How to initialise a second class particle?

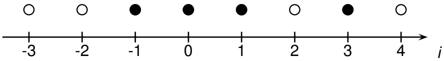
Joint with
Attila László Nagy

Márton Balázs

University of Bristol

Large Scale Stochastic Dynamics
Oberwolfach, 16 November, 2016.

Totally asymmetric simple exclusion



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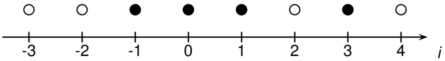

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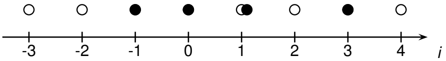

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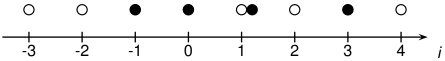

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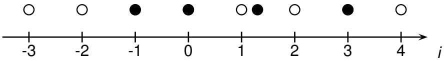

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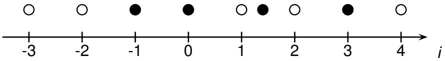

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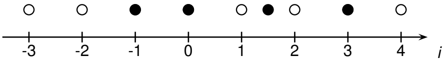

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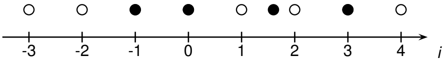

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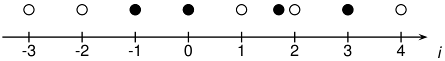

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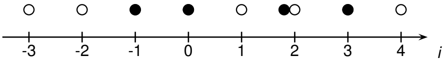

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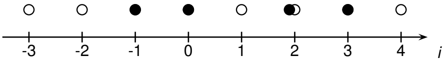

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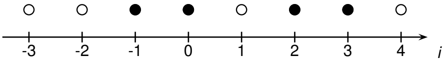

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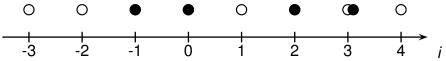

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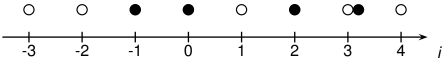

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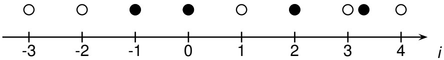

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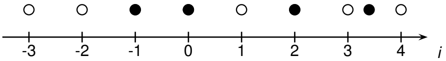

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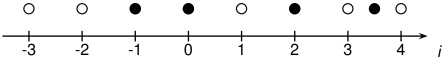

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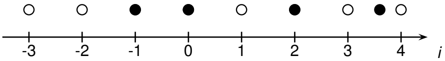

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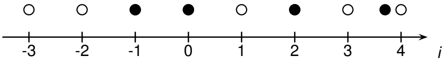

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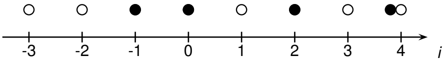

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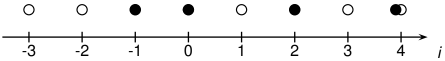

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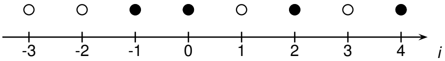

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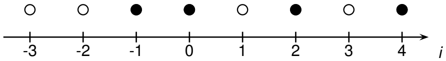

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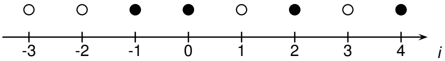

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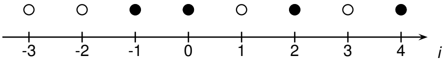

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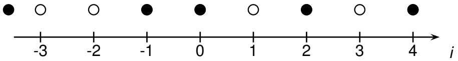

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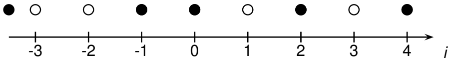

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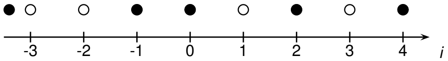

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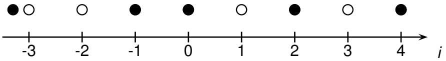

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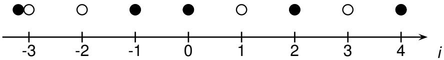

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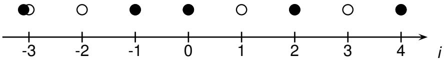

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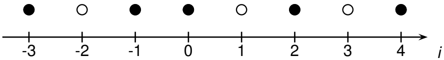

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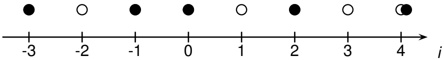

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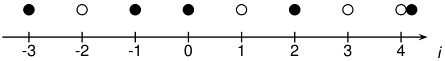

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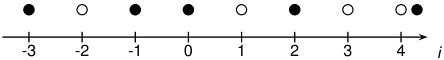

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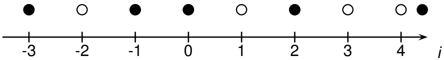

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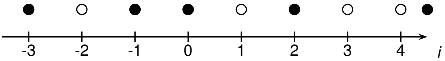

