Electric network for non-reversible Markov chains

Joint work with Áron Folly

Márton Balázs

University of Bristol

Laplacians, Random Walks, Bose Gas, Quantum Spin Systems

Bristol, 19th September 2014.

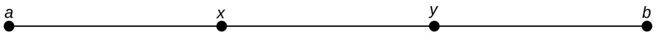
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



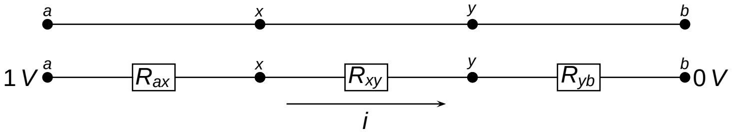
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



Electric resistor network: the voltage u is harmonic ($C = 1/R$):

$$u_x = \sum_y \frac{C_{xy}}{\sum_z C_{xz}} \cdot u_y; \qquad u_a = 1, \quad u_b = 0.$$

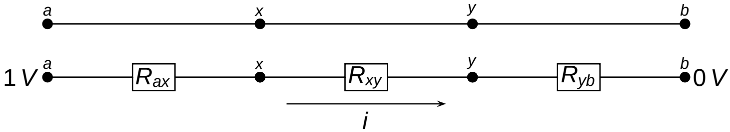
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



Electric resistor network: the voltage u is harmonic ($C = 1/R$):

$$u_x = \sum_y \frac{C_{xy}}{\sum_z C_{xz}} \cdot u_y; \qquad u_a = 1, \quad u_b = 0.$$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}}$$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} =: \frac{C_{xy}}{C_x}.$$

$P_{xy} = C_{xy}/C_x$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} =: \frac{C_{xy}}{C_x}.$$

Stationary distribtuion:

$$\mu_x = \sum_y \mu_y P_{yx} = \sum_y \mu_y \frac{C_{xy}}{C_y}$$

$$C_x = \sum_y C_y \frac{C_{xy}}{C_y}$$

$P_{xy} = C_{xy}/C_x$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} =: \frac{C_{xy}}{C_x}.$$

Stationary distribtuion:

$$\mu_x = \sum_y \mu_y P_{yx} = \sum_y \mu_y \frac{C_{xy}}{C_y}$$

$$C_x = \sum_y C_y \frac{C_{xy}}{C_y}$$

$$\rightsquigarrow \quad C_x = \mu_x.$$

$P_{xy} = C_{xy}/C_x$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} =: \frac{C_{xy}}{C_x}.$$

Stationary distribtuion:

$$\mu_x = \sum_y \mu_y P_{yx} = \sum_y \mu_y \frac{C_{xy}}{C_y}$$

$$C_x = \sum_y C_y \frac{C_{xy}}{C_y}$$

$$\rightsquigarrow \quad C_x = \mu_x.$$

$P_{xy} = C_{xy}/C_x$ $\qquad$ $C_x = \mu_x$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} =: \frac{C_{xy}}{C_x}.$$

Stationary distribtuion:

$$\mu_x = \sum_y \mu_y P_{yx} = \sum_y \mu_y \frac{C_{xy}}{C_y}$$

$$C_x = \sum_y C_y \frac{C_{xy}}{C_y}$$

$$\leadsto \quad C_x = \mu_x.$$

Notice $\mu_x P_{xy} = C_{xy} = C_{yx} = \mu_y P_{yx}$, so the chain is reversible.

