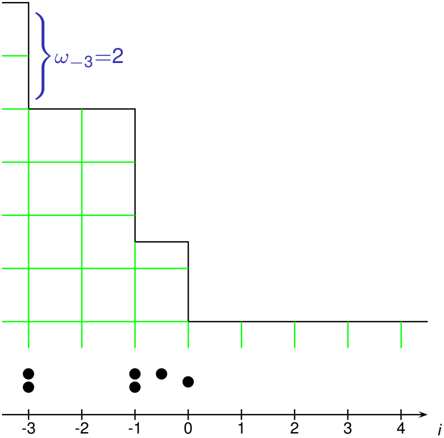

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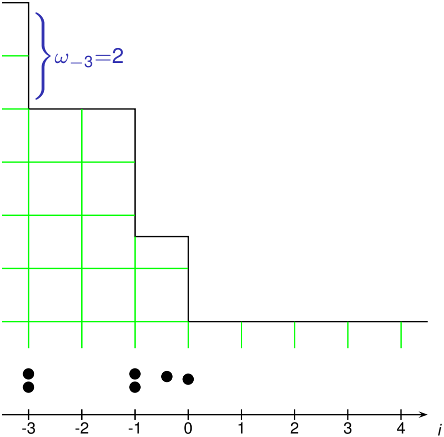

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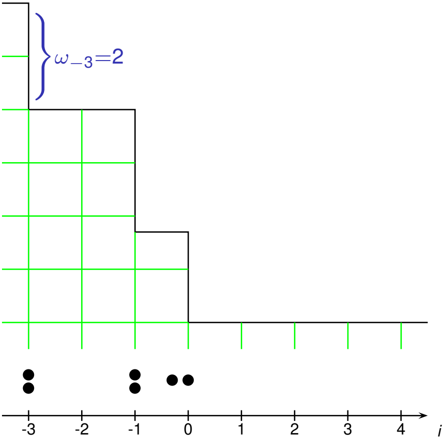

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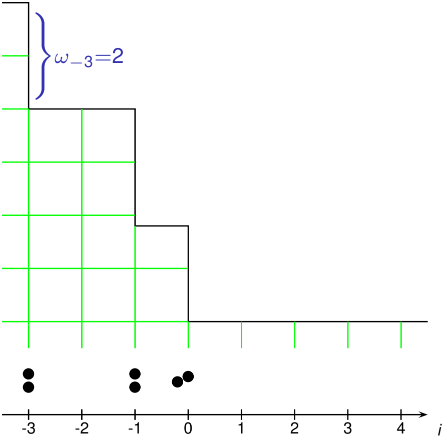

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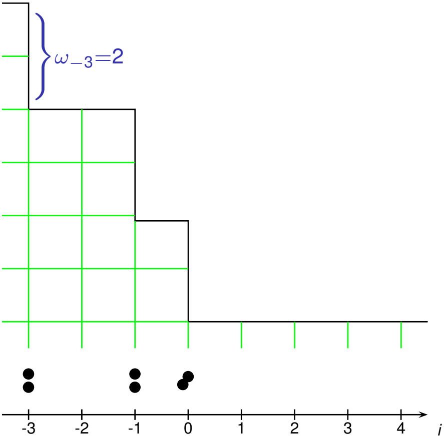

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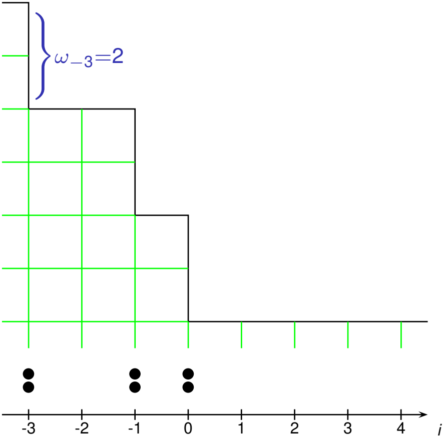

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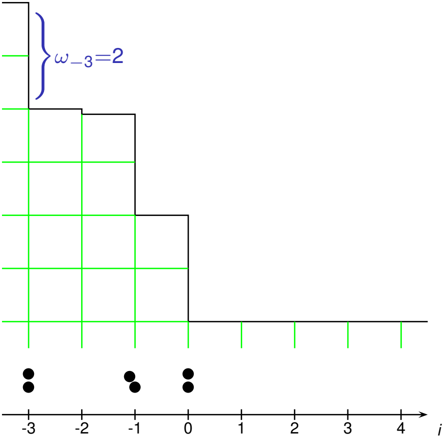

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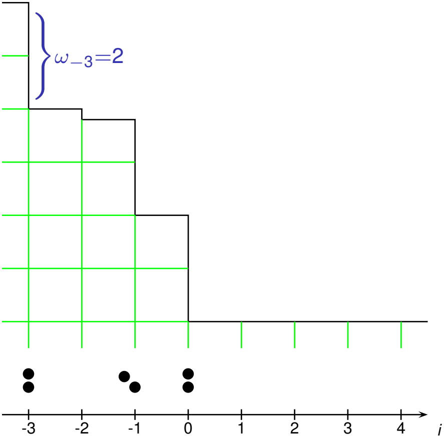

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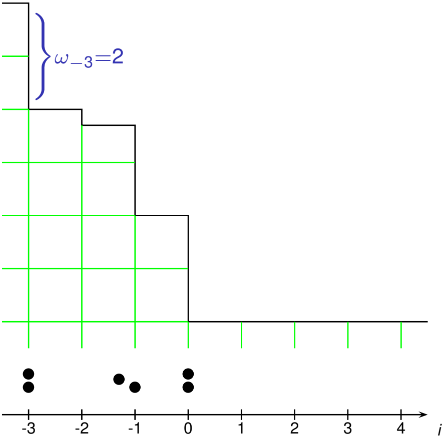

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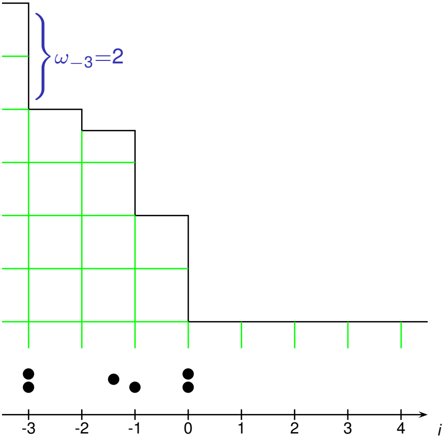

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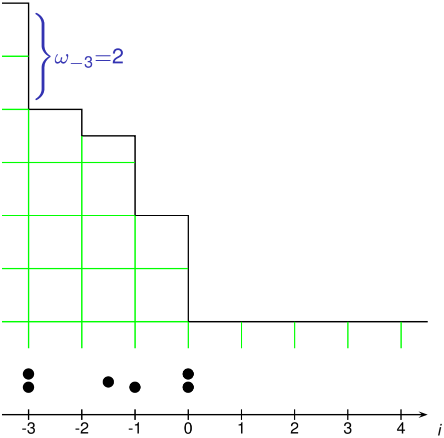

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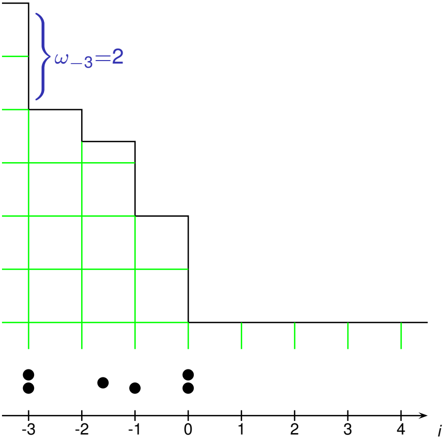

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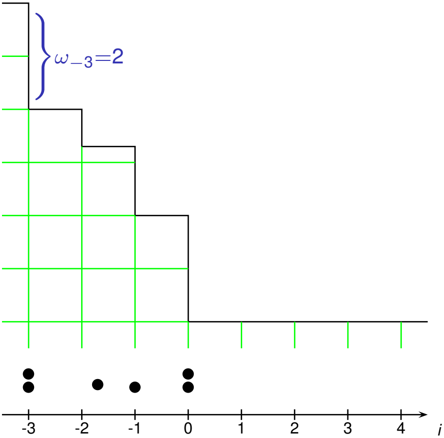

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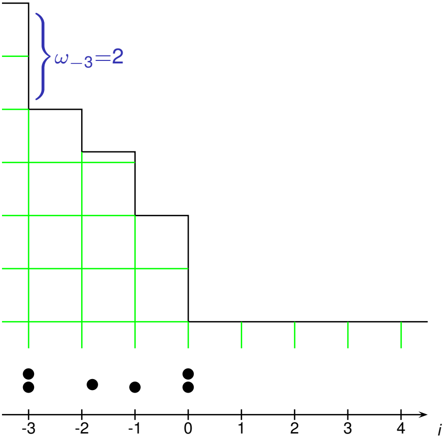

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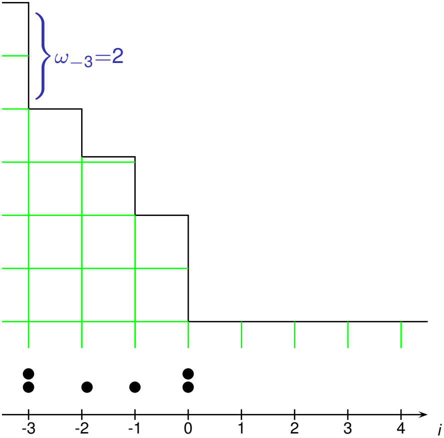

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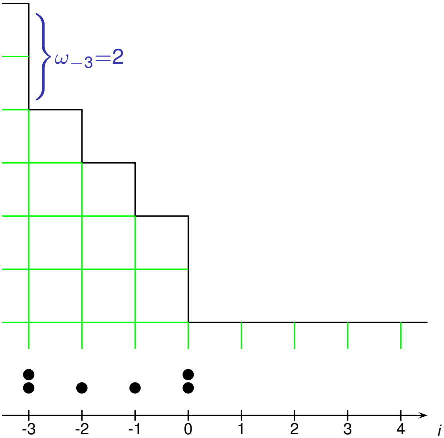

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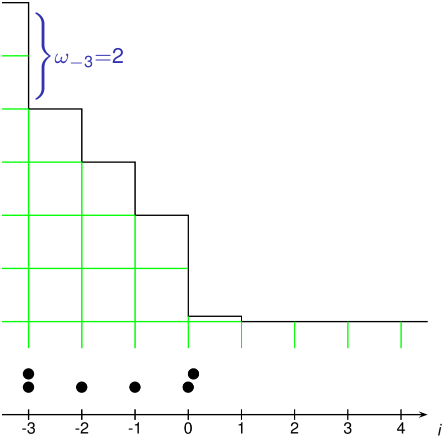

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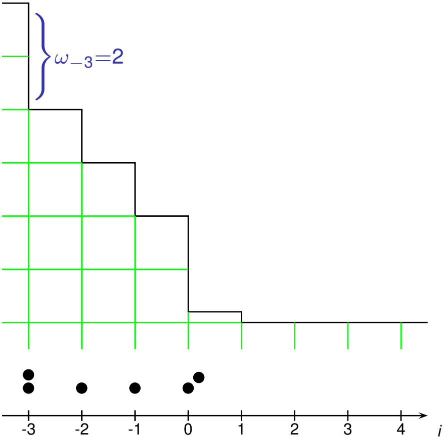

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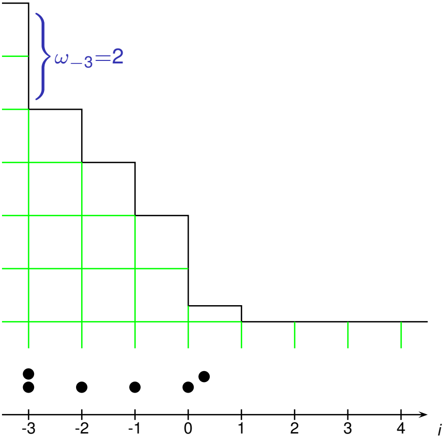

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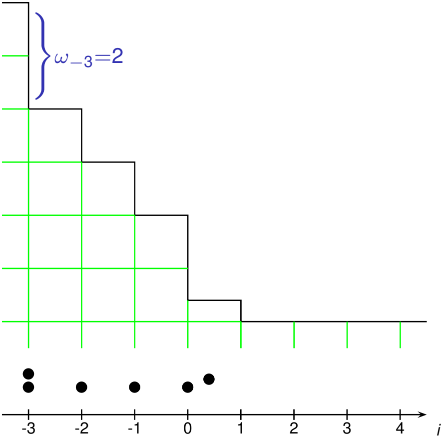

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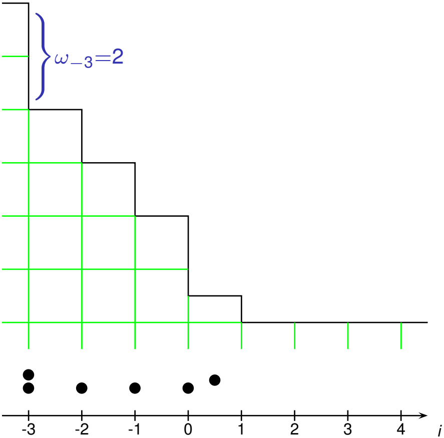

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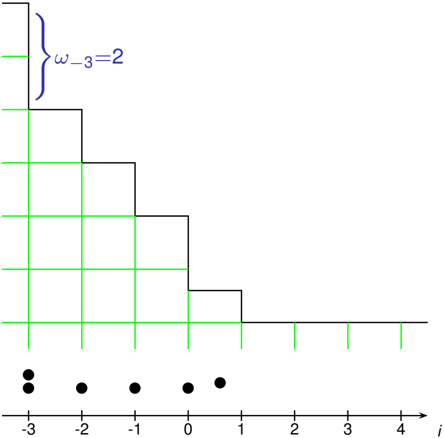

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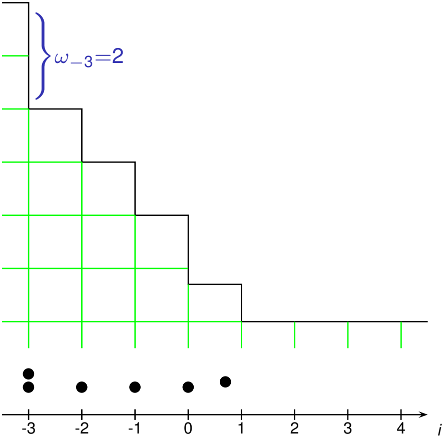

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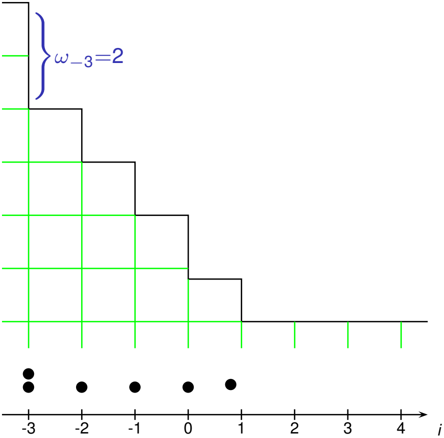

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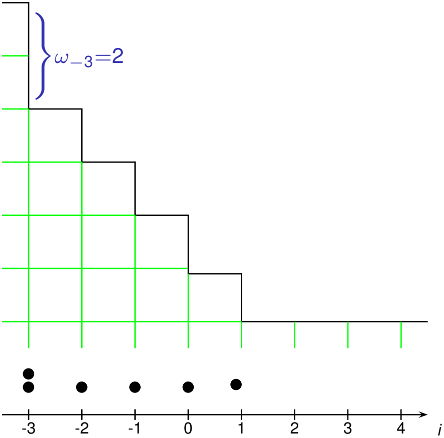

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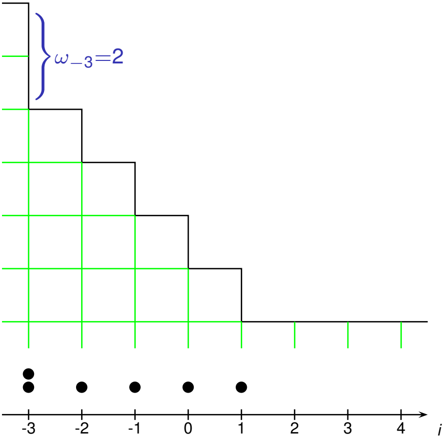