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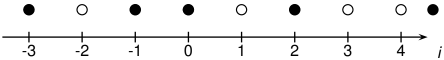

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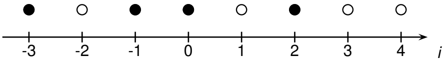

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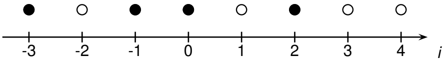

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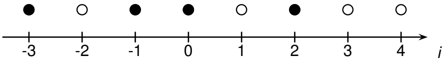

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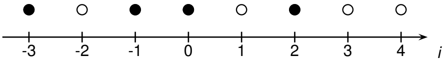

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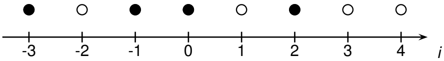

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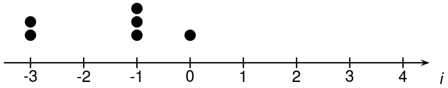
The Bernoulli(ϱ) distribution is time-stationary for any ($0 \leq \varrho \leq 1$). Any translation-invariant stationary distribution is a mixture of Bernoullis.

Totally asymmetric zero range process

$\omega_i \in \mathbb{Z}^+$



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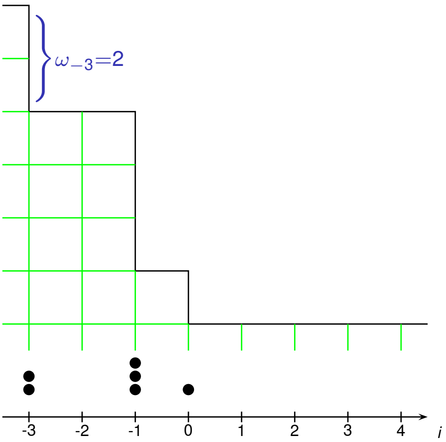

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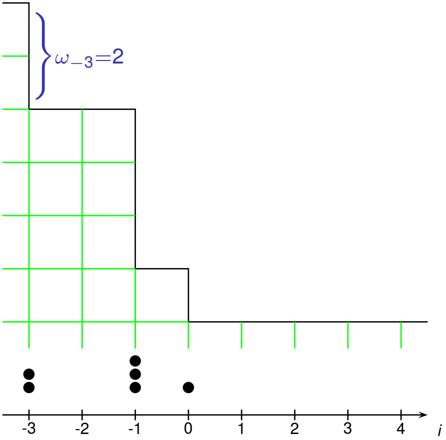

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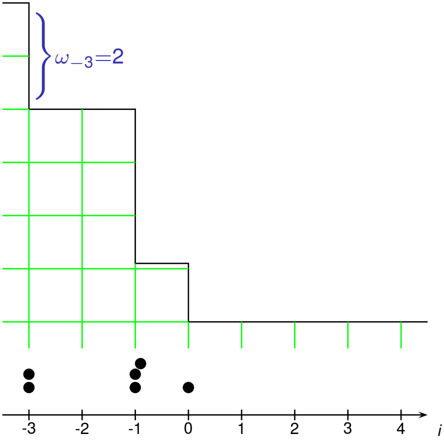

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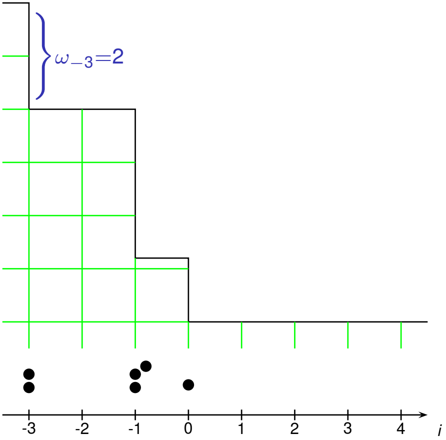

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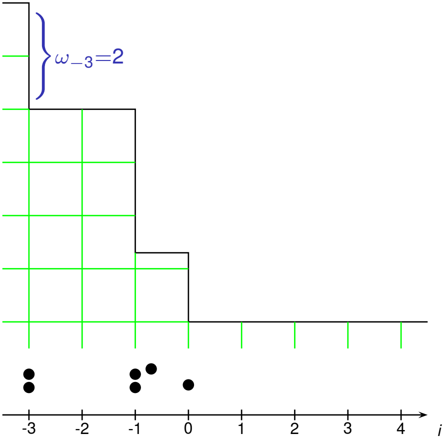

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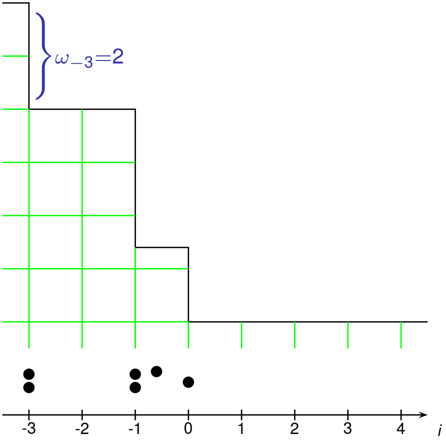