$P_{xy} = C_{xy}/C_x$ $\qquad$ $C_x = \mu_x$

Reversible chains and resistors

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{C_{xy}}{C_y} n_y$$

$P_{xy} = C_{xy}/C_x$ $\qquad\qquad$ $C_x = \mu_x$

Reversible chains and resistors

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{C_{xy}}{C_y} n_y$$

$$u_x = \sum_y \frac{C_{xy}}{C_x} \cdot u_y$$

$P_{xy} = C_{xy}/C_x$ $\qquad$ $C_x = \mu_x$

Reversible chains and resistors

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{C_{xy}}{C_y} n_y$$

$$u_x = \sum_y \frac{C_{xy}}{C_x} \cdot u_y$$

$$u_x C_x = \sum_y \frac{C_{xy}}{C_y} \cdot u_y C_y$$

$P_{xy} = C_{xy}/C_x$ $\qquad$ $C_x = \mu_x$

Reversible chains and resistors

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{C_{xy}}{C_y} n_y$$

$$u_x = \sum_y \frac{C_{xy}}{C_x} \cdot u_y$$

$$u_x C_x = \sum_y \frac{C_{xy}}{C_y} \cdot u_y C_y$$

$$\rightsquigarrow \quad u_x C_x = n_x.$$

Reversible chains and resistors

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{C_{xy}}{C_y} n_y$$

$$u_x = \sum_y \frac{C_{xy}}{C_x} \cdot u_y$$

$$u_x C_x = \sum_y \frac{C_{xy}}{C_y} \cdot u_y C_y$$

$$\rightsquigarrow \quad u_x C_x = n_x.$$

$\mathbf{E}_a$(signed current $x \to y$ before absorbed in b)

$$= n_x P_{xy} - n_y P_{yx} = (u_x - u_y) C_{xy} = i_{xy}.$$ normalisation...

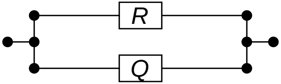
$P_{xy} = C_{xy}/C_x$ $\qquad$ $C_x = \mu_x$

Reducing a network

Series:

$$R_{\text{eff}} = R + Q$$

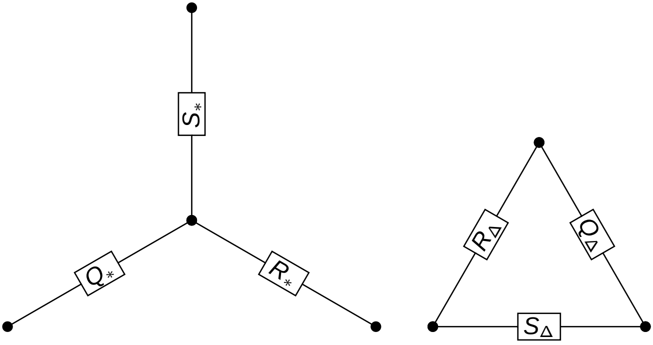
Parallel:



$$\frac{1}{R_{\text{eff}}} = \frac{1}{R} + \frac{1}{Q}$$

Reducing a network

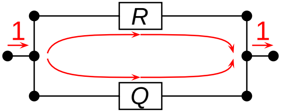
Star-Delta:



$$R_* = \frac{Q_\Delta S_\Delta}{R_\Delta + Q_\Delta + S_\Delta}, \qquad R_\Delta = \frac{R_* Q_* + R_* S_* + Q_* S_*}{R_*}.$$

Thomson, Dirichlet principles

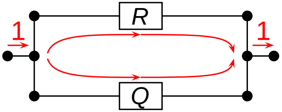
Thomson principle:



The physical unit current is the unit flow that minimizes the sum of the ohmic power losses $\sum i^2 R$.

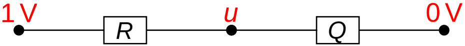
Thomson, Dirichlet principles

Thomson principle:



The physical unit current is the unit flow that minimizes the sum of the ohmic power losses $\sum i^2 R$.

Dirichlet principle:



The physical voltage is the function that minimizes the ohmic power losses $\sum (\nabla u)^2 / R$.

Monotonicity, transience, recurrence

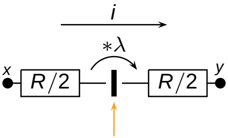
The monotonicity property:
Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

Monotonicity, transience, recurrence

The monotonicity property:
Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

$\rightsquigarrow$ can be used to prove transience-recurrence by reducing the graph to something manageable in terms of resistor networks.