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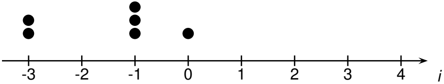

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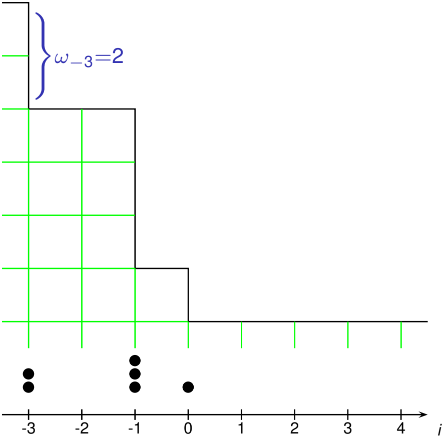

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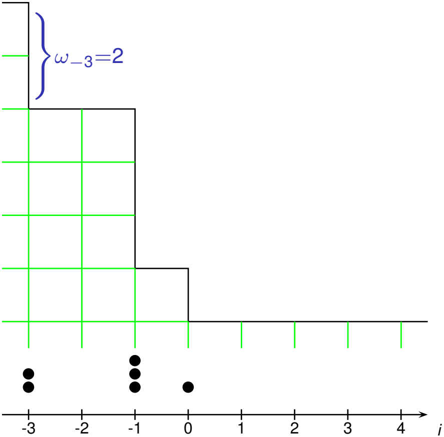

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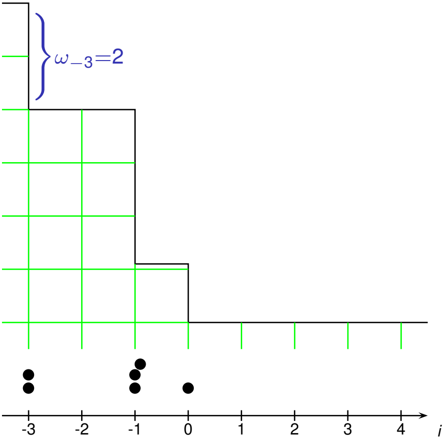

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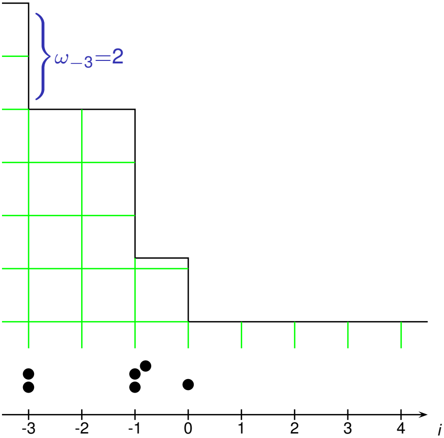

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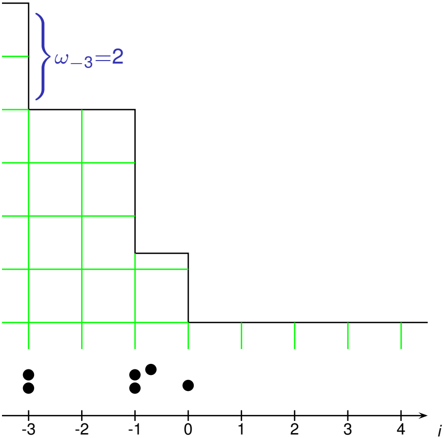

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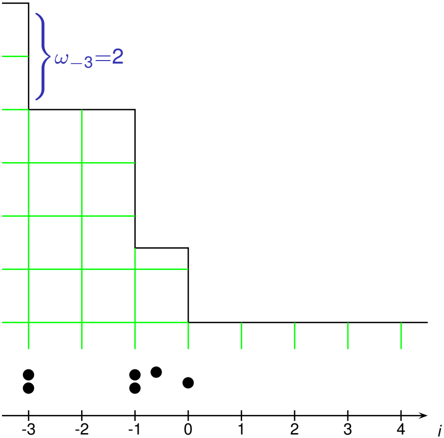

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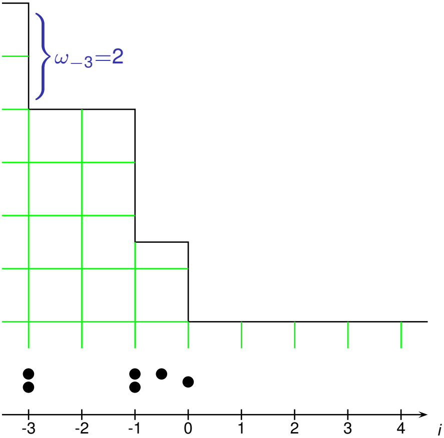

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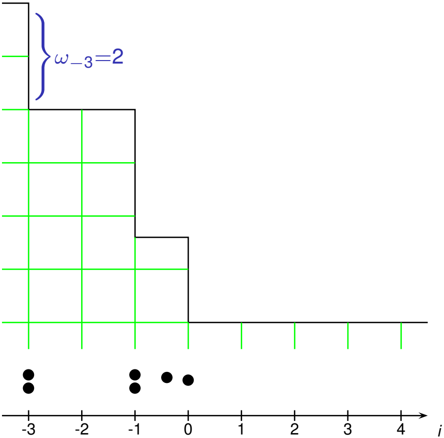

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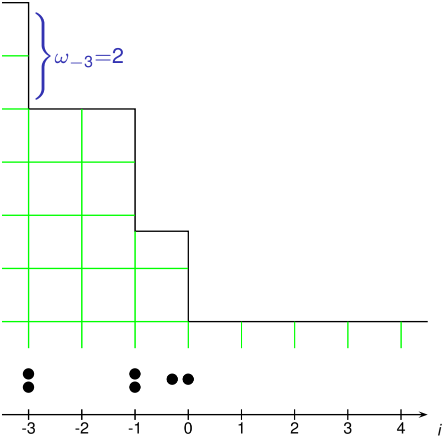

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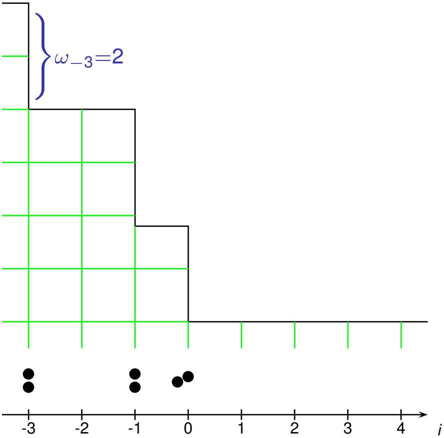

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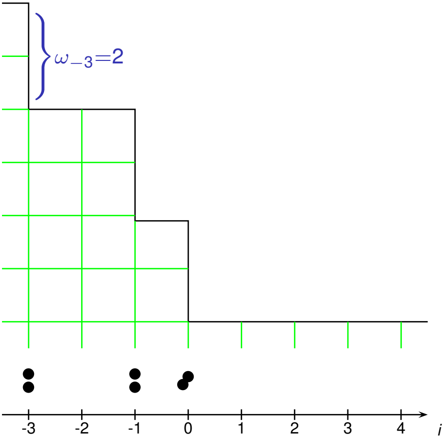

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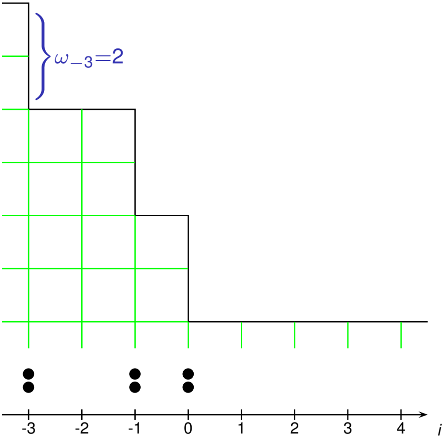

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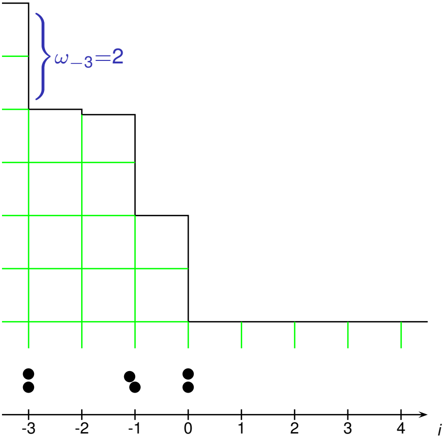

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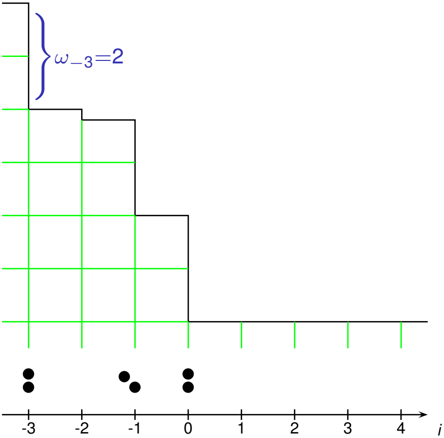

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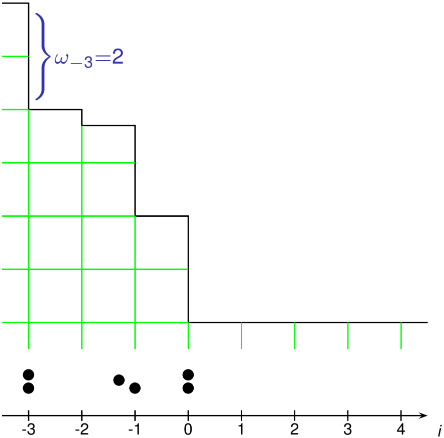

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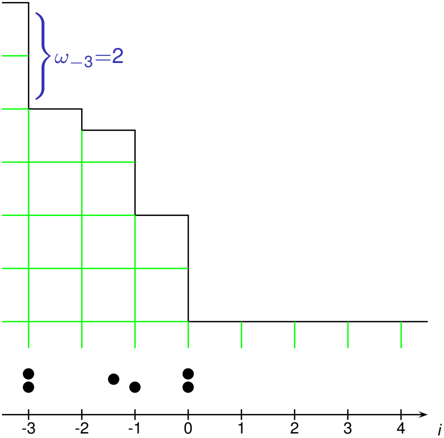

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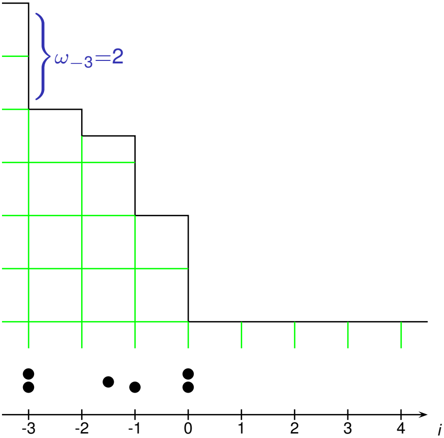

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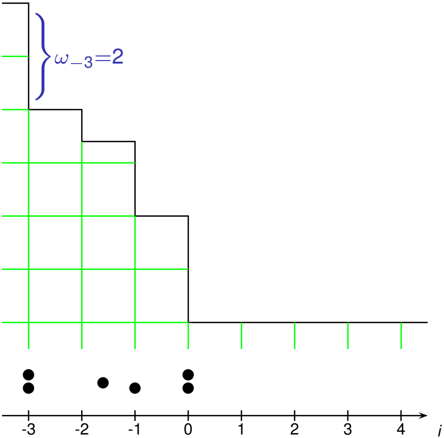

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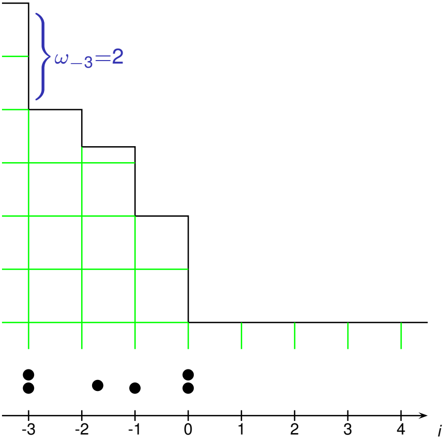

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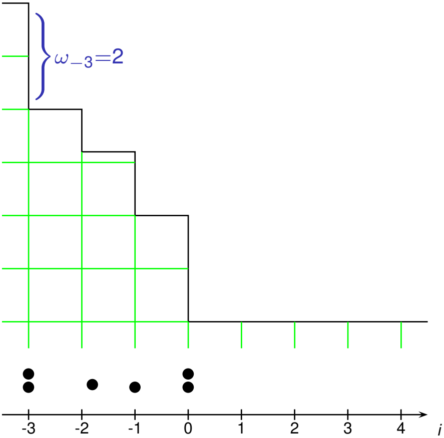

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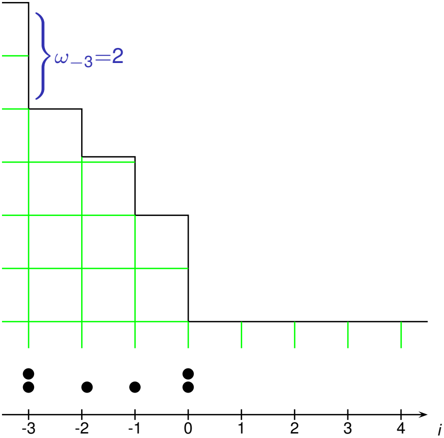

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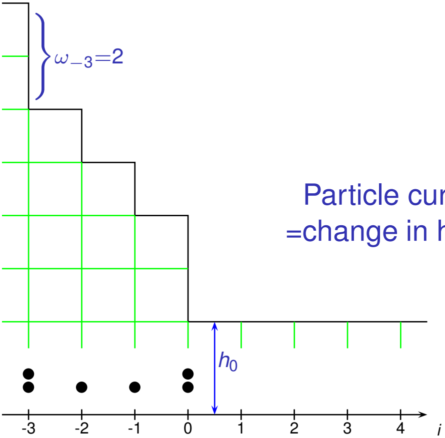

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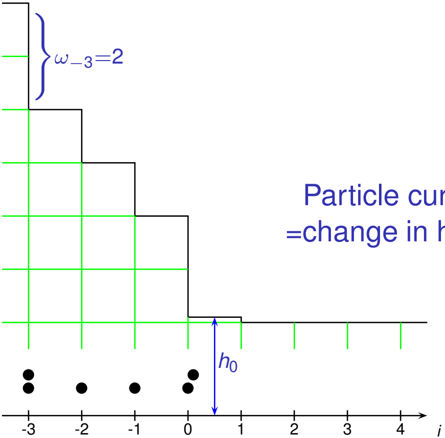

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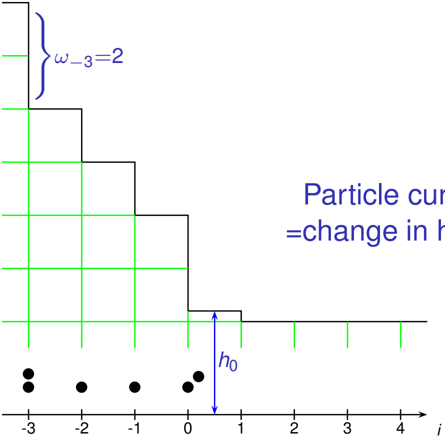

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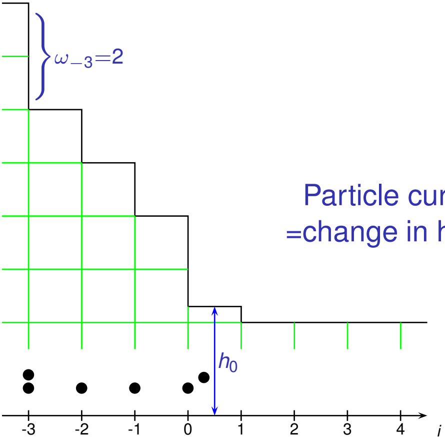

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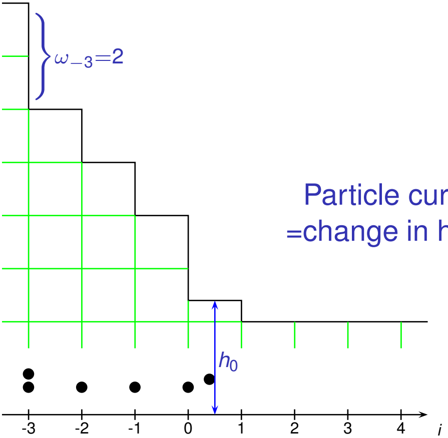

