Road layout in the KPZ class

Growing out of a project that started with
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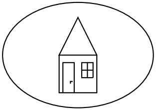
Márton Balázs

University of Bristol

King's College London
21st November 2023.

A naive Poisson model



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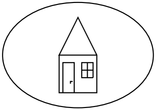

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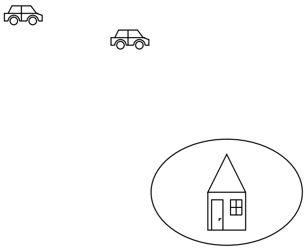

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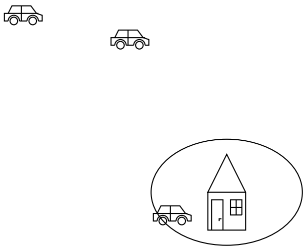

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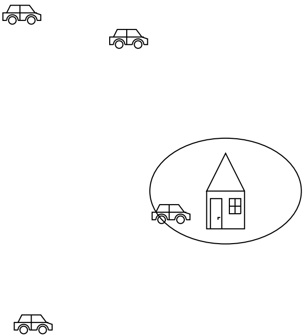

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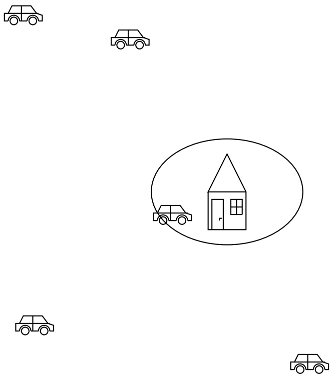

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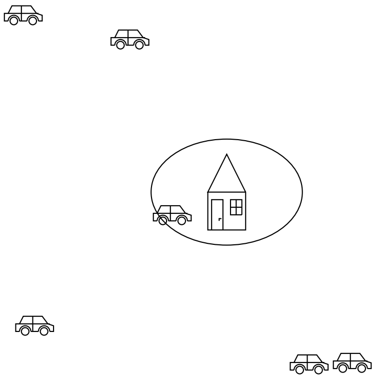

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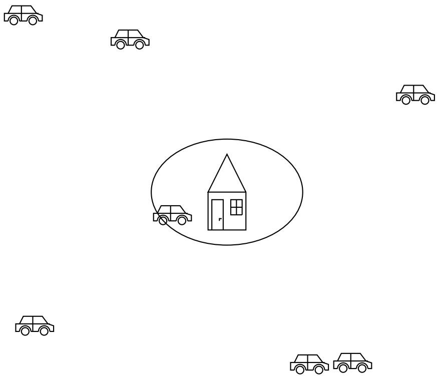

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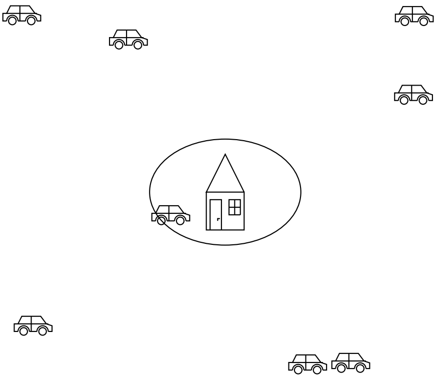

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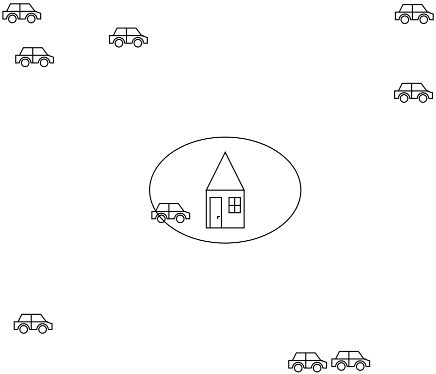

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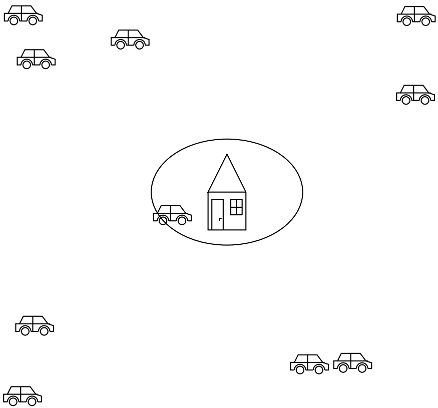