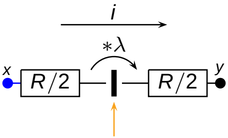
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

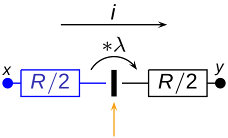
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$\left(u_x - i \cdot \frac{R}{2}\right) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

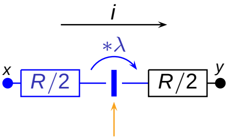
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

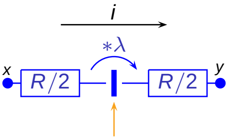
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

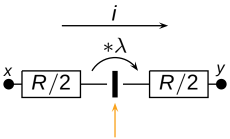
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

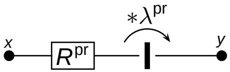
The part



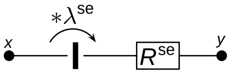
Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

Equivalent:

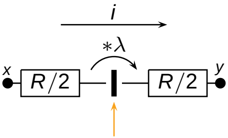


$$(u_x - i \cdot R^{\text{pr}}) \cdot \lambda^{\text{pr}} = u_y$$



$$u_x \cdot \lambda^{\text{se}} - R^{\text{se}} \cdot i = u_y$$

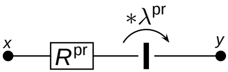
The part



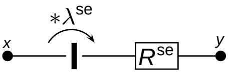
Voltage amplifier: keeps the current, multiplies the potential.

$$\left(u_x - i \cdot \frac{R}{2}\right) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

Equivalent:

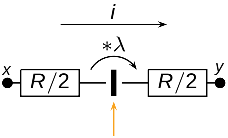


$$(u_x - i \cdot R^{\mathrm{pr}}) \cdot \lambda^{\mathrm{pr}} = u_y$$
$$\lambda^{\mathrm{pr}} = \lambda$$



$$u_x \cdot \lambda^{\mathrm{se}} - R^{\mathrm{se}} \cdot i = u_y$$
$$\lambda^{\mathrm{se}} = \lambda$$

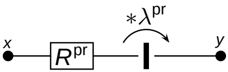
The part



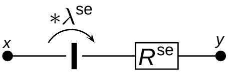
Voltage amplifier: keeps the current, multiplies the potential.

$$\left(u_x - i \cdot \frac{R}{2}\right) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

Equivalent:

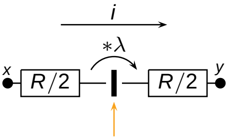


$$(u_x - i \cdot R^{\mathrm{pr}}) \cdot \lambda^{\mathrm{pr}} = u_y$$
$$\lambda^{\mathrm{pr}} = \lambda$$
$$R^{\mathrm{pr}} = \frac{\lambda+1}{2\lambda} \cdot R$$



$$u_x \cdot \lambda^{\mathrm{se}} - R^{\mathrm{se}} \cdot i = u_y$$
$$\lambda^{\mathrm{se}} = \lambda$$
$$R^{\mathrm{se}} = \frac{\lambda+1}{2} \cdot R$$

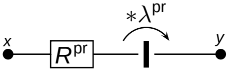
The part



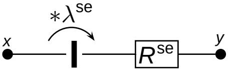
Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

Equivalent:



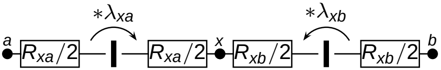
$$(u_x - i \cdot R^{\mathrm{pr}}) \cdot \lambda^{\mathrm{pr}} = u_y$$
$$\lambda^{\mathrm{pr}} = \lambda$$
$$R^{\mathrm{pr}} = \frac{\lambda+1}{2\lambda} \cdot R$$



$$u_x \cdot \lambda^{\mathrm{se}} - R^{\mathrm{se}} \cdot i = u_y$$
$$\lambda^{\mathrm{se}} = \lambda$$
$$R^{\mathrm{se}} = \frac{\lambda+1}{2} \cdot R$$