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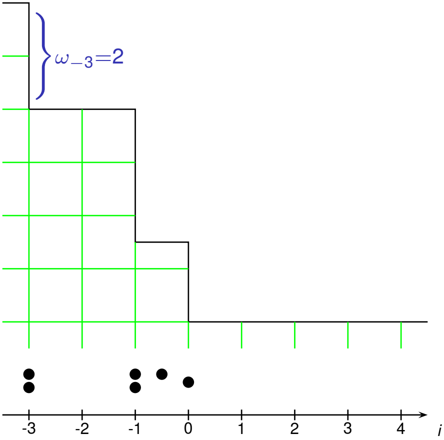

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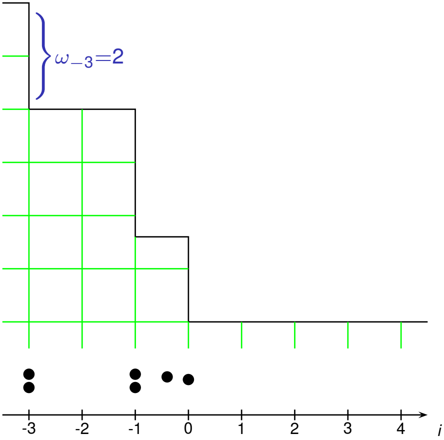
Totally asymmetric zero range process



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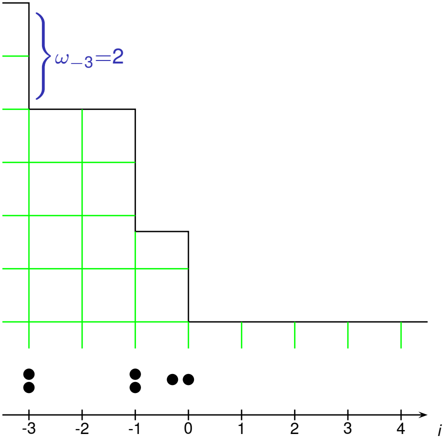

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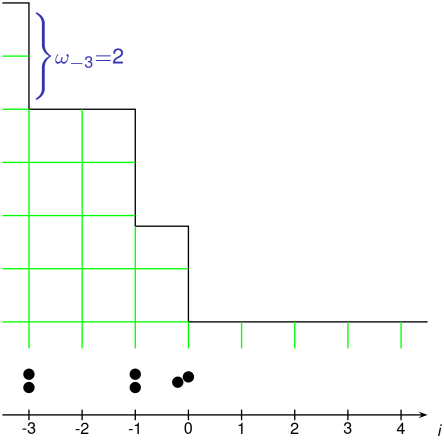

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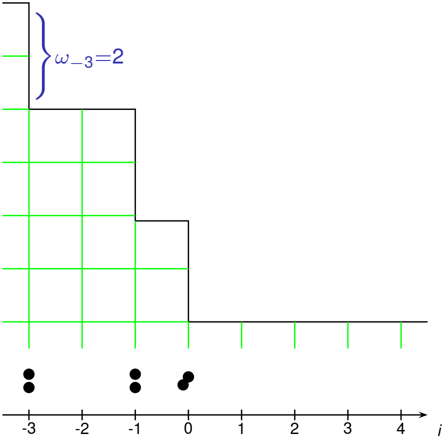

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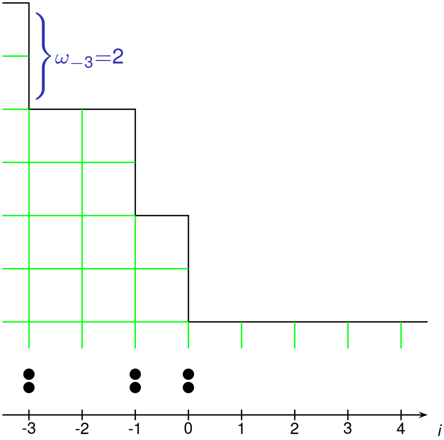

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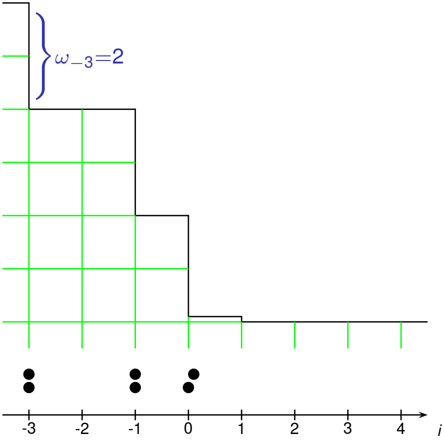

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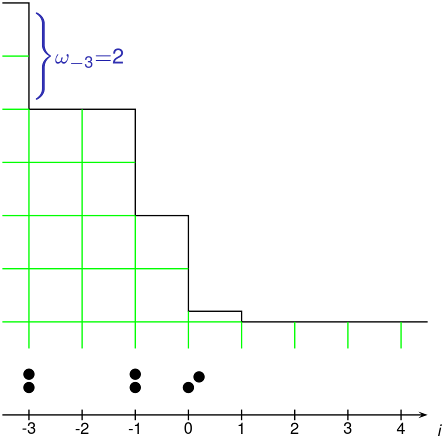