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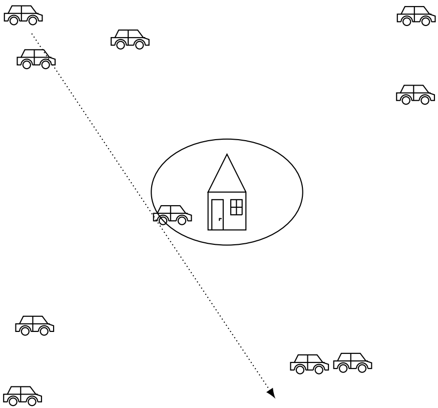

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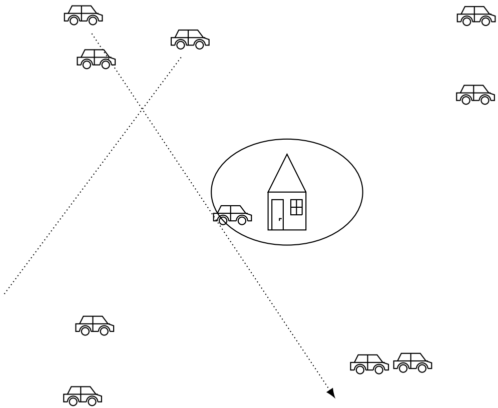

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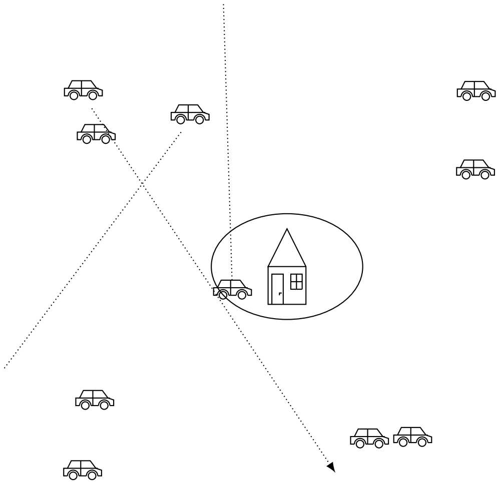

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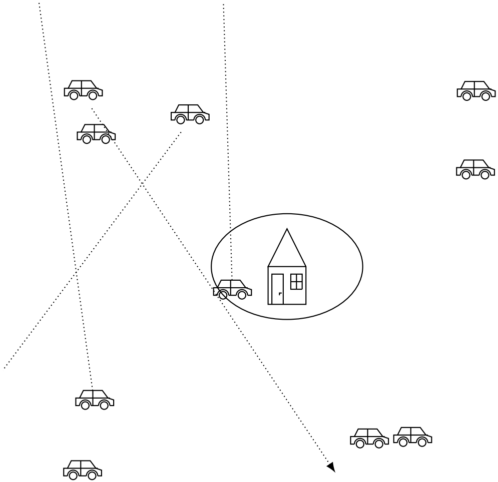

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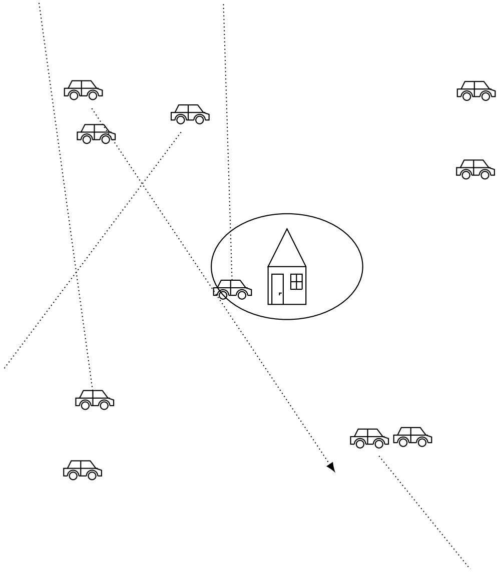

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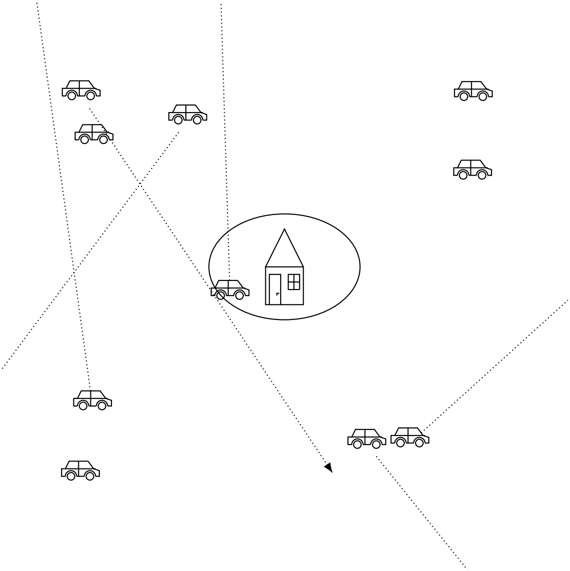