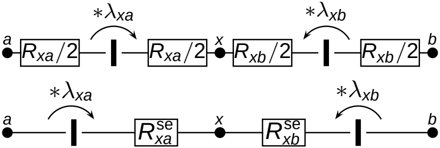
$$R^{\mathrm{pr}} = \frac{\lambda+1}{2\lambda} \cdot R \qquad R^{\mathrm{se}} = \frac{\lambda+1}{2} \cdot R$$

Harmonicity



$R^{\text{pr}} = \frac{\lambda+1}{2\lambda} \cdot R$ $R^{\text{se}} = \frac{\lambda+1}{2} \cdot R$

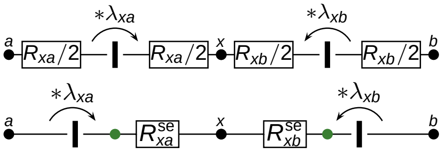
Harmonicity



$R^{\mathrm{pr}} = \frac{\lambda+1}{2\lambda} \cdot R$

$R^{\mathrm{se}} = \frac{\lambda+1}{2} \cdot R$

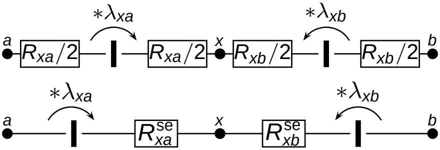
Harmonicity



$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y$$

$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$ $R^{se} = \frac{\lambda+1}{2} \cdot R$

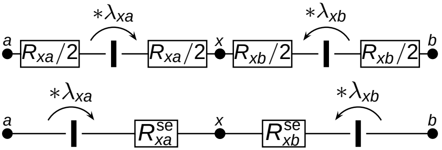
Harmonicity



$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y$$

$R^{\mathrm{pr}} = \frac{\lambda+1}{2\lambda} \cdot R$ $\qquad$ $R^{\mathrm{se}} = \frac{\lambda+1}{2} \cdot R$

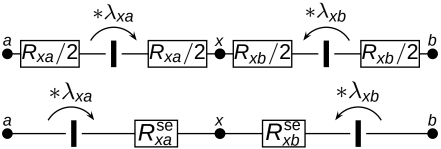
Harmonicity



$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy}\,\frac{2}{\lambda_{xy}+1}}{\sum_z C_{xz}\,\frac{2}{\lambda_{xz}+1}} \cdot \lambda_{xy} u_y$$

$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$ $R^{se} = \frac{\lambda+1}{2} \cdot R$

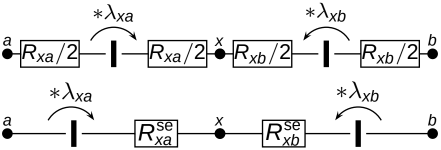
Harmonicity



$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy}\,\frac{2}{\lambda_{xy}+1}}{\sum_z C_{xz}\,\frac{2}{\lambda_{xz}+1}} \cdot \lambda_{xy} u_y$$

$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$ $\qquad$ $R^{se} = \frac{\lambda+1}{2} \cdot R$

Harmonicity



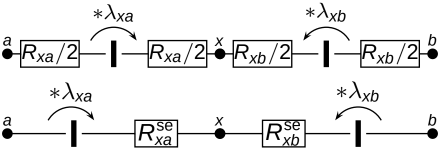
$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy} \frac{2}{\lambda_{xy}+1}}{\sum_z C_{xz} \frac{2}{\lambda_{xz}+1}} \cdot \lambda_{xy} u_y$$

$$= \sum_y \frac{D_{xy}\gamma_{xy}}{\sum_z D_{xz}\gamma_{zx}} \cdot u_y$$

with $\gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}}$, $\qquad D_{xy} = \frac{2\gamma_{xy}C_{xy}}{(\lambda_{xy}+1)} = D_{yx}$.