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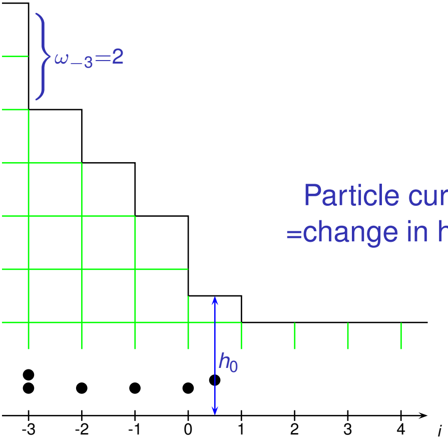

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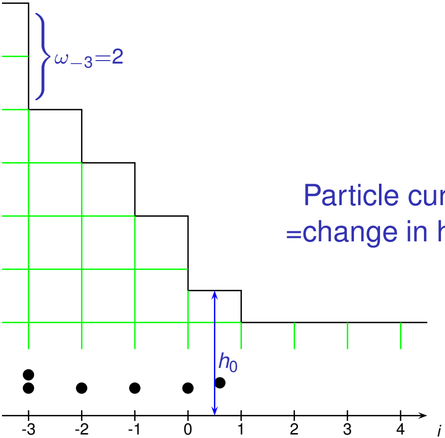

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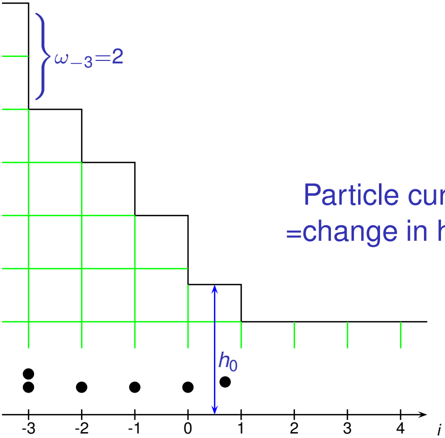

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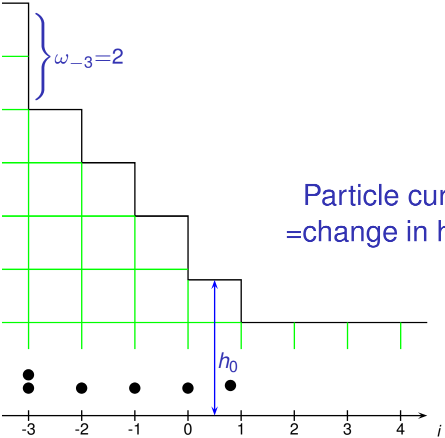

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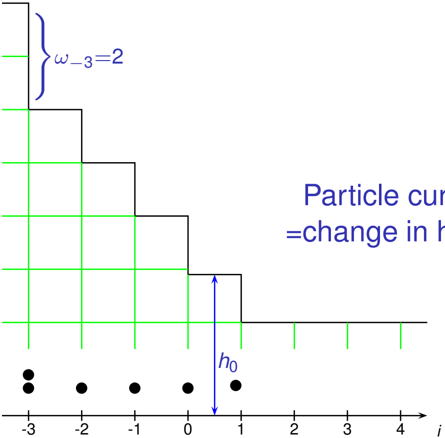

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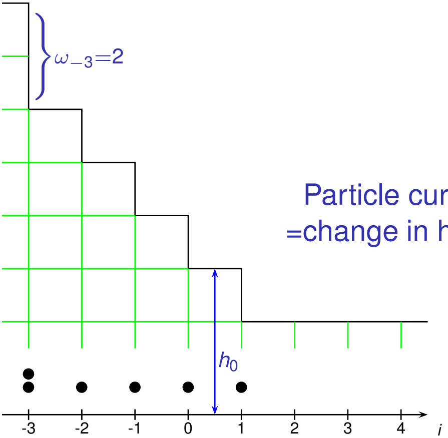

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What is this all good for?

Two ways to look at this:

- Misanthrope particles [C. Cocozza-Thivent ’85]: don’t like each other
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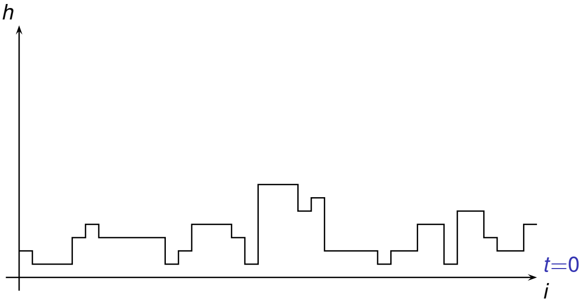
So,

- Cars on the road
- 1-dimensional transport e.g., red blood cells in capillaries
- Infection through crops
- Fire combusting paper or a forest
- . . . more to come.

Integrated particle current



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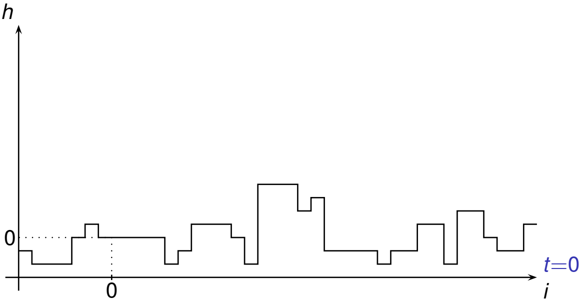

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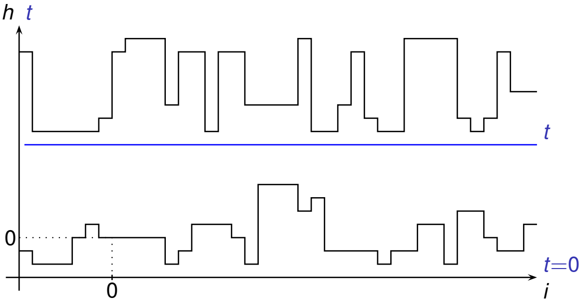

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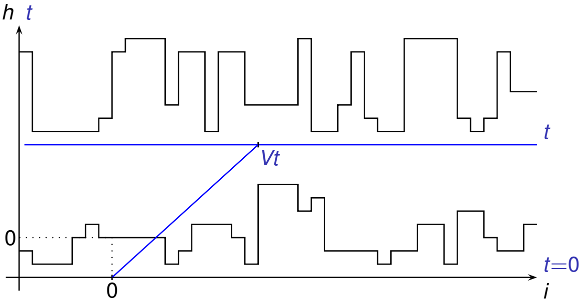

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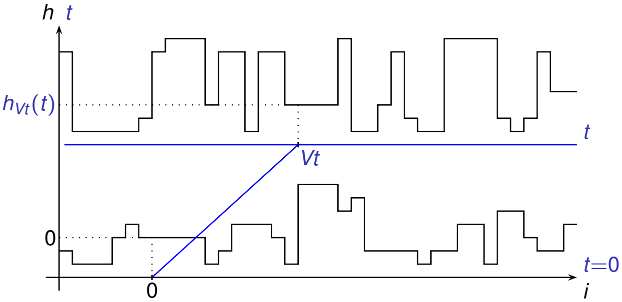

$h_{Vt}(t)$ = height as seen by a moving observer of velocity V.

= net number of particles passing the window $s \mapsto Vs$.

(Remember: particle current=change in height.)

The question

... is the properties of $h_{Vt}(t)$ under a stationary evolution.

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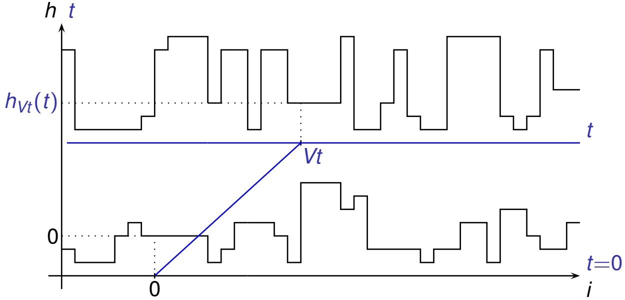
Theorem (CLT [P. A. Ferrari-L. R. Fontes '94 (ASEP)]; [B. '03 (TAZRP)]**)**

$$\lim_{n\to\infty}\frac{h_{Vt}(t)-\mathbb{E}\,h_{Vt}(t)}{t^{1/2}}\to\mathcal{N},\qquad \lim_{t\to\infty}\frac{\mathbb{V}\mathrm{ar}(h_{Vt}(t))}{t}=c\cdot|C-V|$$



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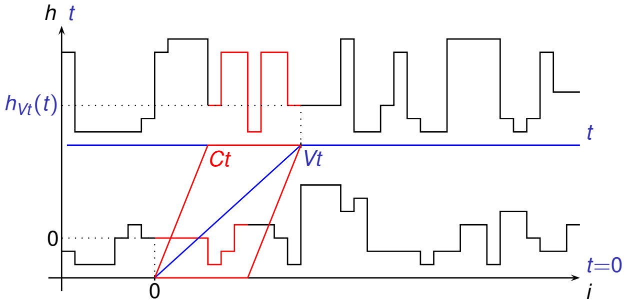

Initial fluctuations are transported along on this scale.

Abnormal fluctuations

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On the line $V = C$,

Theorem (KPZ scaling [B., J. Komjáthy, T. Seppäläinen '08-'12 (ASEP, TAZRP)])

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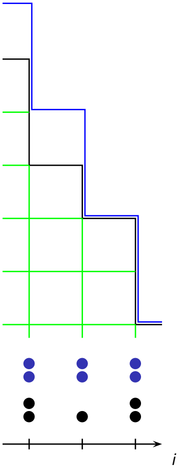
KPZ universality class.

The second class particle

States ω and ω only differ at one site.



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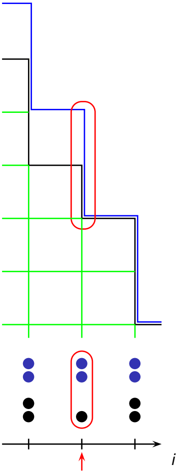

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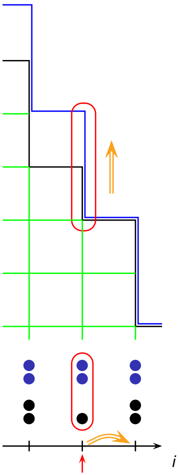

Growth on the right:
rate≤rate

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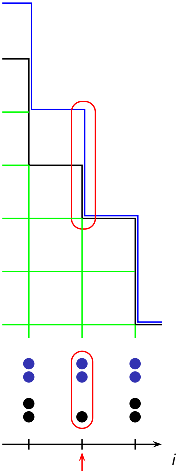

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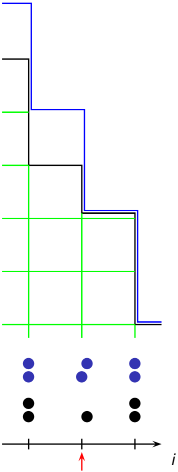

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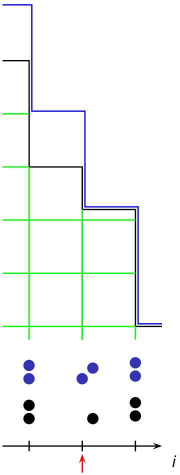

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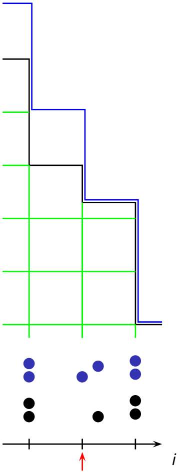

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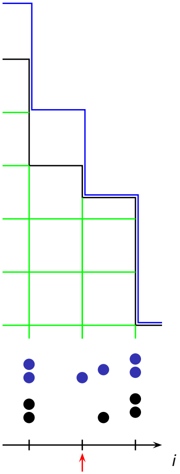

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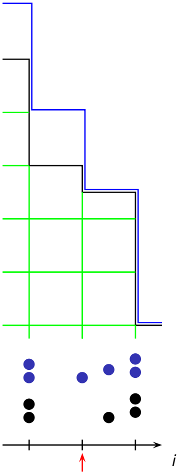

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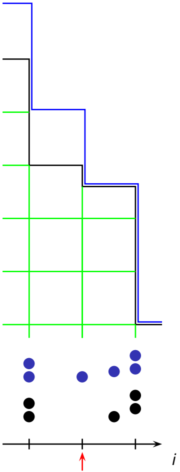

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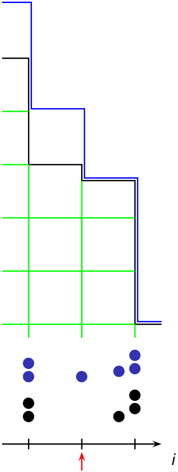

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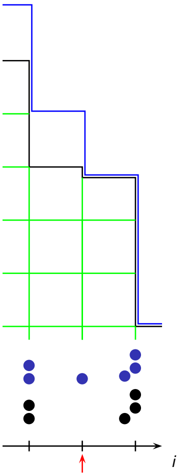

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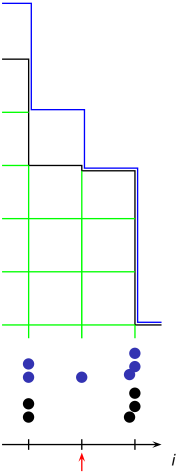

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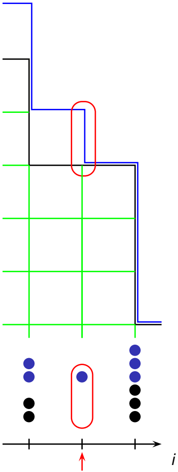

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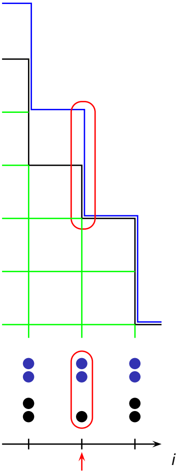

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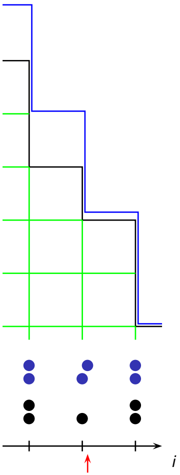

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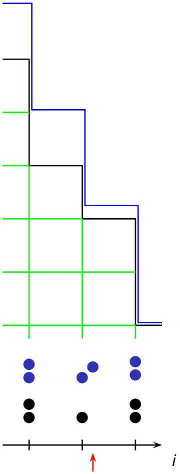

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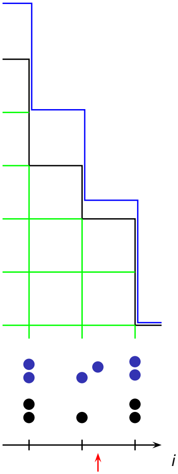

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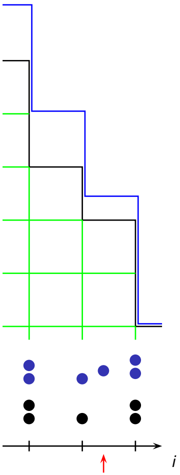

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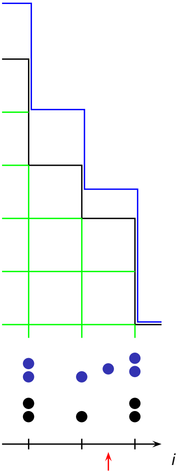

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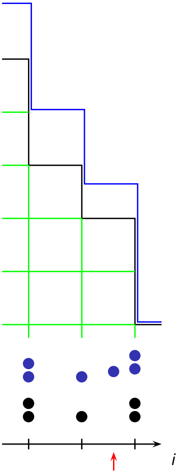

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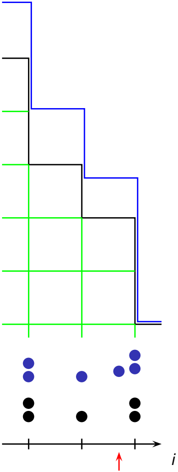

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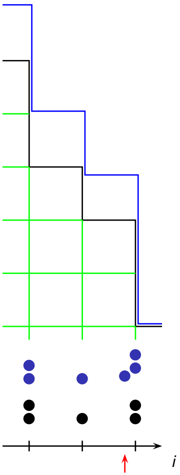

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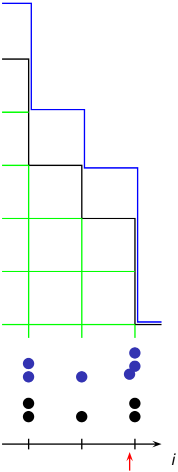