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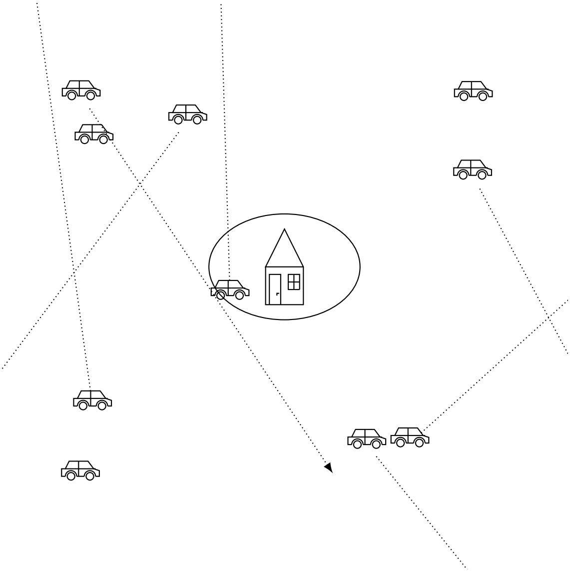

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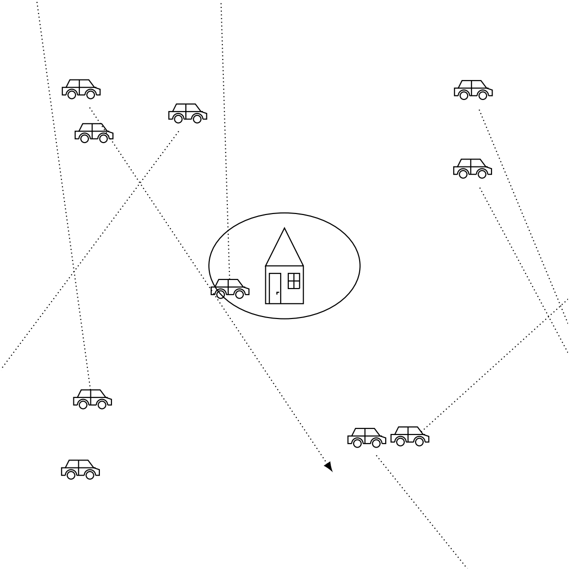

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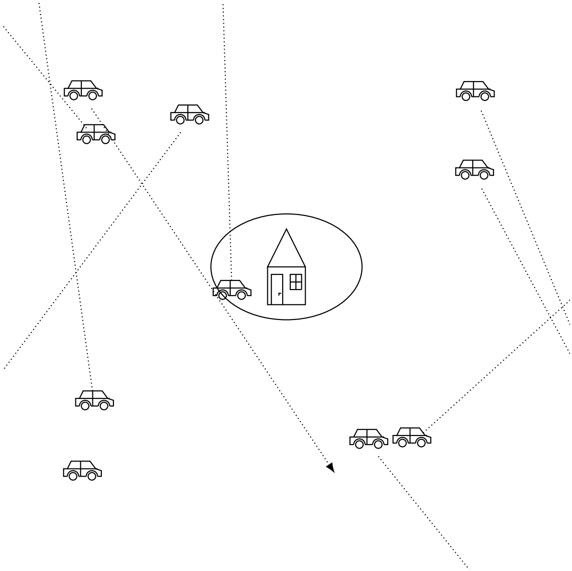

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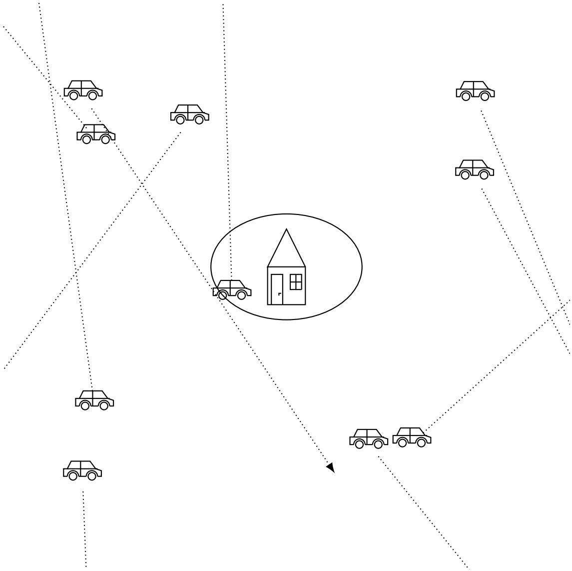

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- Unfortunately $D \gg r$...