$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$ $\qquad$ $R^{se} = \frac{\lambda+1}{2} \cdot R$

Harmonicity



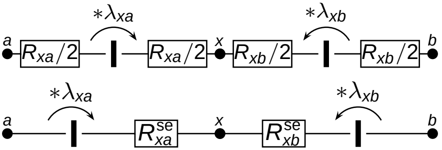
$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy}\,\frac{2}{\lambda_{xy}+1}}{\sum_z C_{xz}\,\frac{2}{\lambda_{xz}+1}} \cdot \lambda_{xy} u_y$$

$$= \sum_y \frac{D_{xy}\gamma_{xy}}{\sum_z D_{xz}\gamma_{zx}} \cdot u_y$$

with $\gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}}$, $\qquad D_{xy} = \frac{2\gamma_{xy} C_{xy}}{(\lambda_{xy}+1)} = D_{yx}$.

$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R \qquad\qquad R^{se} = \frac{\lambda+1}{2} \cdot R$

Harmonicity



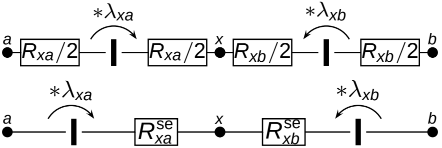
$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy} \frac{2}{\lambda_{xy}+1}}{\sum_z C_{xz} \frac{2}{\lambda_{xz}+1}} \cdot \lambda_{xy} u_y$$

$$= \sum_y \frac{D_{xy}\gamma_{xy}}{\sum_z D_{xz}\gamma_{zx}} \cdot u_y = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$$

with $\gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}}$, $\qquad D_{xy} = \frac{2\gamma_{xy} C_{xy}}{(\lambda_{xy}+1)} = D_{yx}$.

$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R \qquad\qquad R^{se} = \frac{\lambda+1}{2} \cdot R$

Harmonicity



$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy}\frac{2}{\lambda_{xy}+1}}{\sum_z C_{xz}\frac{2}{\lambda_{xz}+1}} \cdot \lambda_{xy} u_y$$

$$= \sum_y \frac{D_{xy}\gamma_{xy}}{\sum_z D_{xz}\gamma_{zx}} \cdot u_y = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$$

with $\gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}}$, $\qquad D_{xy} = \frac{2\gamma_{xy}C_{xy}}{(\lambda_{xy}+1)} = D_{yx}$.

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \qquad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y, \qquad u_a = 1, \quad u_b = 0.$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \qquad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy} C_{xy} / (\lambda_{xy} + 1)$$

From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y, \qquad u_a = 1, \quad u_b = 0.$$

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}.$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \qquad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y, \qquad u_a = 1, \quad u_b = 0.$$

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}.$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \qquad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From network to chain

Stationary distribtuion:

$$\mu_x = \sum_z \mu_z P_{zx} = \sum_z \mu_z \frac{D_{zx}\gamma_{zx}}{D_z}$$

$$D_x = \sum_z D_z \frac{D_{zx}\gamma_{zx}}{D_z}$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \qquad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy} C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From network to chain

Stationary distribtuion:

$$\mu_x = \sum_z \mu_z P_{zx} = \sum_z \mu_z \frac{D_{zx}\gamma_{zx}}{D_z}$$

$$D_x = \sum_z D_z \frac{D_{zx}\gamma_{zx}}{D_z}$$

$$\rightsquigarrow \quad D_x = \mu_x .$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \qquad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy} C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

"Markovian" property

$$u_x = \sum_z P_{xz} u_z; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} \qquad\qquad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

"Markovian" property

$$u_x = \sum_z P_{xz} u_z; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

$$\sum_z P_{xz} = \sum_z \frac{D_{xz}\gamma_{xz}}{D_x} = 1$$

$$D_x := \sum_z D_{xz}\gamma_{zx}$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy} C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

"Markovian" property

$$u_x = \sum_z P_{xz} u_z; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

$$\sum_z P_{xz} = \sum_z \frac{D_{xz}\gamma_{xz}}{D_x} = 1$$

$$D_x := \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz}.$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} \qquad D_{xy} = 2\gamma_{xy} C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

"Markovian" property

$$u_x = \sum_z P_{xz} u_z; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

$$\sum_z P_{xz} = \sum_z \frac{D_{xz}\gamma_{xz}}{D_x} = 1$$

$$D_x := \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz}.$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy} C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From chain to network