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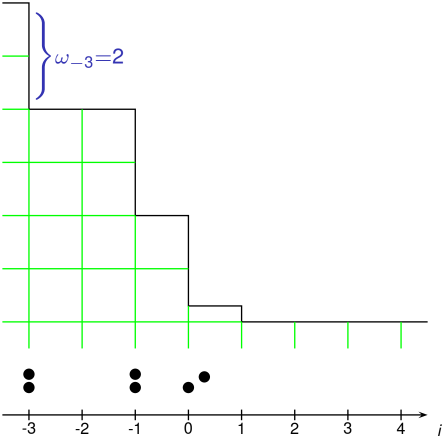

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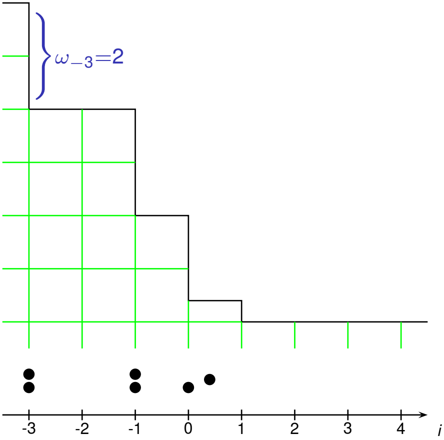

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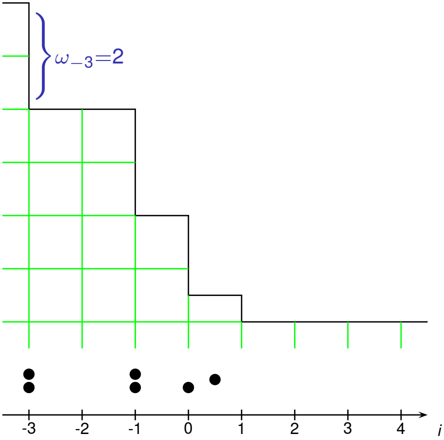

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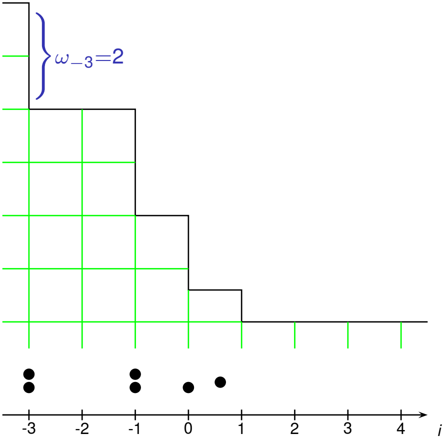

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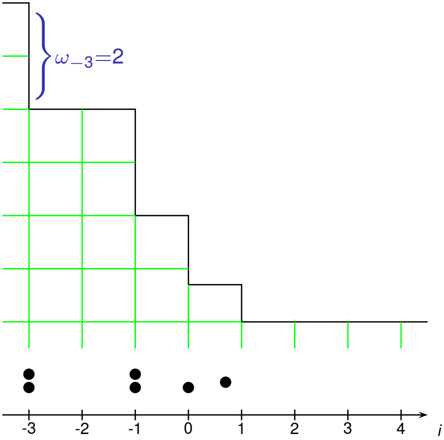

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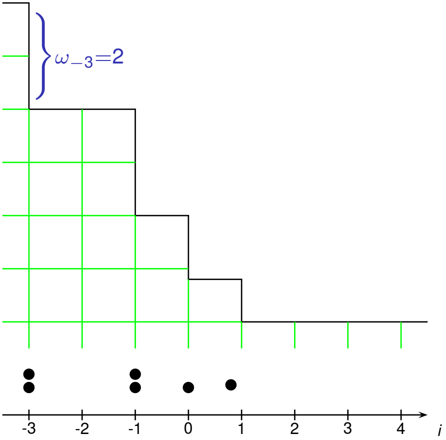

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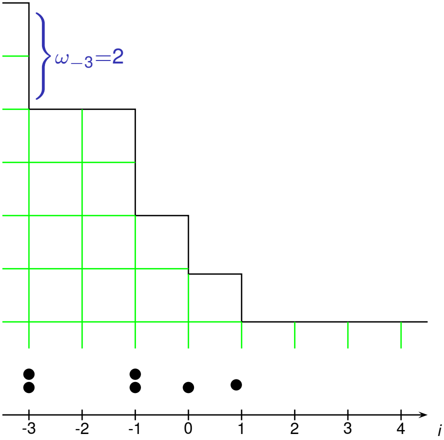