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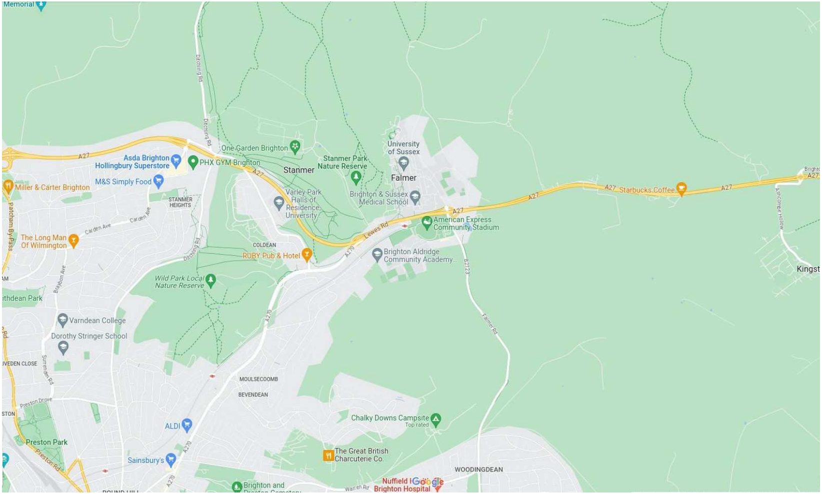

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- People who first beat a path; horse drawn carriages; road builders try to minimise obstacles. Gradients, built-up objects, etc.
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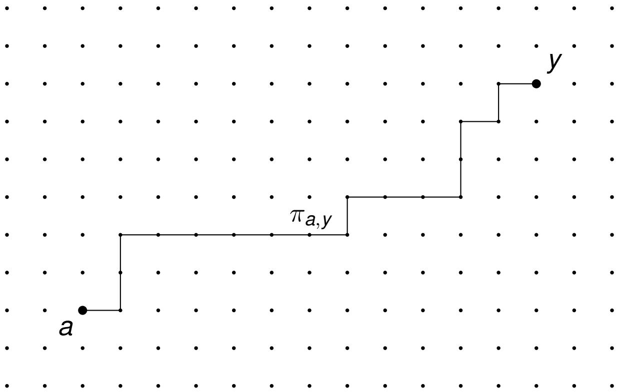
- More tools in Exponential last passage percolation (LPP). Should behave similarly to FPP.

Last passage percolation

- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{a,y}$ from a to y is the a.s. unique heaviest up-right path from a to y. Its weight is $G_{a,y}$.



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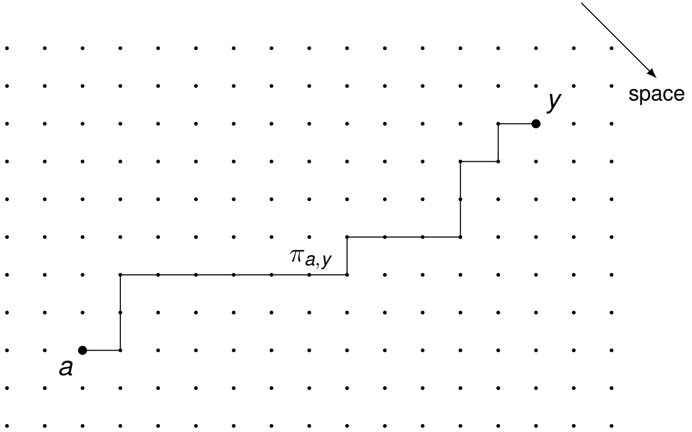

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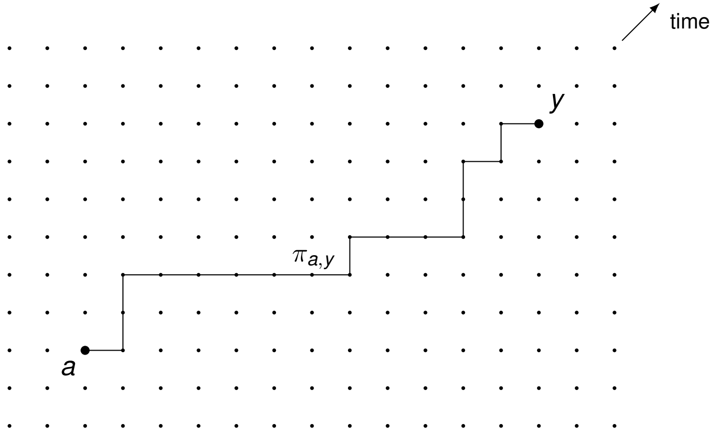
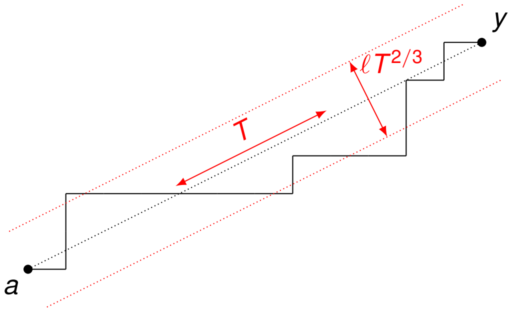