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x} = \frac{D_{xy}\gamma_{xy}}{\mu_x}$$

$$\mu_x P_{xy} \cdot \mu_y P_{yx} = D_{xy}^2;$$

$$\frac{\mu_x P_{xy}}{\mu_y P_{yx}} = \gamma_{xy}^2 = \lambda_{xy}.$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From chain to network

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x} = \frac{D_{xy}\gamma_{xy}}{\mu_x}$$

$$\mu_x P_{xy} \cdot \mu_y P_{yx} = D_{xy}^2;$$

$$\frac{\mu_x P_{xy}}{\mu_y P_{yx}} = \gamma_{xy}^2 = \lambda_{xy}.$$

Reversed chain: Replace P_{xy} by $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$.

$\rightsquigarrow$ D_{xy} stays, λ_{xy} reverses to λ_{yx}.

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From chain to network

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{D_{yx}\gamma_{yx}}{D_y} n_y$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From chain to network

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{D_{yx}\gamma_{yx}}{D_y} n_y$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$$

$\gamma_{xy} = \sqrt{\lambda_{xy}}$ $\quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz}$ $\quad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$

$P_{xy} = D_{xy}\gamma_{xy}/D_x$

From chain to network

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{D_{yx}\gamma_{yx}}{D_y} n_y$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$$

$$u_x D_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_y} \cdot u_y D_y$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From chain to network

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$n_x = \sum_y n_y P_{yx} = \sum_y \frac{D_{yx}\gamma_{yx}}{D_y} n_y$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$$

$$u_x D_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_y} \cdot u_y D_y$$

$$\rightsquigarrow \quad \hat{u}_x D_x = n_x$$

in the reversed chain.

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

From chain to network

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

$$\rightsquigarrow \quad \hat{u}_x D_x = n_x$$

in the reversed chain.

$$\mathbf{E}_a(\text{signed current } x \to y \text{ before absorbed in } b)$$
$$= n_x P_{xy} - n_y P_{yx} = (\hat{u}_x \gamma_{xy} - \hat{u}_y \gamma_{yx}) D_{xy} = \hat{i}_{xy}. \quad \text{normalisation...}$$

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy} C_{xy}/(\lambda_{xy}+1)$$
$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

$\leadsto$The "Markovian" property has another solution: constant u_b potentials with zero external currents.

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

$\rightsquigarrow$The ”Markovian” property has another solution: constant u_b potentials with zero external currents.

$\rightsquigarrow$ The difference of these two: $\{u_x - u_b\}_{x\in\Omega}$ is a solution too, with i_a flowing in the network.

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

$\rightsquigarrow$The "Markovian" property has another solution: constant u_b potentials with zero external currents.

$\rightsquigarrow$ The difference of these two: $\{u_x - u_b\}_{x\in\Omega}$ is a solution too, with i_a flowing in the network.

$\rightsquigarrow$ Going backwards from $u_b - u_b = 0$ at b, all currents and potentials are proportional to $u_a - u_b$ at a.

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

$\rightsquigarrow$The "Markovian" property has another solution: constant u_b potentials with zero external currents.

$\rightsquigarrow$ The difference of these two: $\{u_x - u_b\}_{x\in\Omega}$ is a solution too, with i_a flowing in the network.

$\rightsquigarrow$ Going backwards from $u_b - u_b = 0$ at b, all currents and potentials are proportional to $u_a - u_b$ at a.

$\rightsquigarrow$ In particular, i_a is proportional to $u_a - u_b$. We have effective resistance.

What works

... the analogy with $\mathbf{P}\{\tau_a < \tau_b\}$.

What works

... the analogy with $\mathbf{P}\{\tau_a < \tau_b\}$.

Modulo normalisation...

$$\mathbf{E}_a(\text{signed current } x \to y \text{ before absorbed in } b) = \hat{i}_{xy}.$$

in the reversed network!

What works

... the analogy with $\mathbf{P}\{\tau_a < \tau_b\}$.