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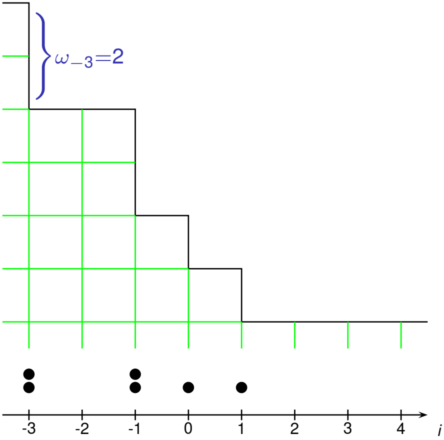

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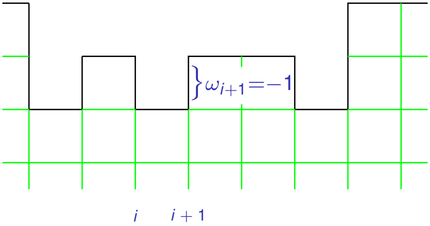
Totally asymmetric zero range process

Extremal translation-invariant stationary distributions are still product, and rather explicit in terms of $r(\cdot)$.

Two special cases:

- $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$: classical zero range; $\omega_i \sim \text{Geom}(\theta)$.
- $r(\omega_i) = \omega_i$: independent walkers; $\omega_i \sim \text{Poi}(\theta)$.

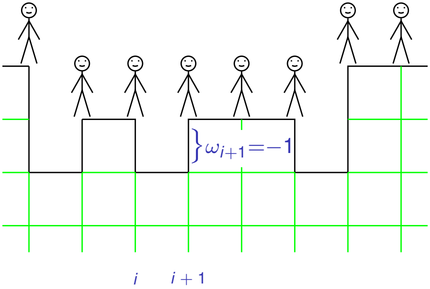
Totally asymmetric bricklayers process



$$\omega_i \in \mathbb{Z}$$

$$\begin{pmatrix} \omega_i \\ \omega_{i+1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Totally asymmetric bricklayers process



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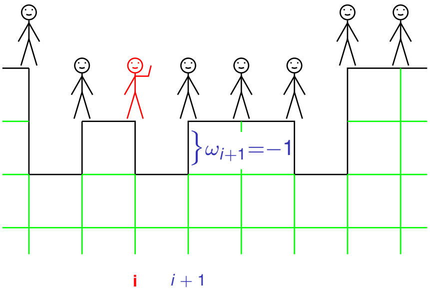
$(r(\omega) \cdot r(1-\omega) = 1; \quad r$ non-decreasing$)$.

Totally asymmetric bricklayers process



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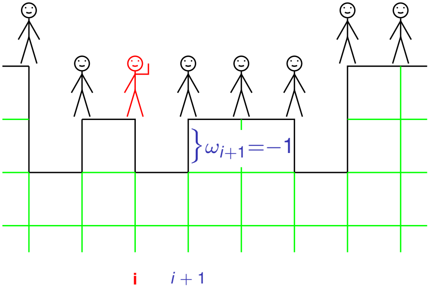

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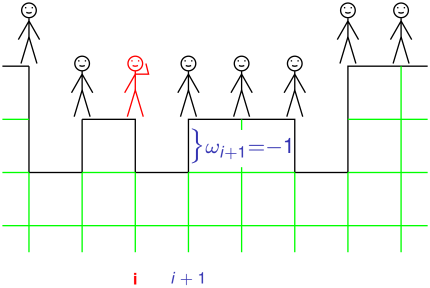

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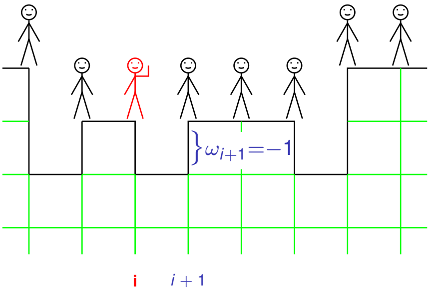

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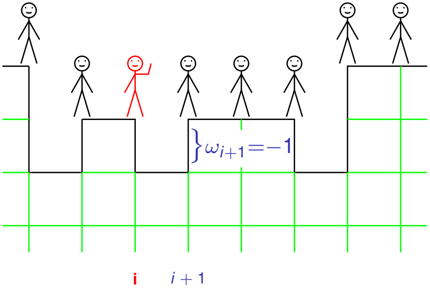

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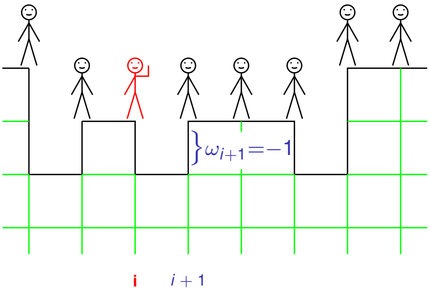

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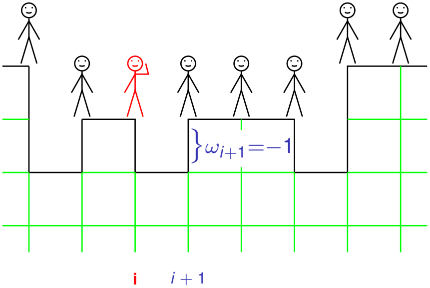