Last passage percolation: properties



$\mathbb{P}\{$geodesic exits width $\ell T^{2/3}\} \leq$ const $\cdot\, \mathrm{e}^{-C\ell^3}$ [Basu, Sarkar, Sly '19]

(KPZ transversal fluctuations).

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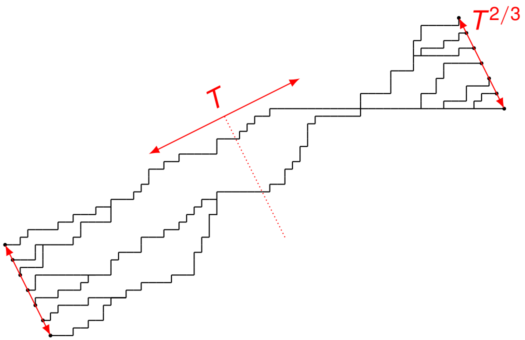

$\mathbb{P}\{\text{more than } \ell \text{ geodesics at mid-line}\} \leq \text{const} \cdot \mathrm{e}^{-C\ell^{1/128}}$

[Basu, Hoffman, Sly '22]

(Midpoint problem).

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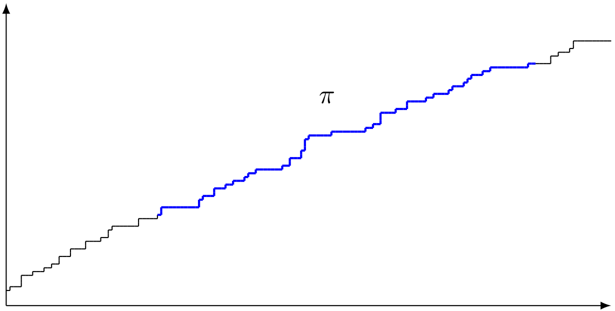

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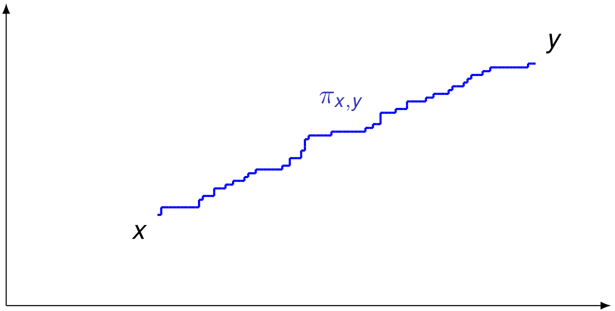

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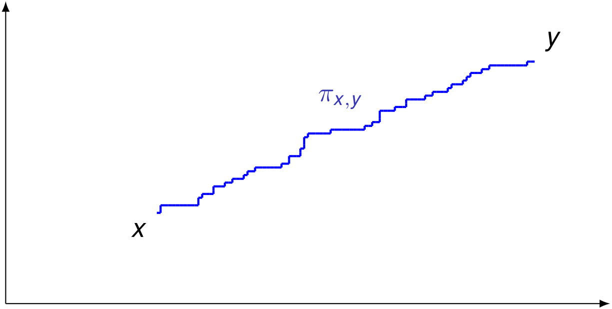
For any fixed direction this a.s. exists and is unique. [Newman (et al) '96]; [Wüthricht '02]; [Ferrari, Pimentel '05]; [Coupier '11]; [Janjigian, Rassoul-Agha, Seppäläinen '23]