Modulo normalisation. . .

$$\mathbf{E}_a(\text{signed current } x \to y \text{ before absorbed in } b) = \hat{i}_{xy}.$$

in the reversed network!

Theorem (Chandra, Raghavan, Ruzzo, Smolensky and Tiwari '96 for reversible)

Commute time $= R_{\mathit{eff}} \cdot$ *all conductances.*

What works

For all sets A, B, capacity$\sim$escape probability.

$$\operatorname{cap}(A, B) = C^{\text{eff}}_{AB} = \frac{1}{R^{\text{eff}}_{AB}}$$

What works

For all sets A, B, capacity$\sim$escape probability.

$$\mathrm{cap}(A, B) = C_{AB}^{\mathrm{eff}} = \frac{1}{R_{AB}^{\mathrm{eff}}} = \frac{1}{2} \sum_{x \sim y \in V} C_{xy} (u_x - u_y)^2.$$

This is non-physical!

What works

For all sets A, B, capacity$\sim$escape probability.

$$\mathrm{cap}(A, B) = C^{\mathrm{eff}}_{AB} = \frac{1}{R^{\mathrm{eff}}_{AB}} = \frac{1}{2} \sum_{x \sim y \in V} C_{xy} (u_x - u_y)^2 .$$

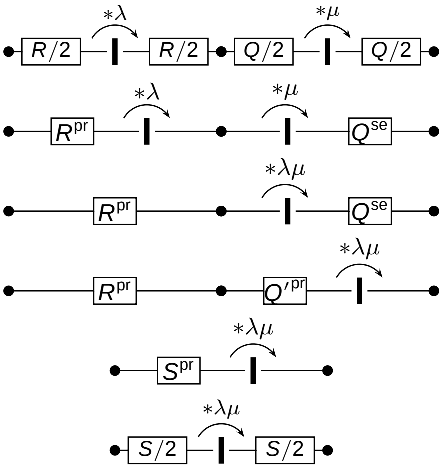
This is non-physical!

In particular, symmetrising the chain ($P_{xy} \to \frac{P_{xy}+\hat{P}_{xy}}{2}$) cannot increase escape probabilities:

- symmetrising leaves C_{xy} unchanged;
- the above sum is minimised by the symmetric voltages, not $\{u_x\}$ (Classical Dirichlet principle).

The electric network

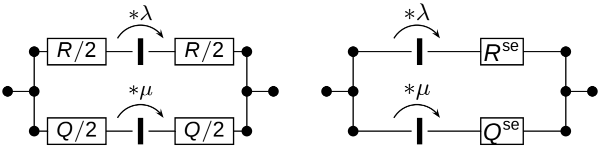
Series:



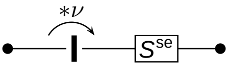
$$S = R\frac{(\lambda+1)\mu}{\lambda\mu+1} + Q\frac{\mu+1}{\lambda\mu+1}.$$

The electric network

Parallel:



Compare this with



$$S = \frac{RQ}{R+Q}$$

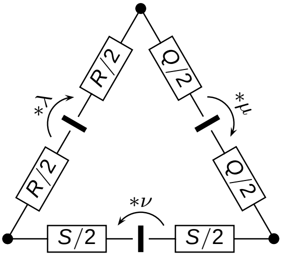
$$\nu = \frac{Q\lambda(\mu+1)+R\mu(\lambda+1)}{Q(\mu+1)+R(\lambda+1)}.$$

The electric network

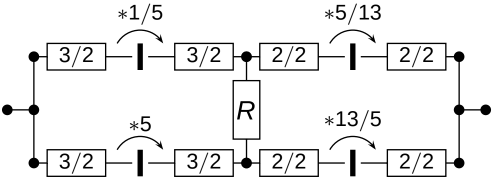
Star-Delta:

Star to Delta works,

Delta to Star only works if Delta does not produce a circular current by itself ($\lambda\mu\nu = 1$).