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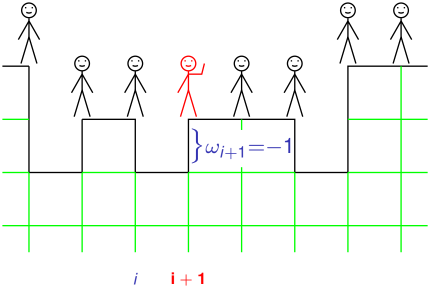

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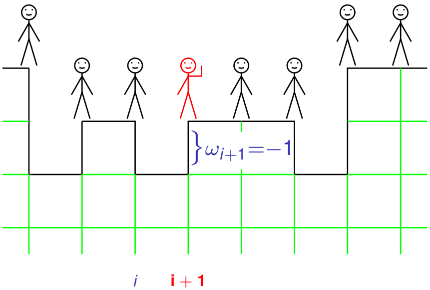

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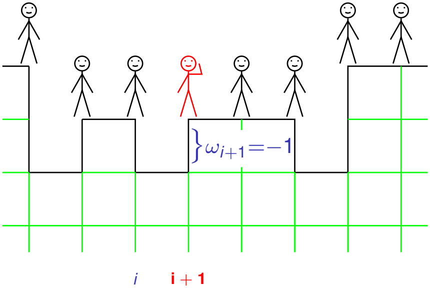

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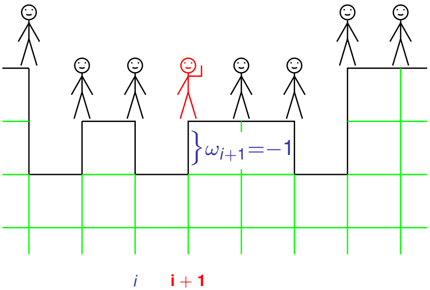

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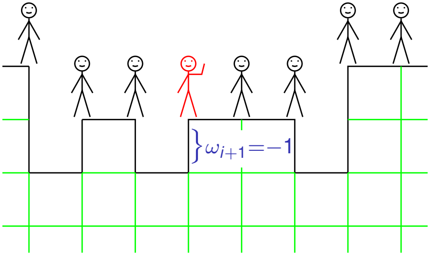

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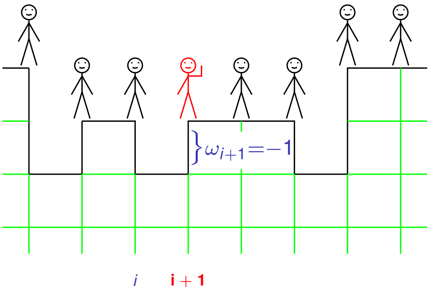

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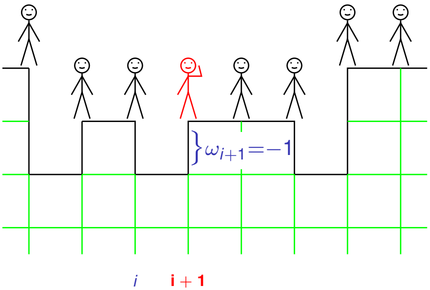

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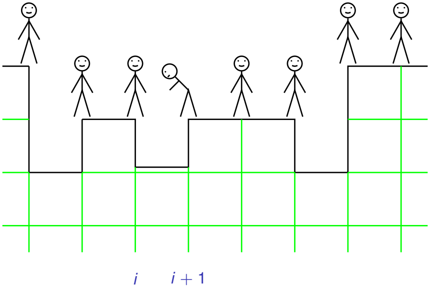

$$\omega_i \in \mathbb{Z}$$

$$\begin{pmatrix}\omega_i \\ \omega_{i+1}\end{pmatrix} \to \begin{pmatrix}\omega_i - 1 \\ \omega_{i+1} + 1\end{pmatrix}$$

$$\| \qquad\qquad \|$$

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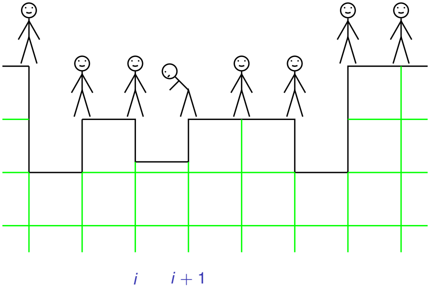

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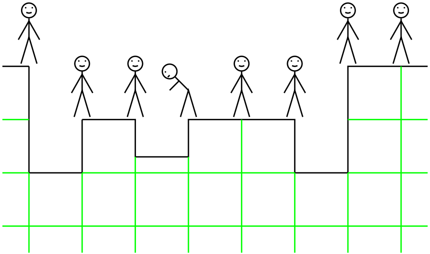

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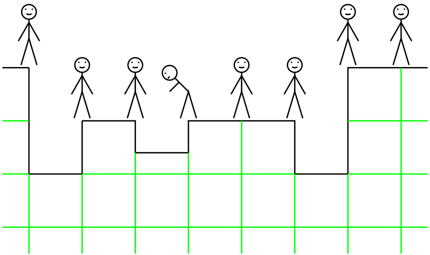