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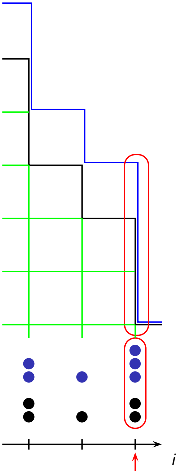

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Growth on the left:
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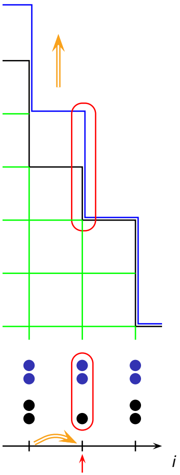

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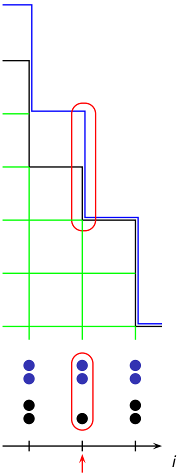

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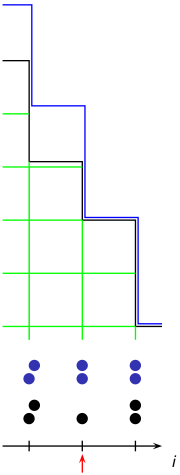

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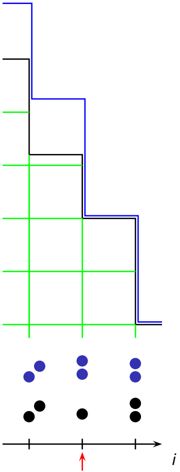

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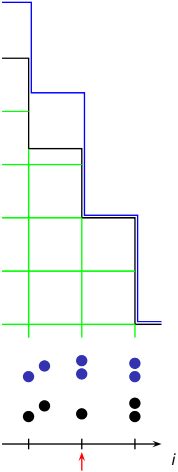

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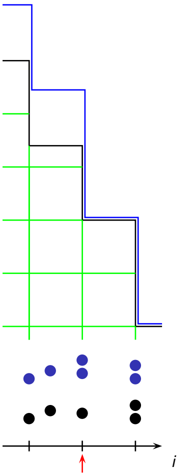

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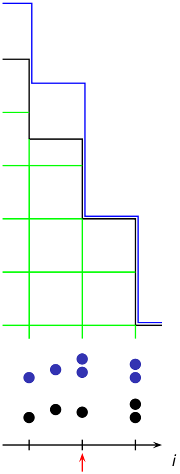

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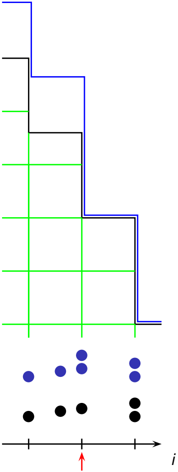

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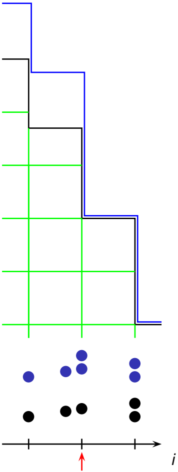

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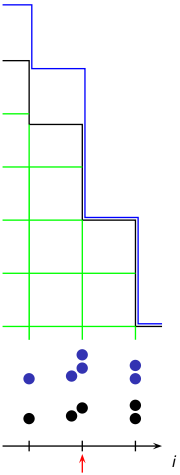

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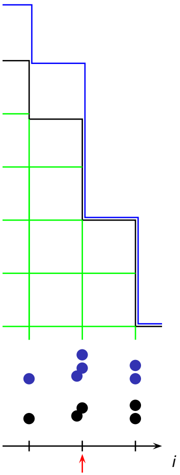

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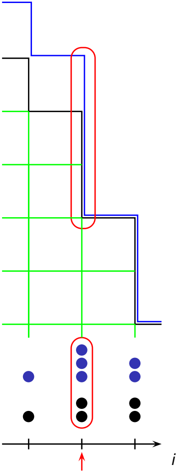

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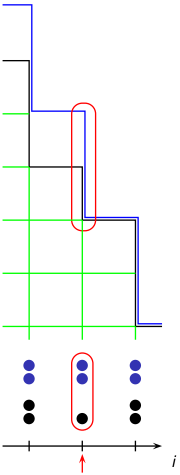

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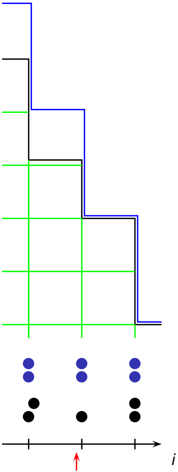

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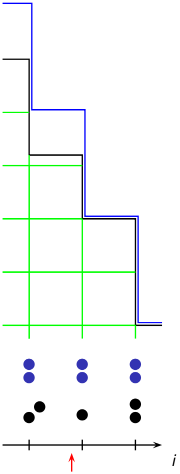

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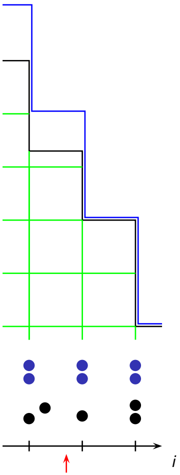

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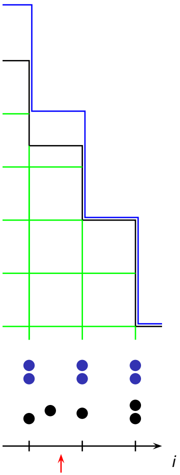

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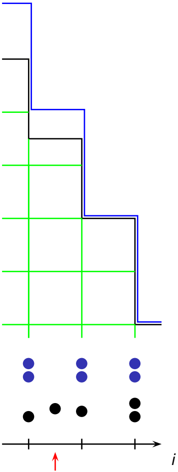

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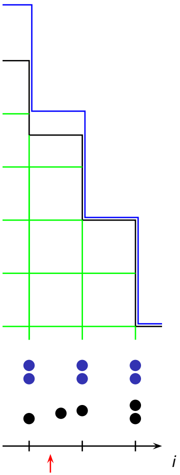

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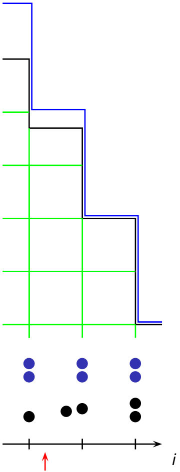

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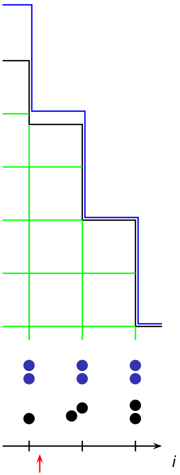

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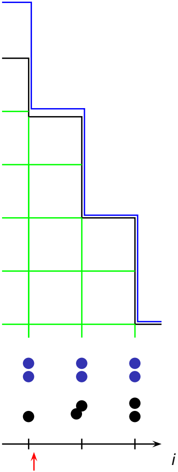

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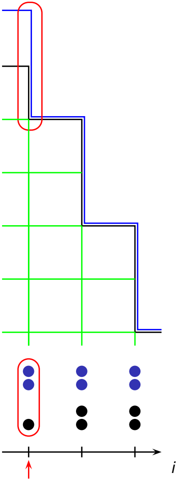

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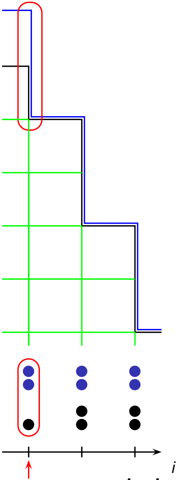

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with rate-rate:

A single discrepancy ↑, the *second class particle*, is conserved.

The second class particle

Under some conditions, flat initial state, for the location $Q(t)$ of the second class particle,

Theorem (KPZ scaling [B., J. Komjáthy, T. Seppäläinen '08-'12 (ASEP, TAZRP)])

$$0 < \liminf_{t\to\infty} \frac{\mathbb{E}\,|Q(t) - \mathbb{E}\,Q(t)|}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbb{E}\,|Q(t) - \mathbb{E}\,Q(t)|}{t^{2/3}} < \infty.$$

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Strong correlations in time, highly nontrivial motion.

The second class particle

However,

- Place it in a shock, in some cases $Q(t)$ becomes a simple random walk (with CLT)! [B., L. Duffy, Gy. Farkas, P. Kovács, A. Rákos, D. Pantelli ’16-’19]

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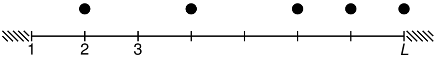
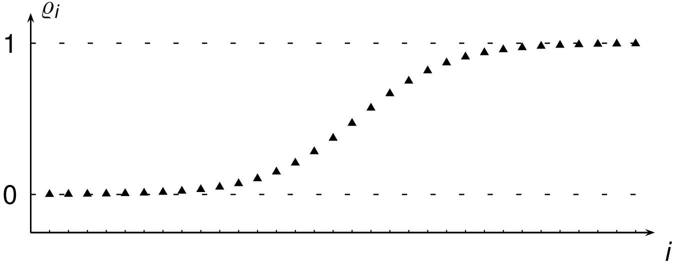
$$\lim_{t\to\infty} \frac{Q(t)}{t} \to \text{something random.}$$

Blocking ASEP [T. Liggett '76]

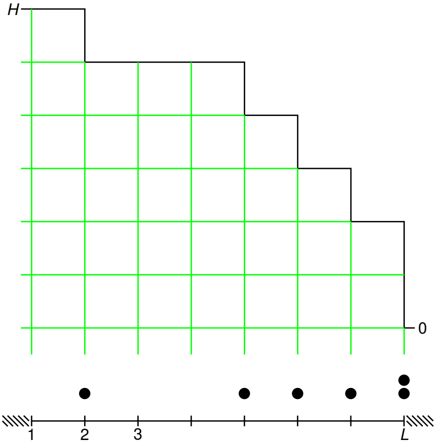


$p > q$, but particles are blocked. The resulting density profile:



Blocking AZRP

$p > q$: convex



Particles jump to the right with rate $p \cdot r(\omega_i)$
to the left with rate $q \cdot r(\omega_i)$.

Blocking AZRP

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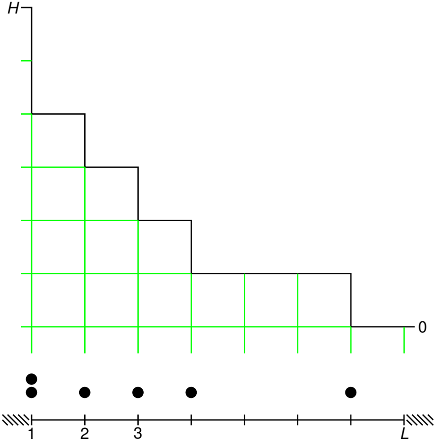

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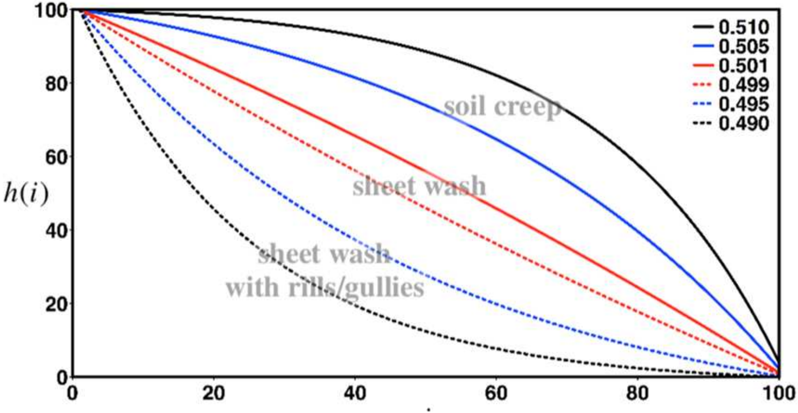
Hills [J. Calvert, B., K. Michaelides '18]

Rescaling this surface with weak aymmetry ($p \simeq q$) results in a *convection-diffusion type equation with boundary conditions*. And this explains a lot of things about hillslope evolution.



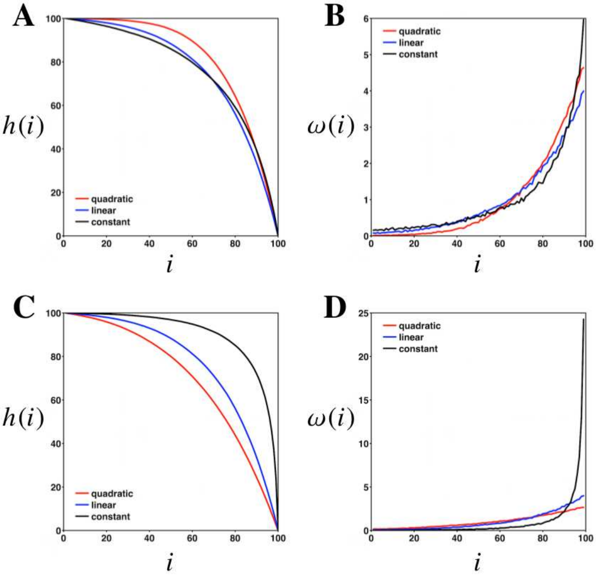
Convex hills

Wikipedia

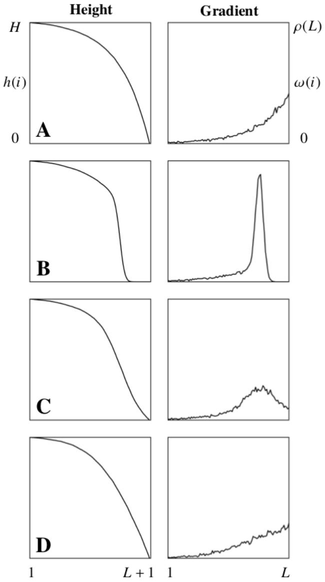
Concave hills

Stockphotos4free

The stationary slope



Dynamics



The OMG slides: blocking ASEP [D. Adams, B., J. Jay '24+]

Theorem (Euler's identity)

$$\sum_{i=0}^{\infty} \frac{q^{\frac{i(i-1)}{2}} z^i}{(q;q)_i} = (-z;q)_\infty$$

$(a;q)_n = \prod_{i=0}^{n}(1 - aq^i)$

The OMG slides: blocking ASEP [D. Adams, B., J. Jay '24+]

Theorem (q-binomial theorem)

$$\sum_{i=0}^{m} q^{\frac{i(i-1)}{2}} z^i \begin{bmatrix} m \\ i \end{bmatrix}_q = (-z;q)_m$$

$(a;q)_n = \prod_{i=0}^{n}(1-aq^i)$

$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m \cdot (q;q)_{n-m}}$

The OMG slides: blocking ASEP [D. Adams, B., J. Jay '24+]

Theorem (Durfee rectangles identity)

$$\sum_{i=n^-}^{\infty} \frac{q^{i(n+i)}}{(q;q)_{n+i} \cdot (q;q)_i} = \frac{1}{(q;q)_\infty}$$

$(a;q)_n = \prod_{i=0}^{n} (1 - aq^i)$

The OMG slides: blocking ASEP and AZRP [B., R. Bowen '18]

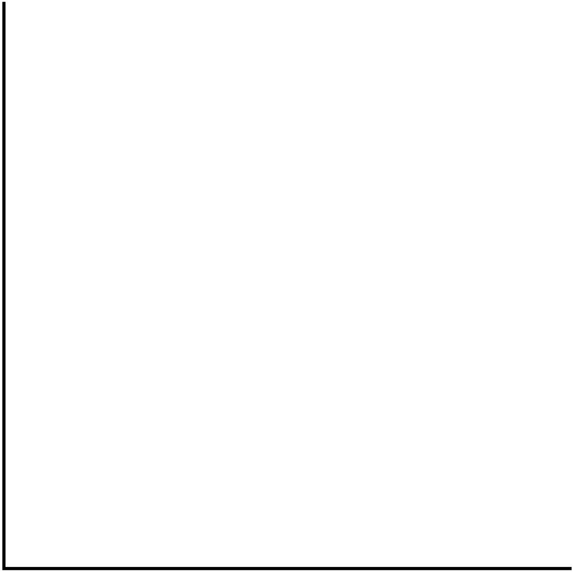
Theorem (Jacobi triple product)

$$\sum_{i=-\infty}^{\infty} q^{\frac{i(i+1)}{2}} z^i = (q;q)_\infty \cdot (-qz;q)_\infty \cdot \left(-\frac{1}{z};q\right)_\infty$$