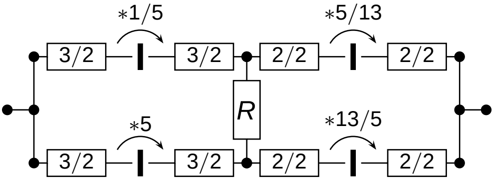


Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}.$$

Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}. \qquad ☹$$

Dirichlet principle

Classical case:

Dirichlet principle

Classical case:

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\mathrm{Ohm}}(i_u) = \sum_{x \sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$C^{\text{eff}}_{ab} = \qquad\qquad E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)^2_{xy} \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y} \big(i_u^* - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$C_{ab}^{\text{eff}} = \qquad E_{\text{Ohm}}(i_u^* - \Psi),$$

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y} \big(i_u^* - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$C_{ab}^{\text{eff}} = \min_{\Psi:\text{ flow}} E_{\text{Ohm}}(i_u^* - \Psi),$$

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y} \big(i_u^* - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} \; \min_{\Psi:\,\text{flow}} E_{\text{Ohm}}(i_u^* - \Psi),$$

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y} \big(i_u^* - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy}\cdot\big(u(x)-u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y}(i_u)_{xy}^2\cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$C_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0}\ \min_{\Psi:\text{ flow}} E_{\text{Ohm}}(i_u^* - \Psi),$$

$$(i_u^*)_{xy} = D_{xy}\cdot\big(\gamma_{xy}u(x)-\gamma_{yx}u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y}\big(i_u^* - \Psi_{xy}\big)^2\cdot R_{xy}.$$

Thank you.

A trivial statement

Theorem (Well Known Theorem)

A Markov chain is reversible if and only if for every closed cycle $x_0, x_1, x_2, \ldots, x_n = x_0$ in Ω we have

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}.$$

In particular, any Markov chain on a finite connected tree G is necessarily reversible.

A trivial statement

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}$$

$$\gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} = \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0} \quad \text{, or}$$

$$\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1.$$

- Total multiplication factor along any loop is one.

A trivial statement

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}$$

$$\gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} = \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0} \qquad \text{, or}$$

$$\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1.$$

- Total multiplication factor along any loop is one.
- Zero current and free vertices is a solution.

A trivial statement

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}$$

$$\gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} = \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0} \qquad \text{, or}$$

$$\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1.$$

- Total multiplication factor along any loop is one.
- Zero current and free vertices is a solution.
- It's the only solution.

A trivial statement

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}$$

$$\gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} = \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0} \quad \text{, or}$$

$$\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1.$$

- Total multiplication factor along any loop is one.
- Zero current and free vertices is a solution.
- It's the only solution.
- The network is "Markovian": potential is constant.

A trivial statement

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}$$

$$\gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} = \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0} \quad \text{, or}$$

$$\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1.$$

- Total multiplication factor along any loop is one.
- Zero current and free vertices is a solution.
- It's the only solution.
- The network is ”Markovian”: potential is constant.
- All λ's are 1, and the chain is reversible.

A trivial statement

Electrical proof.

Repeat for trees:

- There are no loops.

□

A trivial statement

Electrical proof.

Repeat for trees:

- There are no loops.
- Zero current and free vertices is a solution.

□

A trivial statement

Electrical proof.

Repeat for trees:

- There are no loops.
- Zero current and free vertices is a solution.
- It's the only solution.

□

A trivial statement

Electrical proof.

Repeat for trees:

- There are no loops.
- Zero current and free vertices is a solution.
- It's the only solution.
- The network is "Markovian": potential is constant.

□

A trivial statement

Electrical proof.

Repeat for trees:

- There are no loops.
- Zero current and free vertices is a solution.
- It's the only solution.
- The network is "Markovian": potential is constant.
- All λ's are 1, and the chain is reversible.

$\square$

A trivial statement

Electrical proof.

Repeat for trees:

- There are no loops.
- Zero current and free vertices is a solution.
- It's the only solution.
- The network is "Markovian": potential is constant.
- All λ's are 1, and the chain is reversible.

$\square$

Second thank you.