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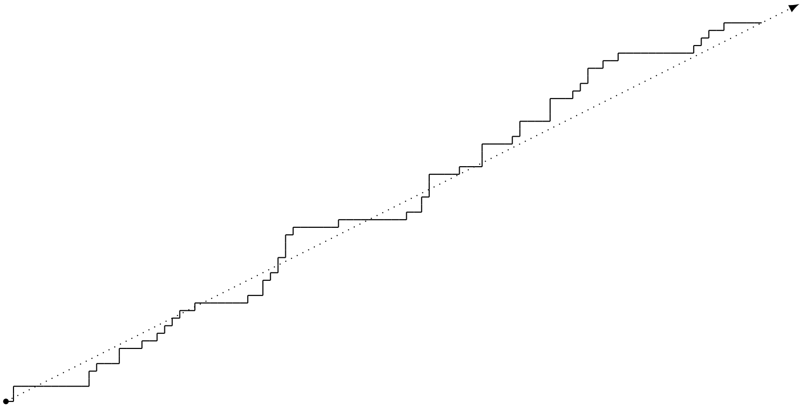
- Draw the a.s. semi-infinite geodesic for each point to its chosen direction. Many of these will coalesce when the angles are close. That's our road map with traffic data on it. A road segment is *busy* when many geodesics use that edge.

Our model



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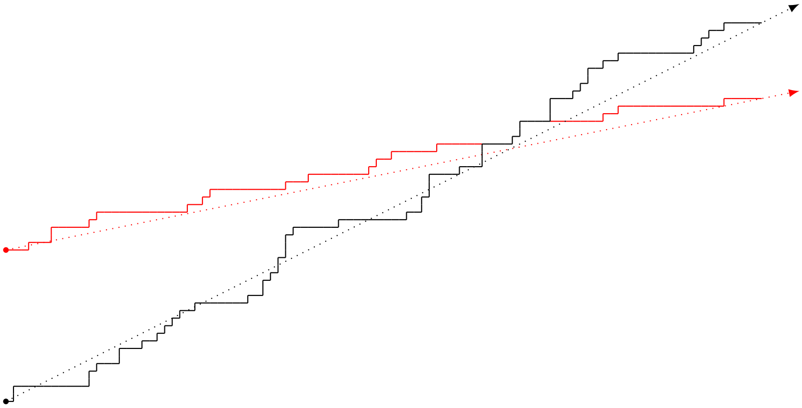

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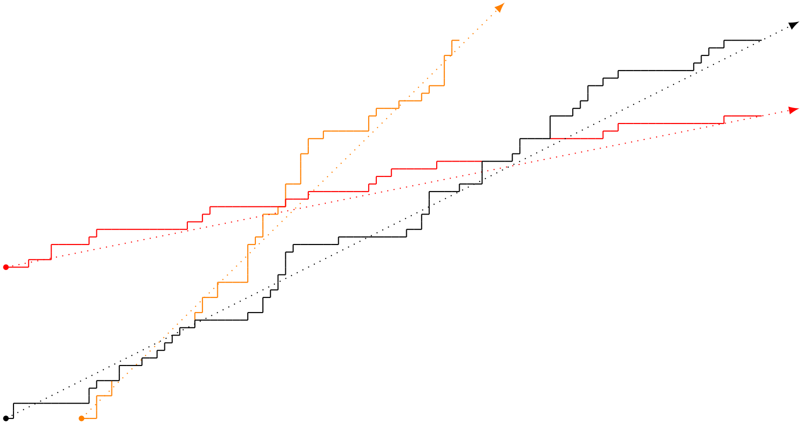

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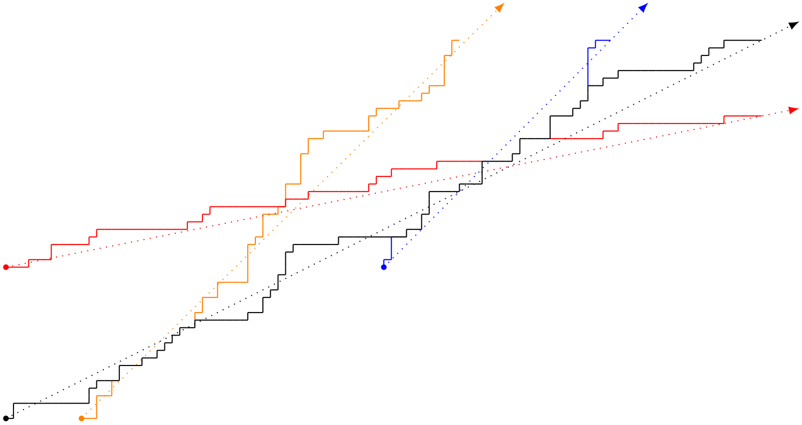