Plus: generalisations to the *fun model* and more [B., D. Fretwell, J. Jay '22]

$$(a;q)_n = \prod_{i=0}^{n} (1 - aq^i)$$

Last passage percolation [H. Rost '81]



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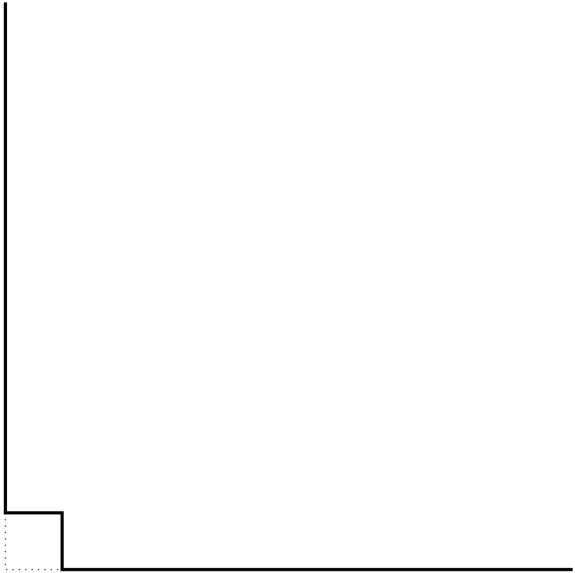

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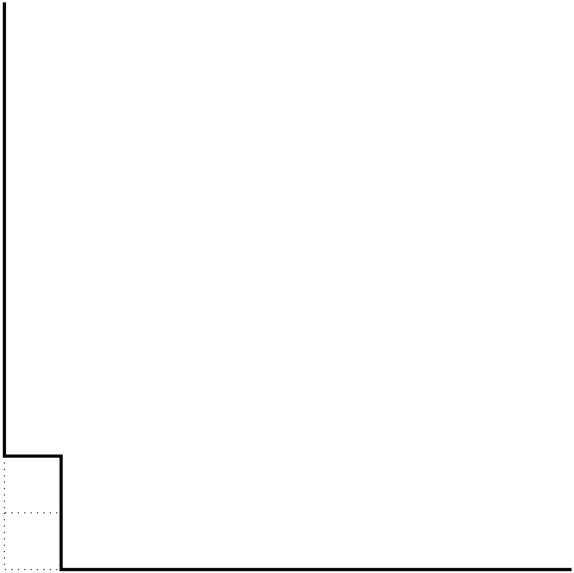

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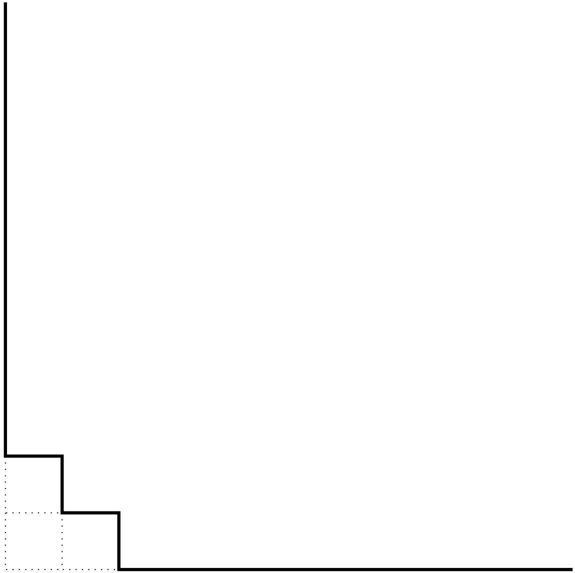

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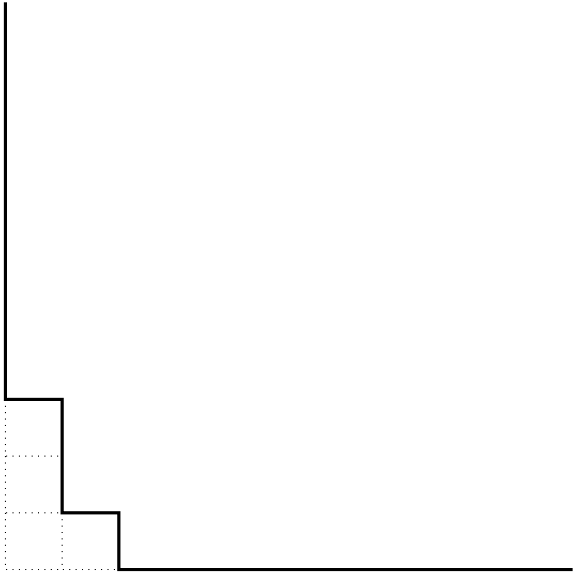

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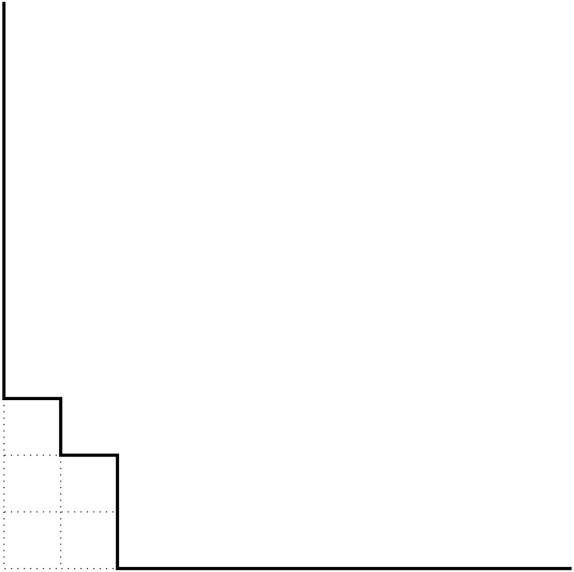

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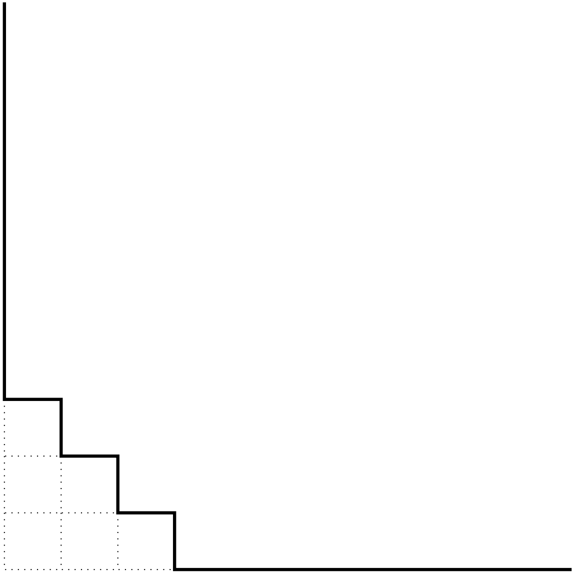

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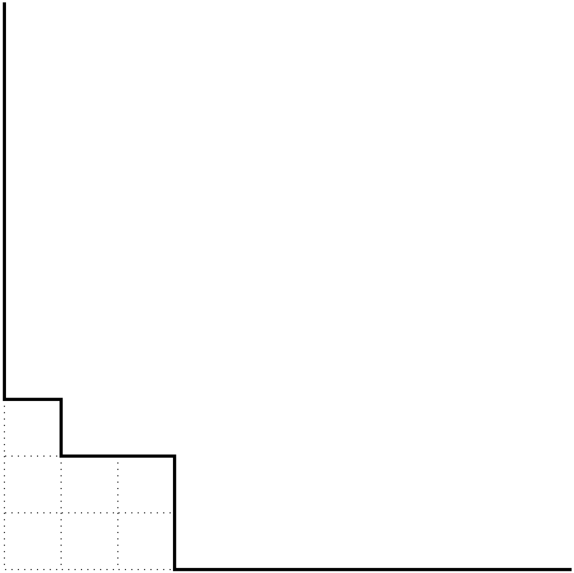

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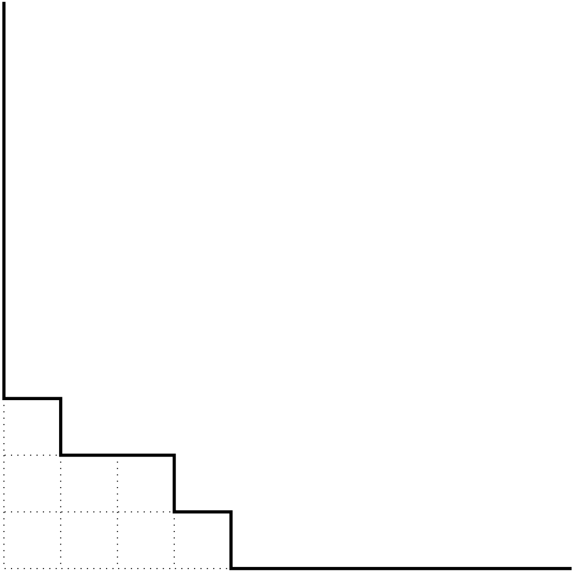

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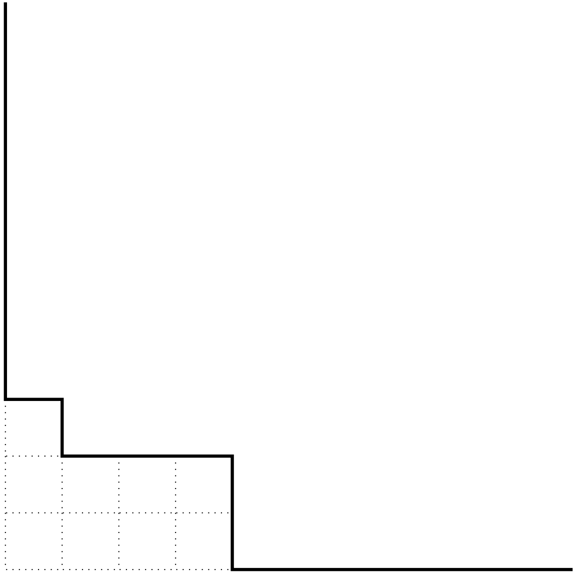

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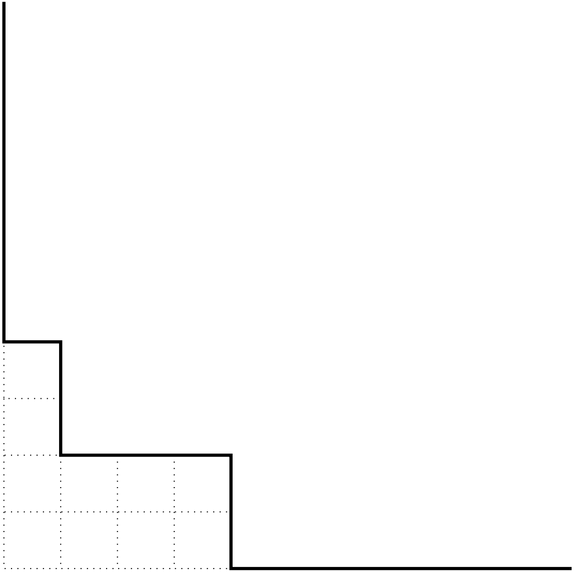

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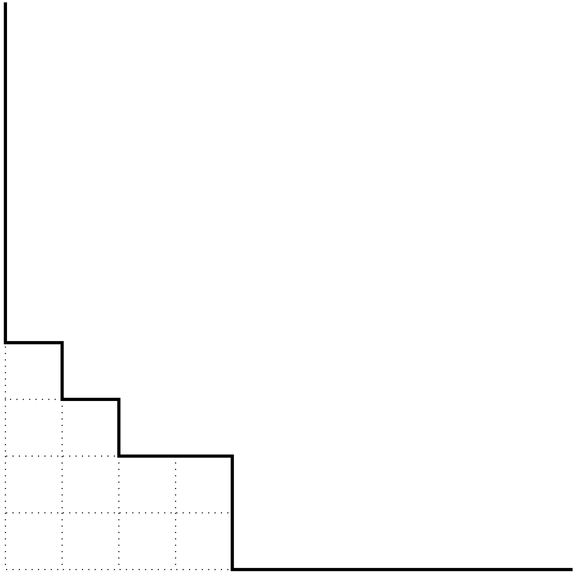

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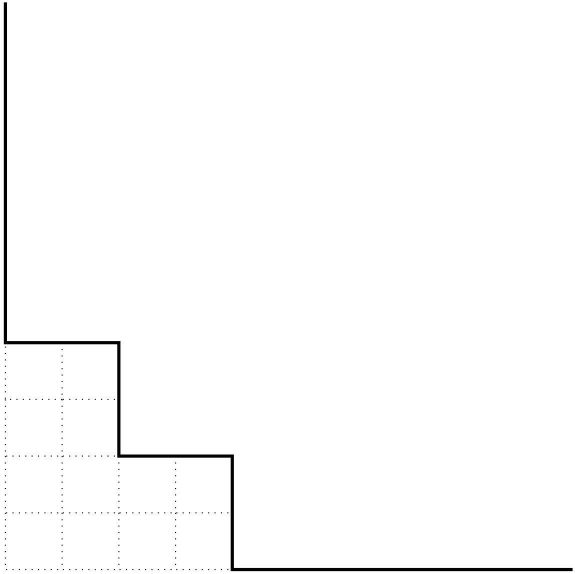

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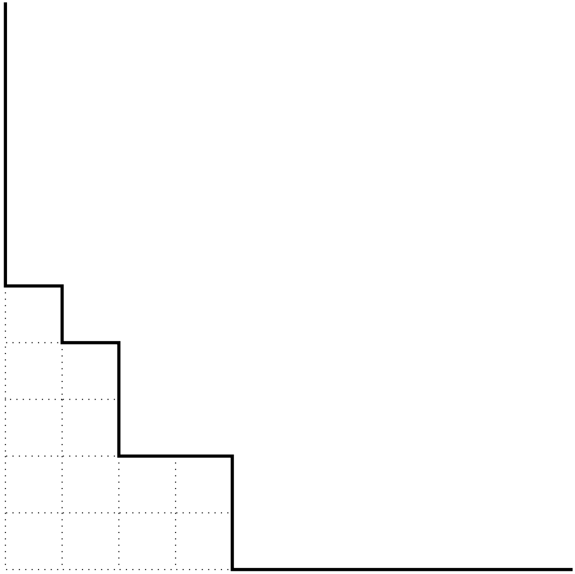

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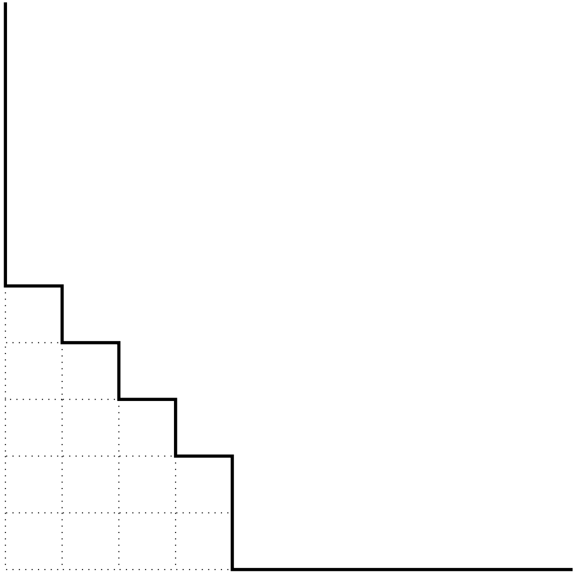

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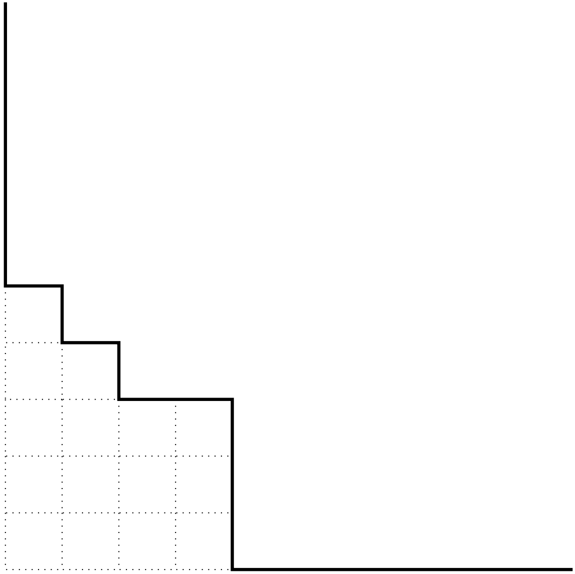