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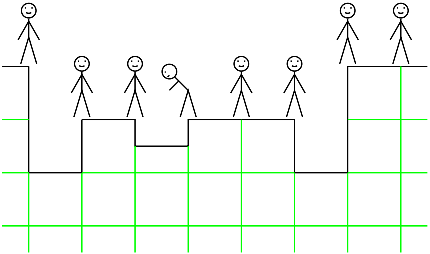

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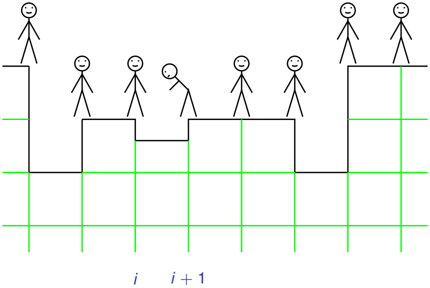

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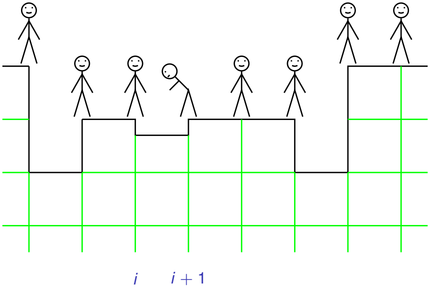

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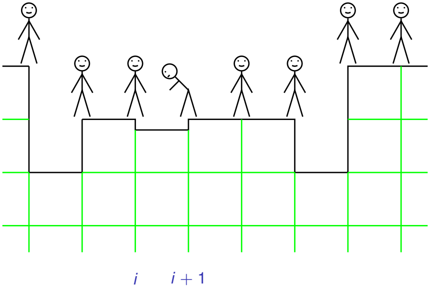

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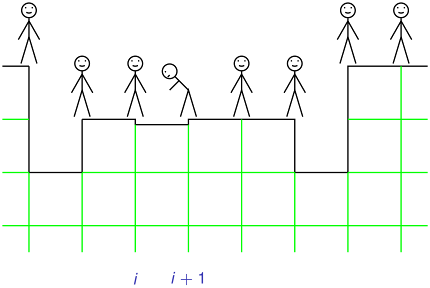

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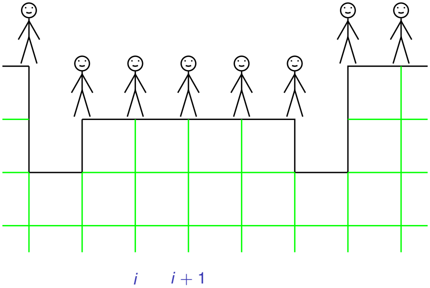

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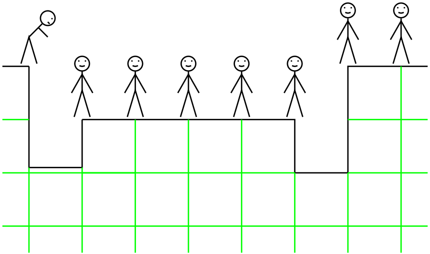

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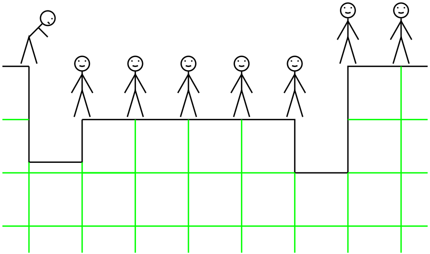

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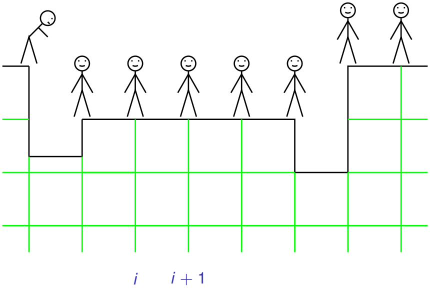

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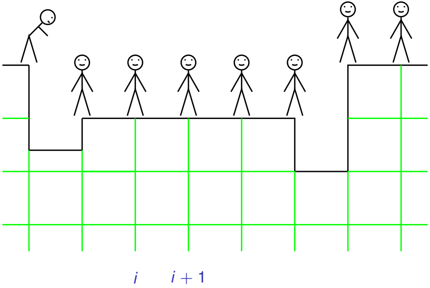

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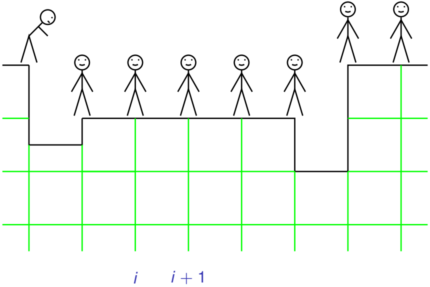