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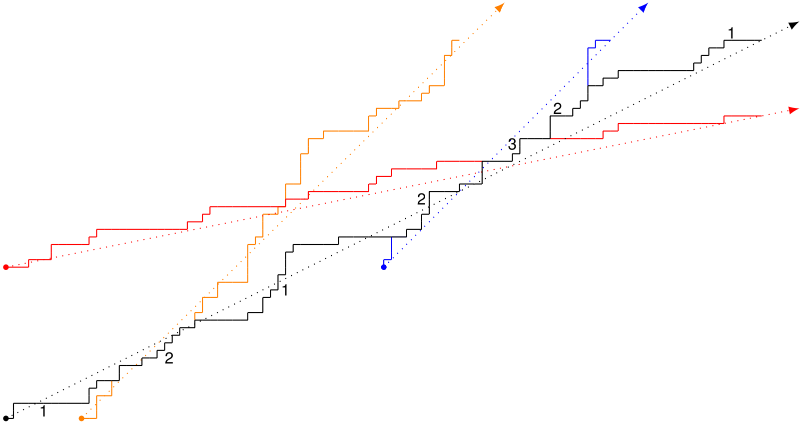

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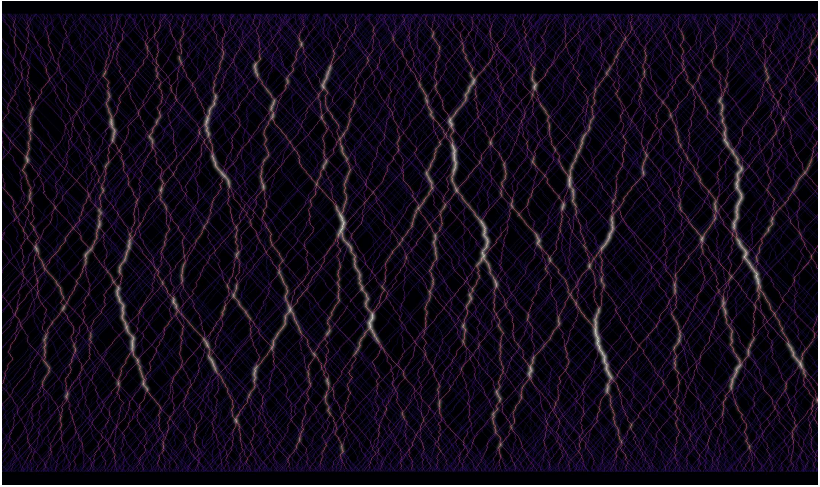

Simulation by David Harper

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- How far is the nearest busy road? I.e., within distance k in the space-direction, what is the probability to find an edge with k^α geodesics on it (and what is the interesting α)?
- Is this actually a good model of real road networks out there?

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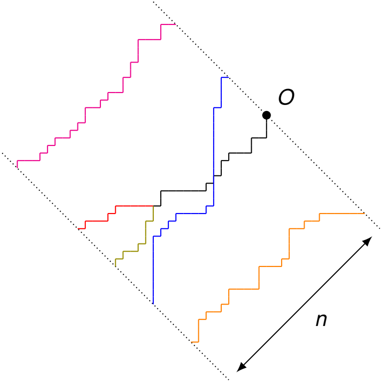
The number N_n that start from distance n and go through O is mean 1 (mass transport principle).



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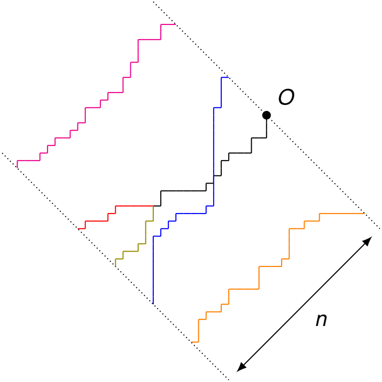

From all layers: $N = \sum_{n=1}^{\infty} N_n$ is of infinite mean.

Answers

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We don't believe the log.

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- We already know they cannot come from too far.

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Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
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Busy road close by?

With similar methods,

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$$\mathbb{P}\Big\{\textit{no road with} \geq k^4 \textit{ cars within distance } \frac{\delta k}{\log k}\Big\} \geq 1 - C\delta.$$