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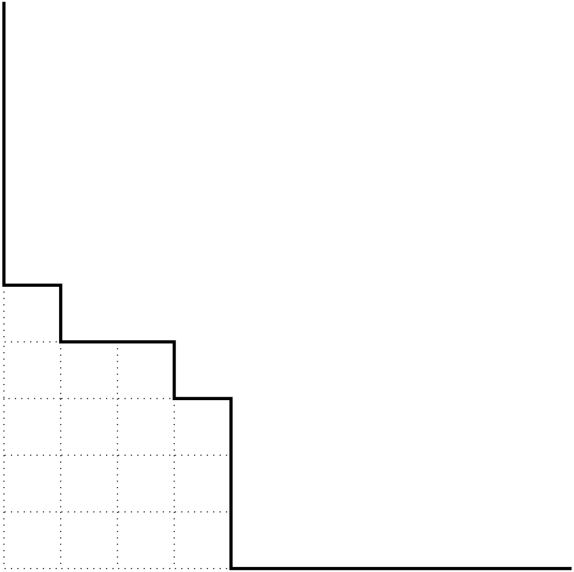

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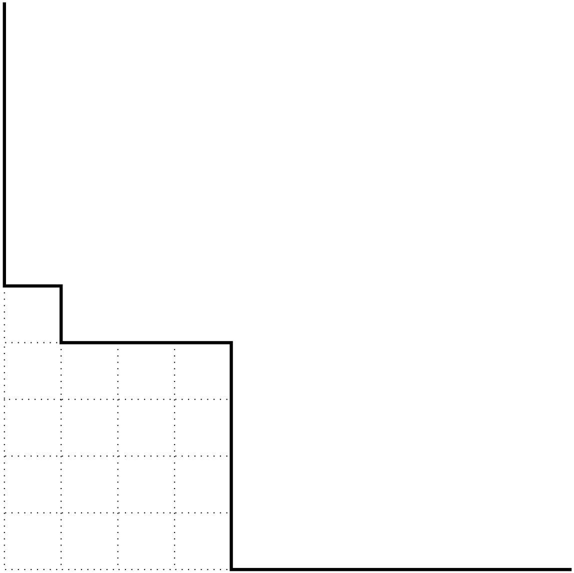

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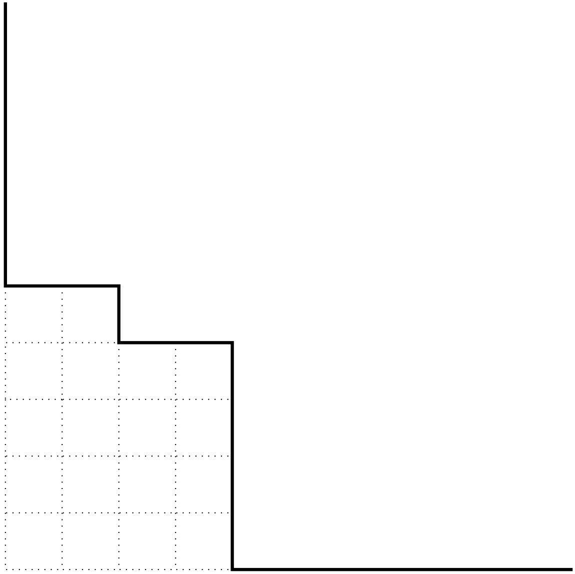

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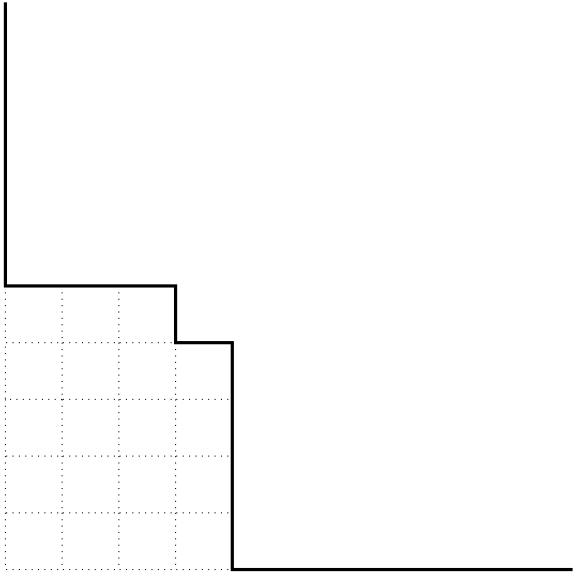

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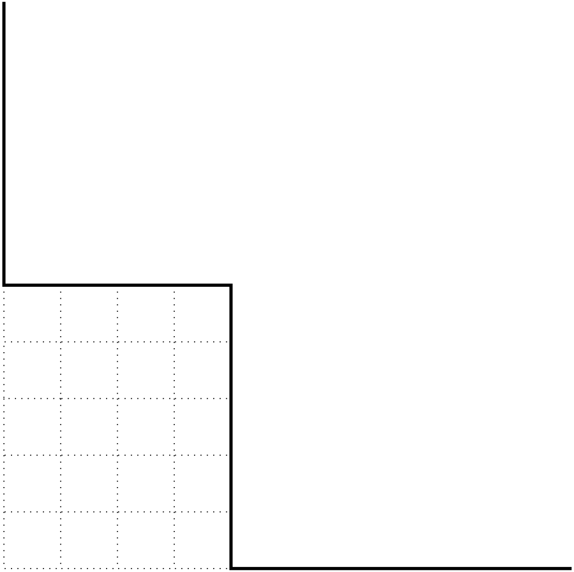

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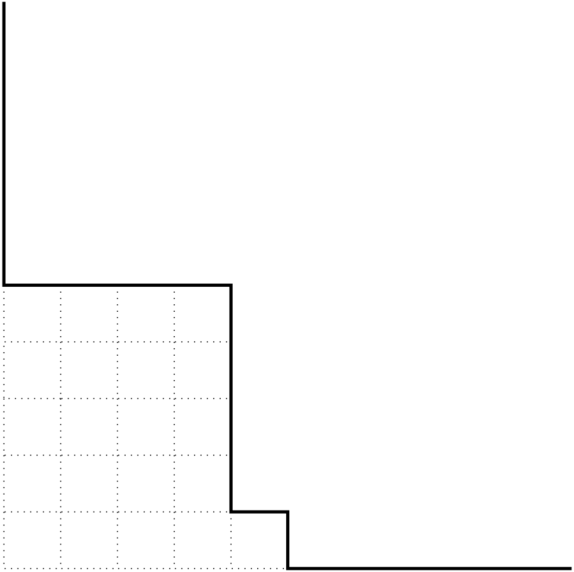

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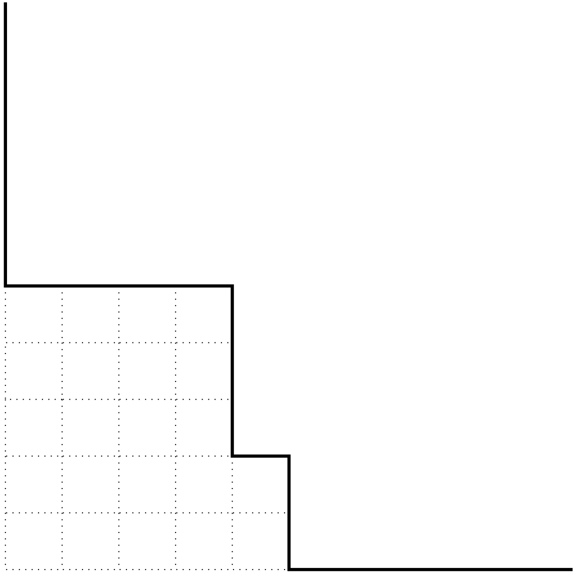

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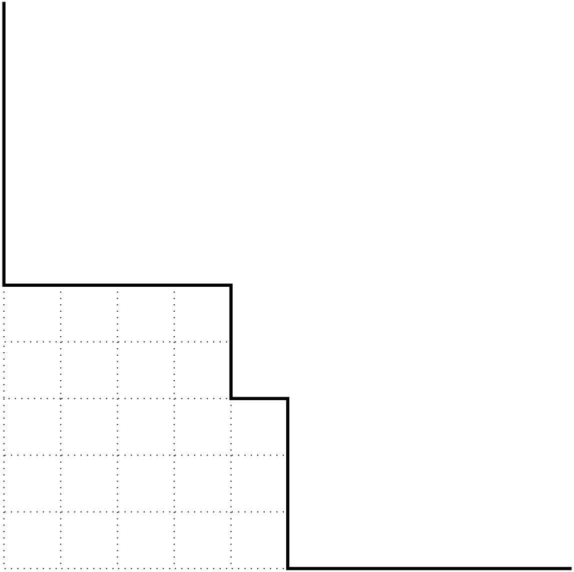

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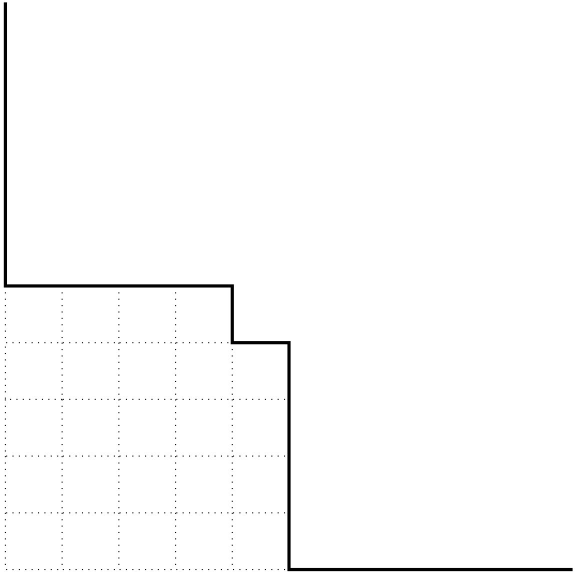

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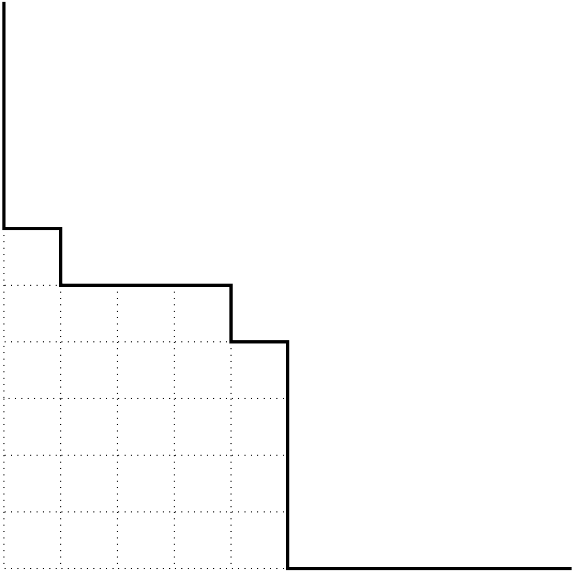

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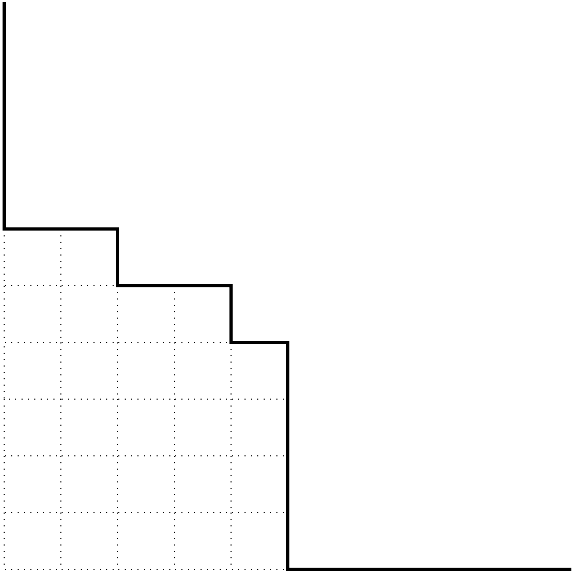

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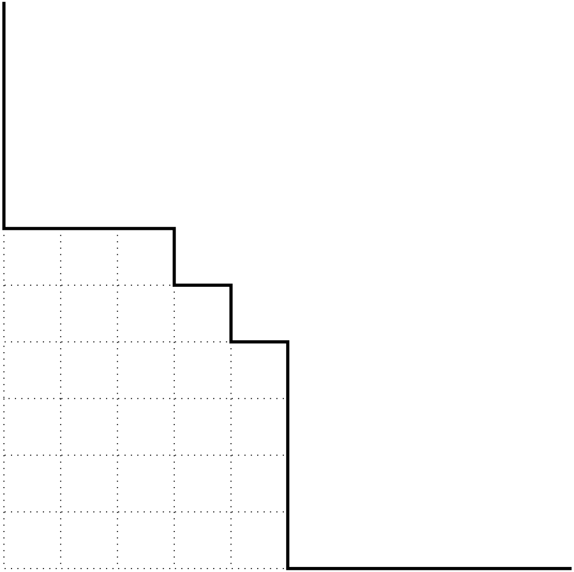

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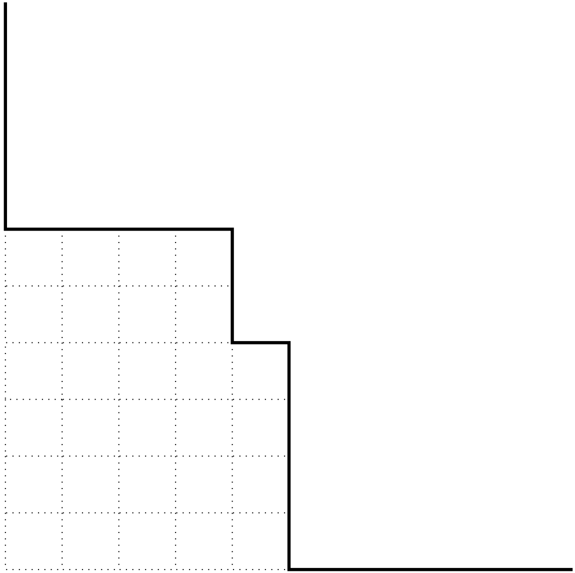

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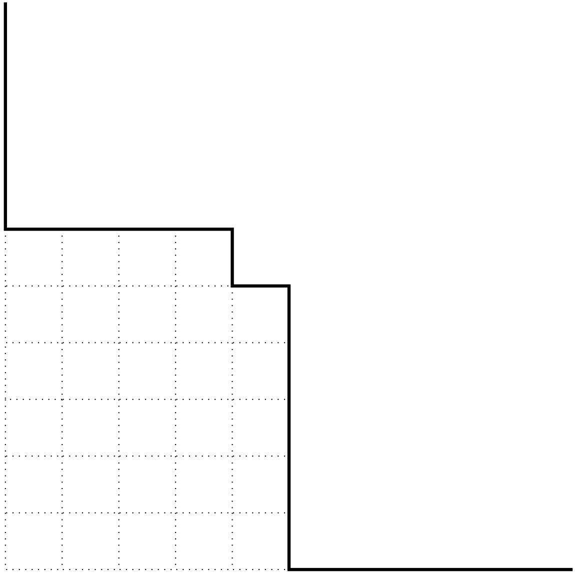

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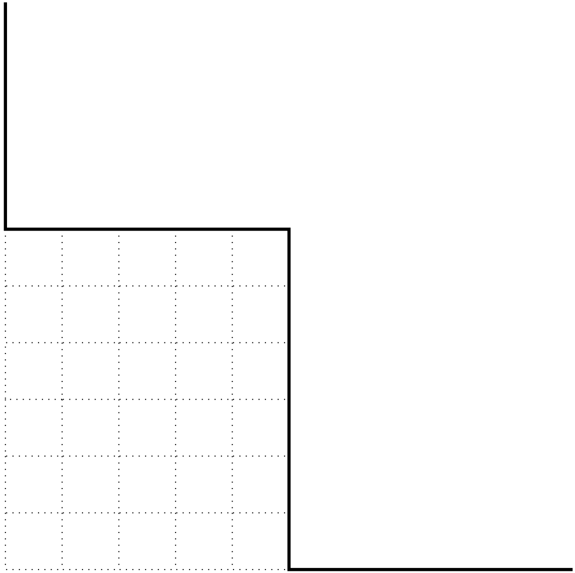

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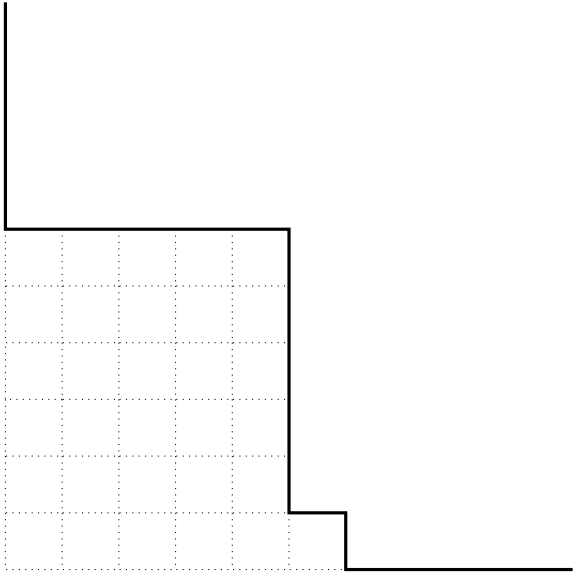

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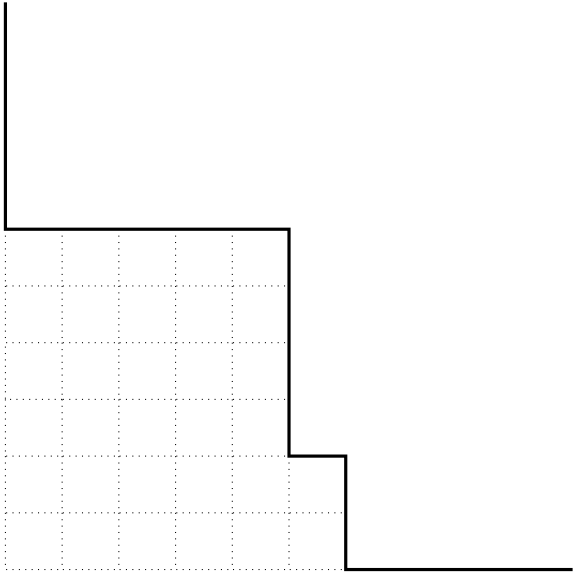

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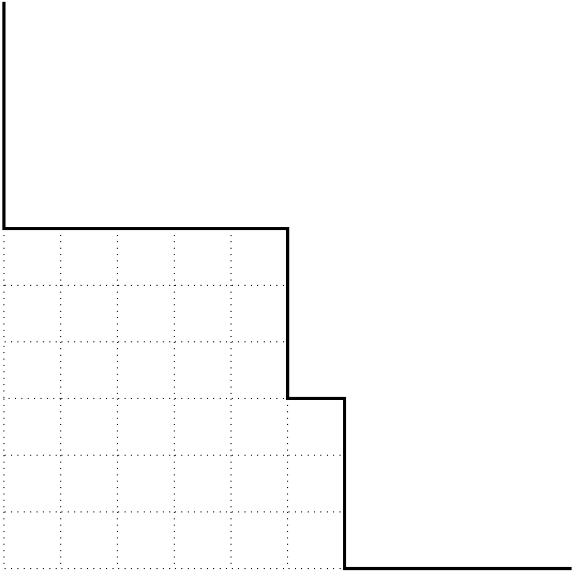

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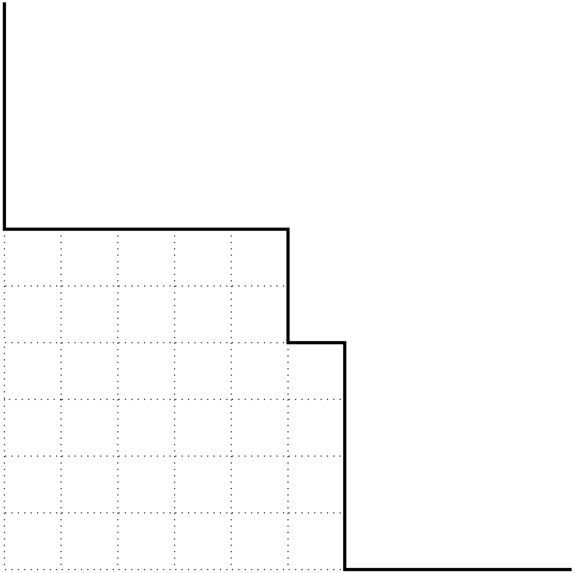

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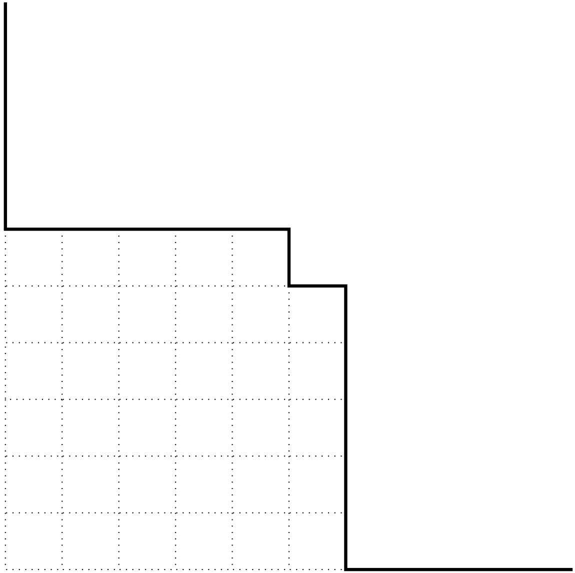

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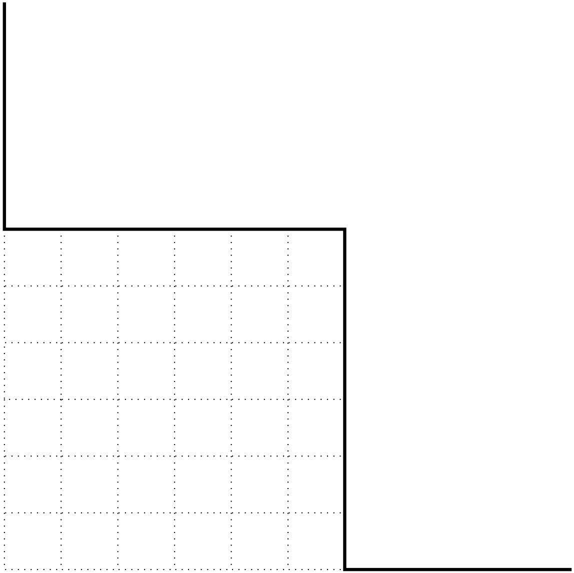

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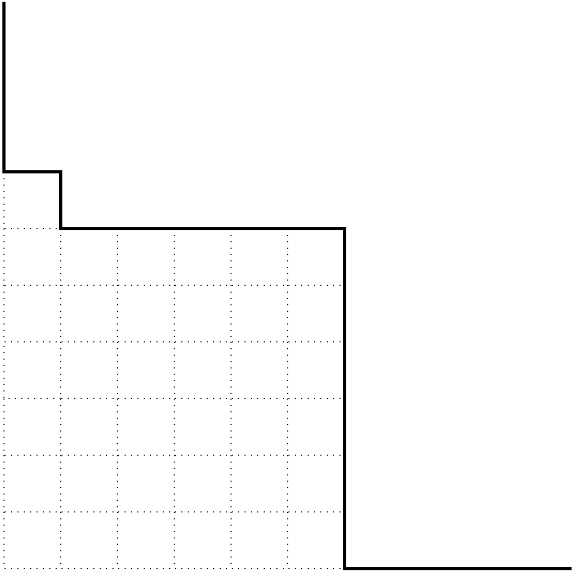

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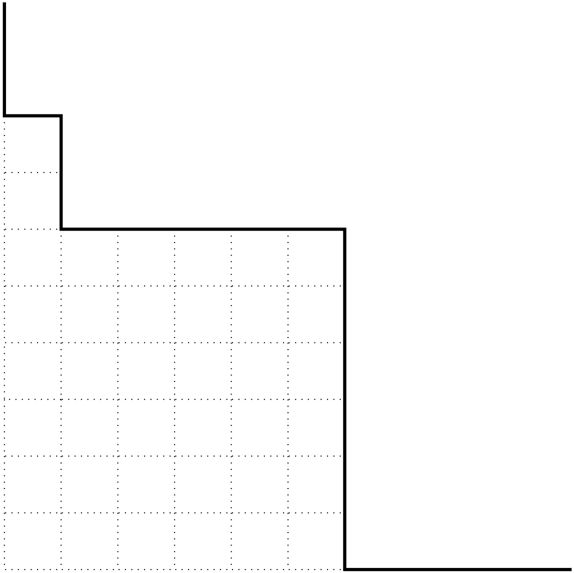

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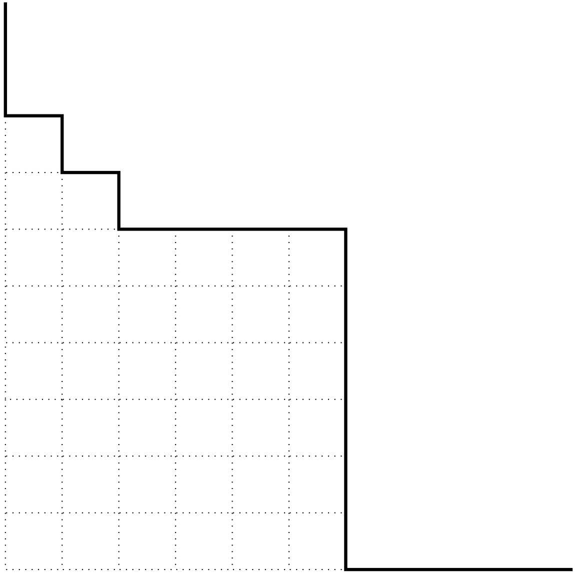

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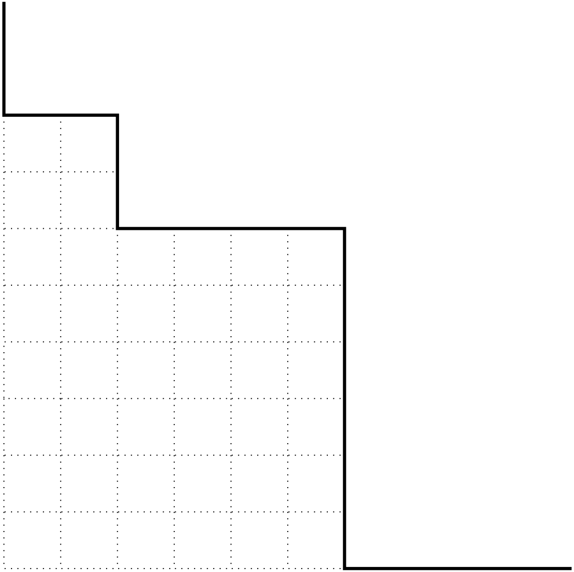

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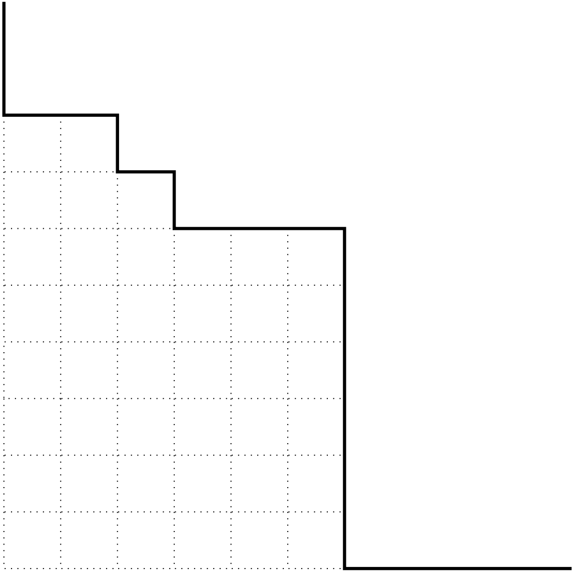

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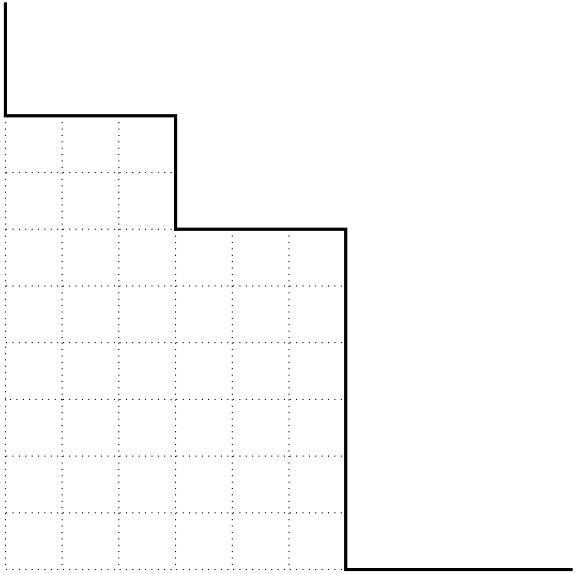

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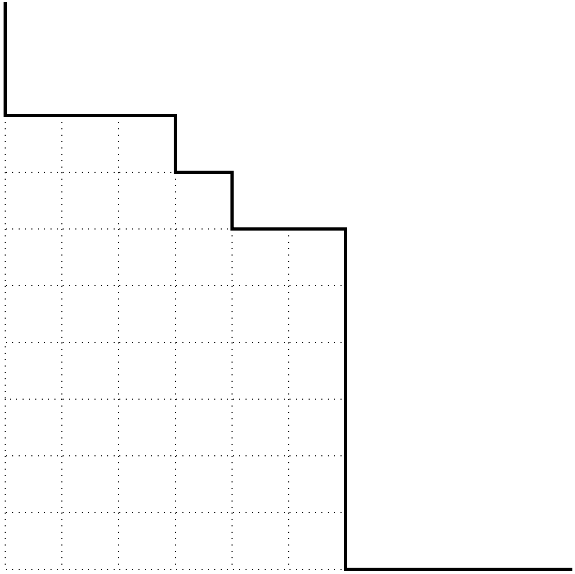

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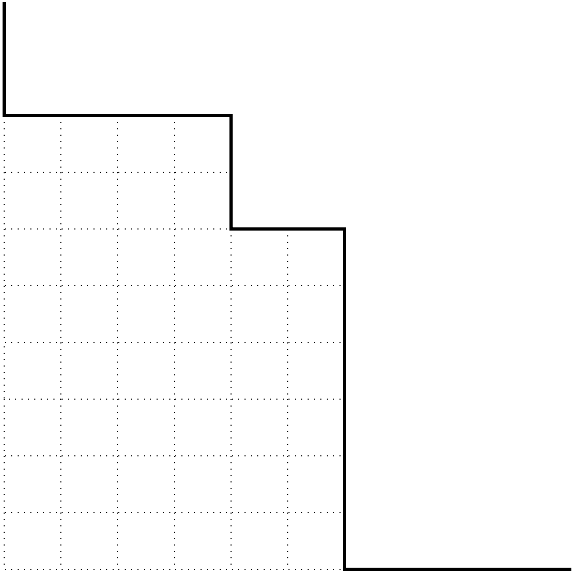

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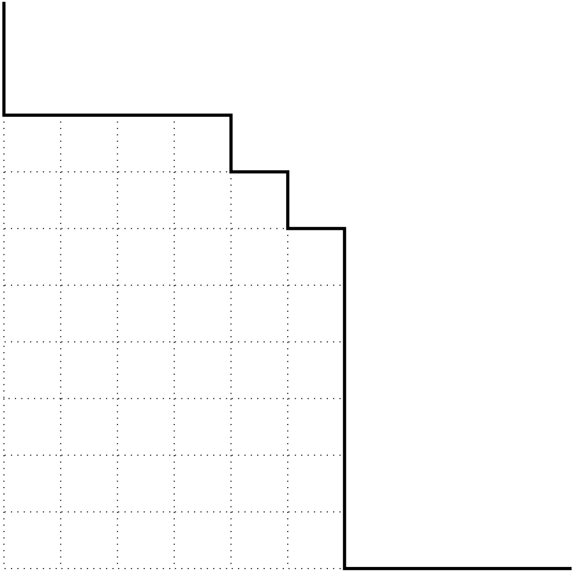