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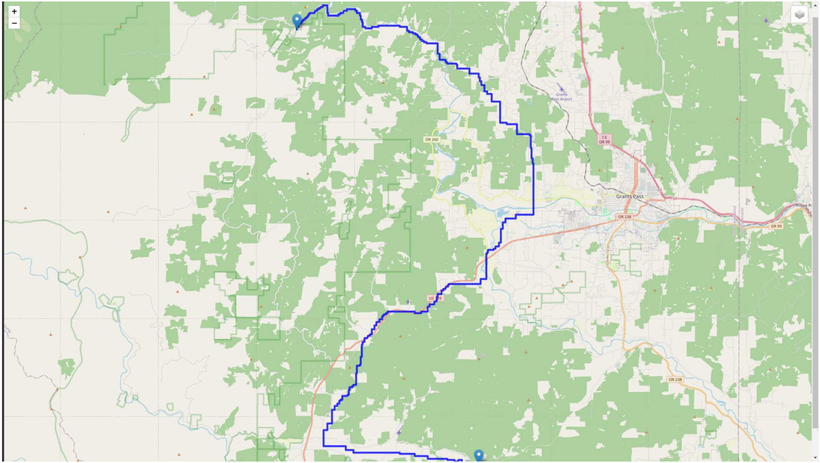
Theorem

$$\mathbb{P}\{\textit{yes, road with} \geq \textit{const} \cdot k^4 \textit{ cars within distance } k\} \geq c.$$

Is this all any good?



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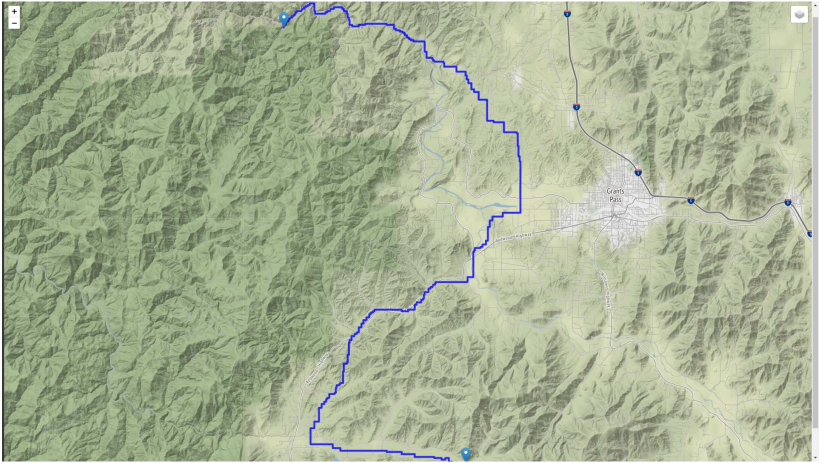

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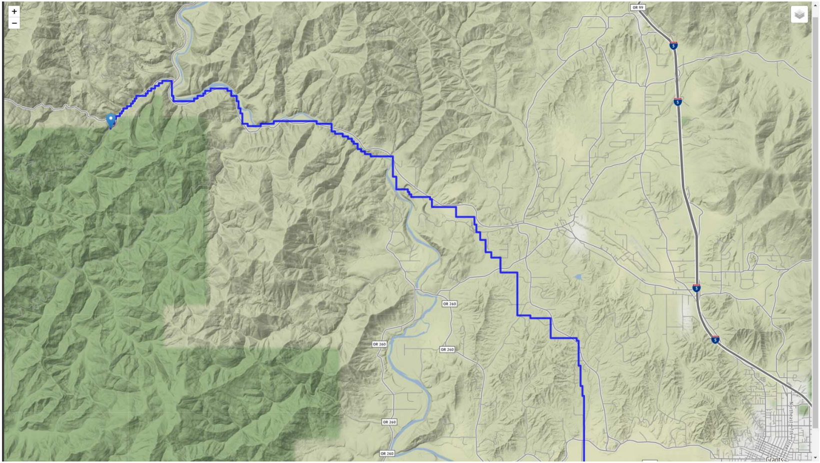

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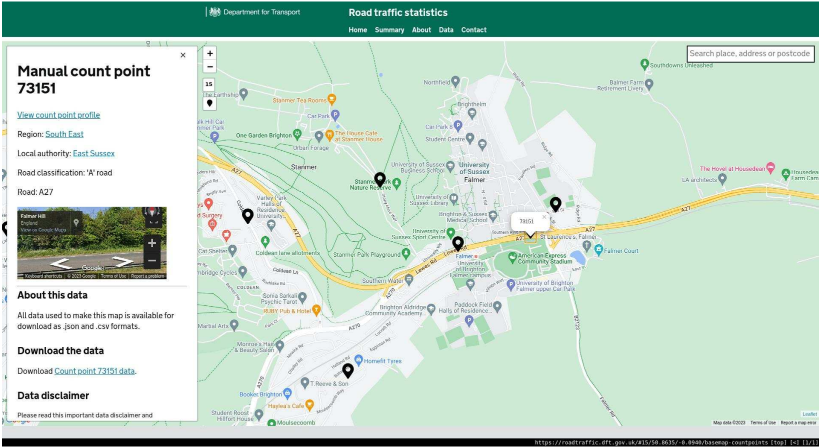

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Thank you.