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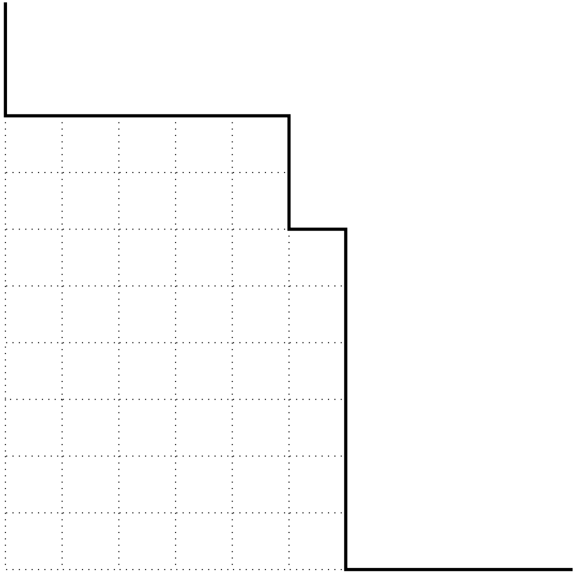

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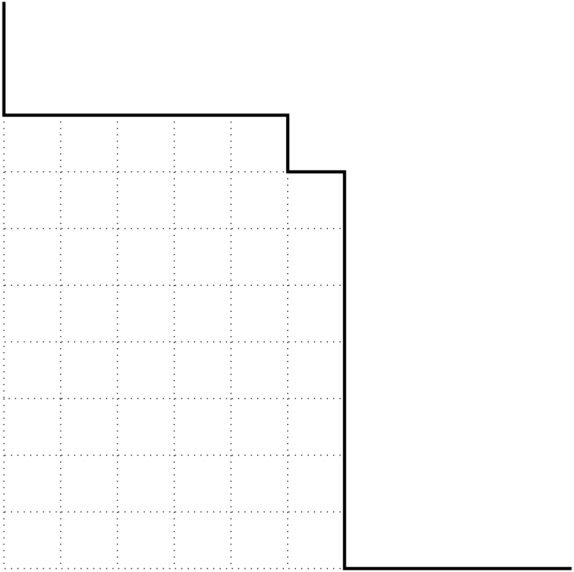

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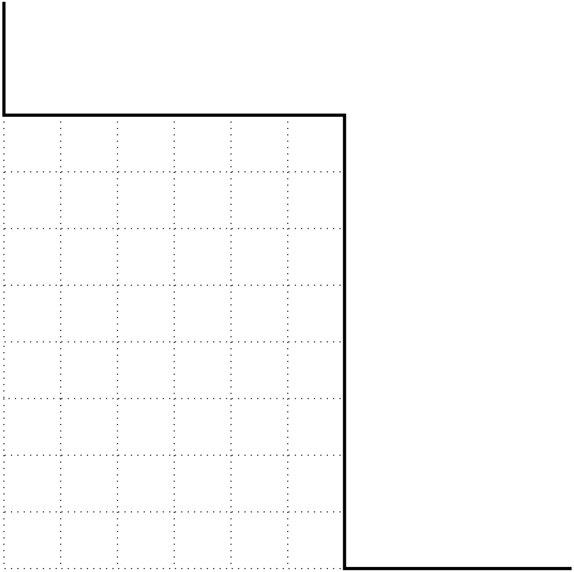

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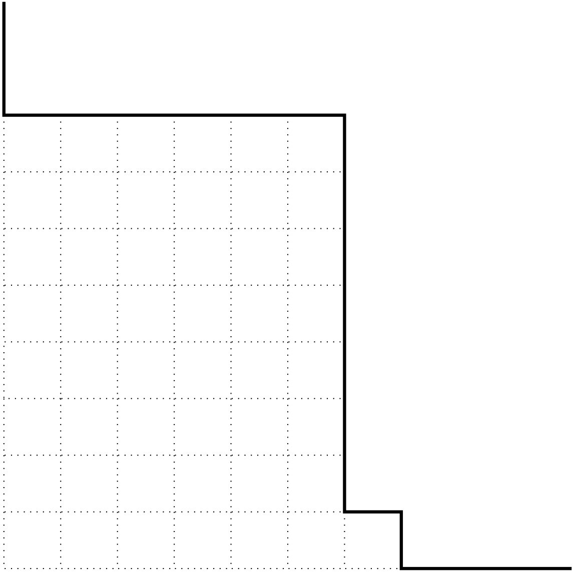

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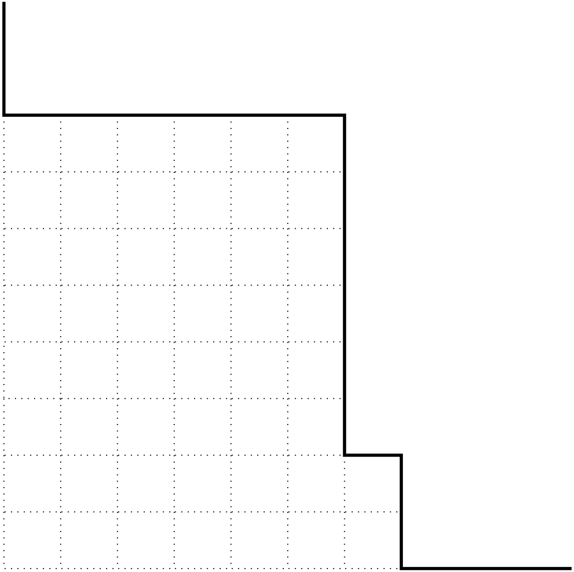

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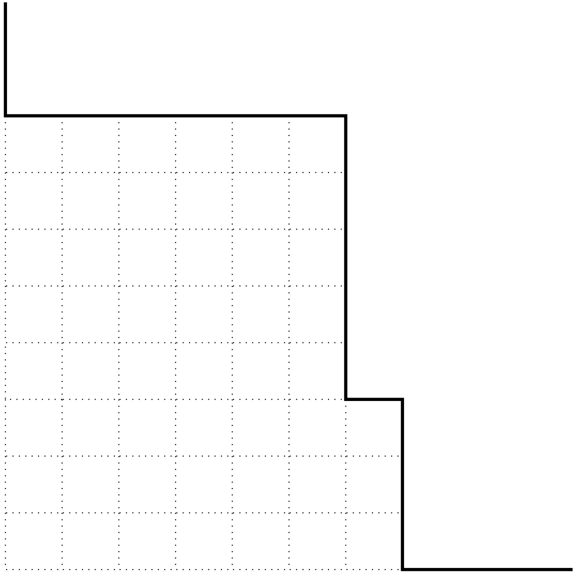

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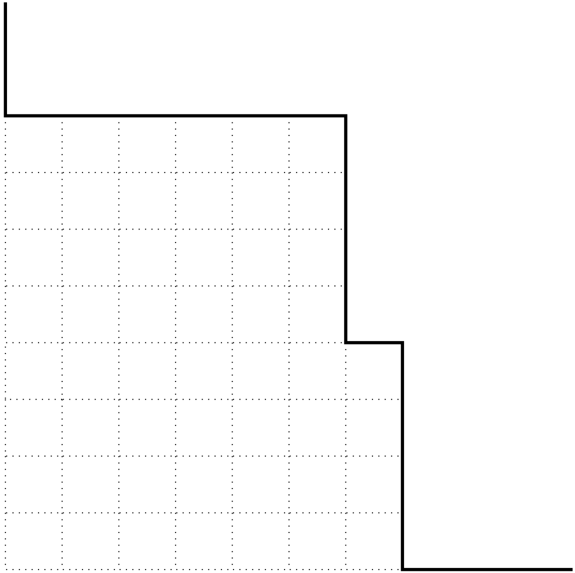

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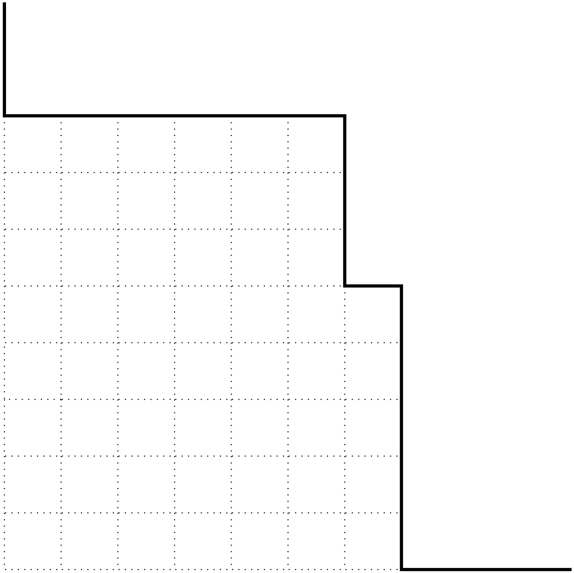

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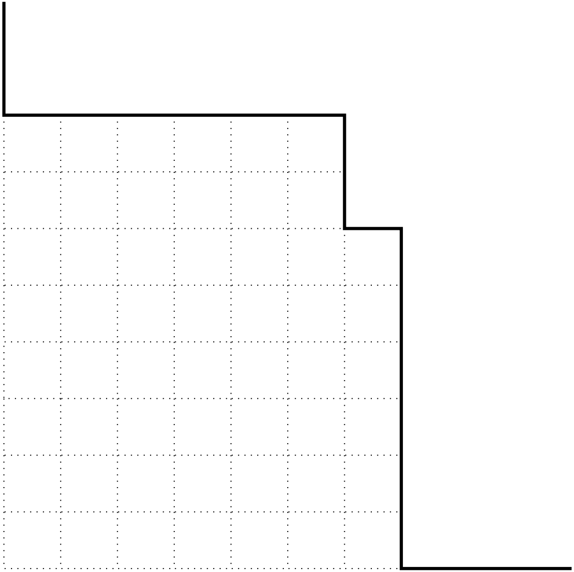

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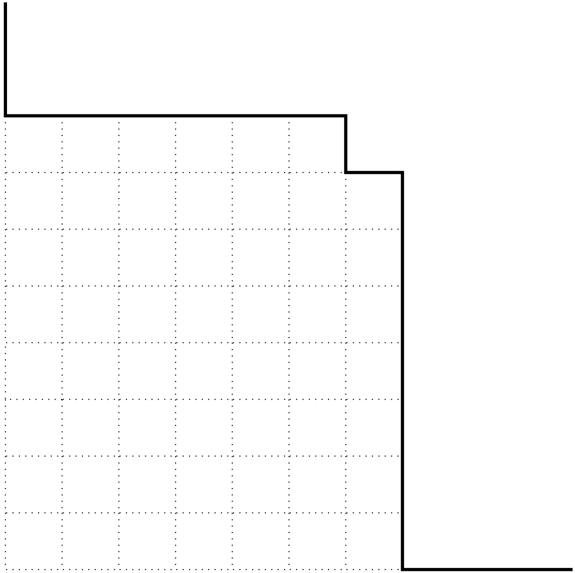
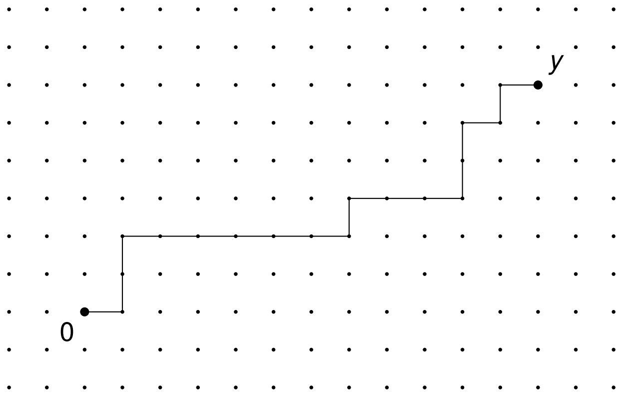


Last passage percolation

- Place i.i.d. random weights on $\mathbb{Z}^2$.
- The *geodesic* from 0 to y is the heaviest up-right path from 0 to y. Its weight is $G_{0,y}$, the time when square y becomes occupied.



Last passage percolation

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[B., T. Seppäläinen ’06]

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[B., T. Seppäläinen '06]

There is a huge literature now on limit distribution results

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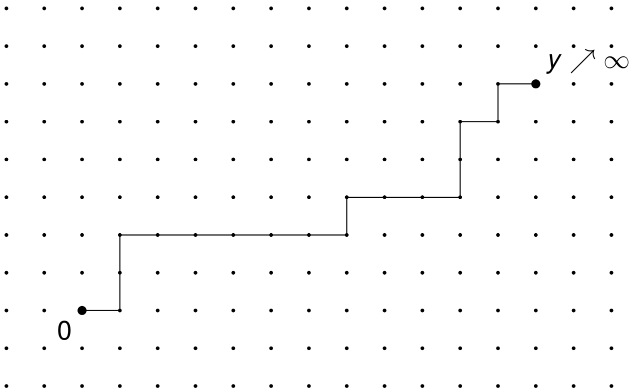
KPZ universality class.

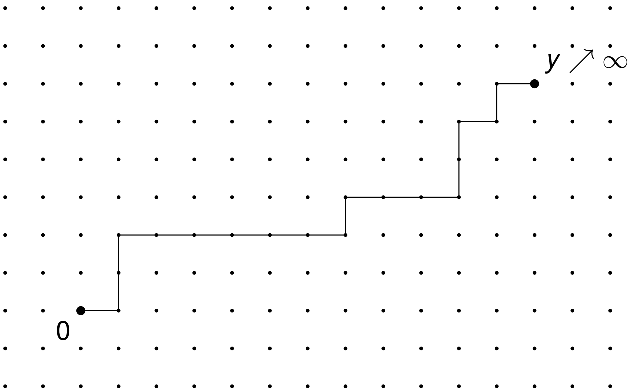
Last passage percolation

- Half-infinite geodesics exist, things stabilise [B., O. Busani, T. Seppäläinen '21]

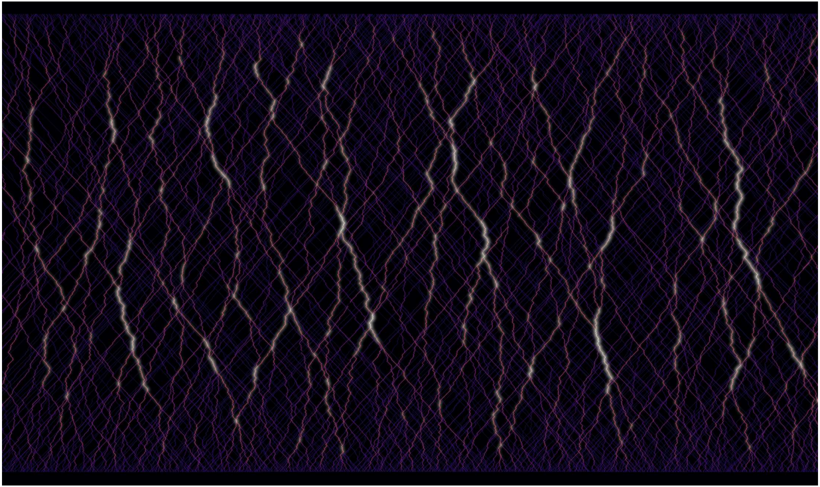


Last passage percolation

- Half-infinite geodesics exist, things stabilise [B., O. Busani, T. Seppäläinen '21]
- But there are no doubly infinite geodesics [B., O. Busani, T. Seppäläinen '20]



LPP: Road network [B., R. Basu, S. Battacharjee, D. Harper, K. Das ’23-’24+]



Simulation by David Harper

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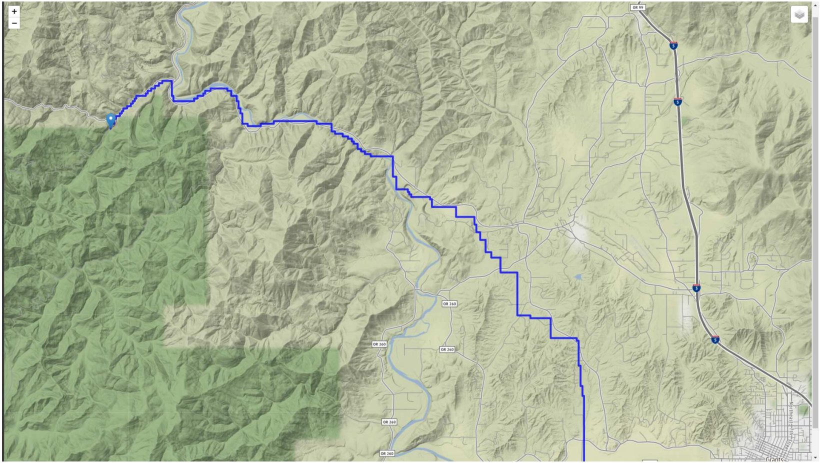

Simulation by David Harper

Conclusion

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Exclusion, its friends and relatives are absolutely **everywhere.**

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And they are pretty interesting.

Thank you.