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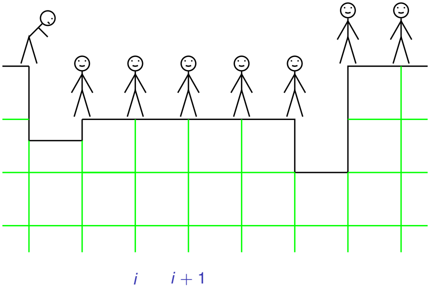

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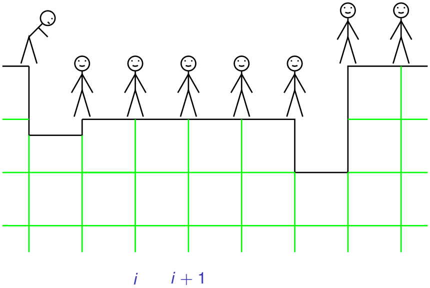

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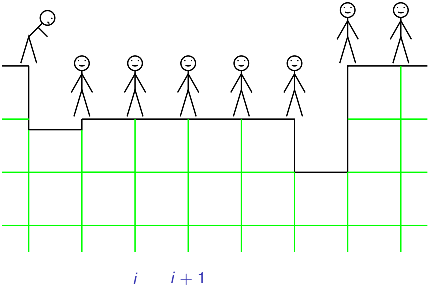

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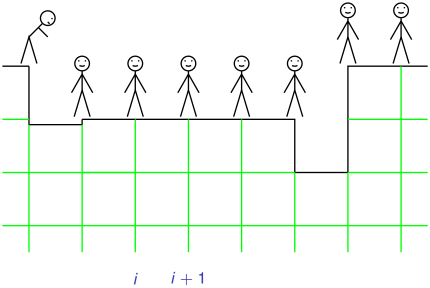

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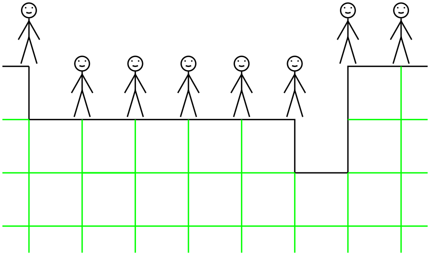

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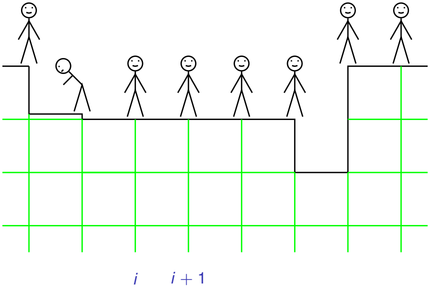

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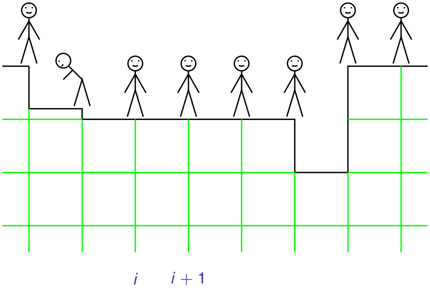

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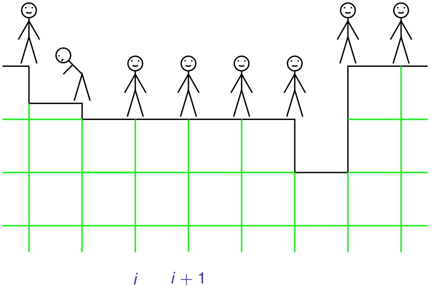

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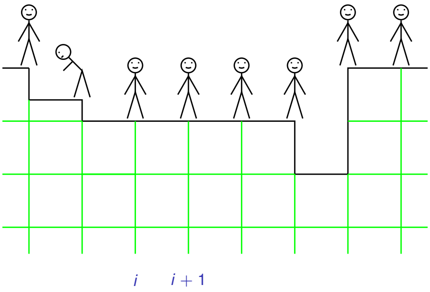

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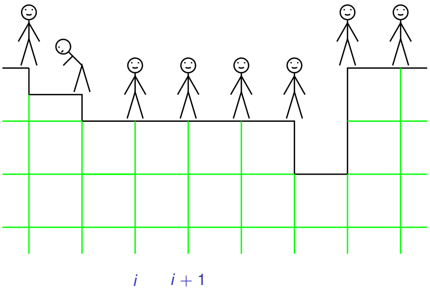

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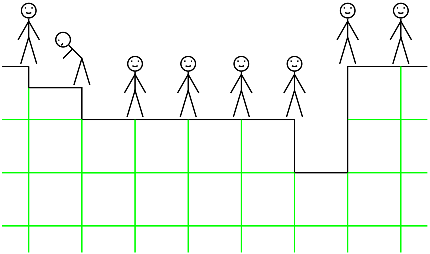

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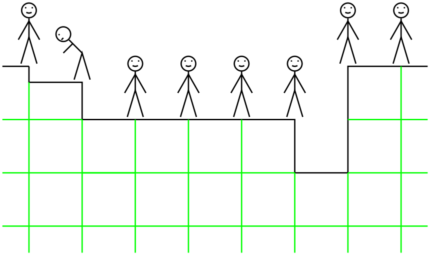

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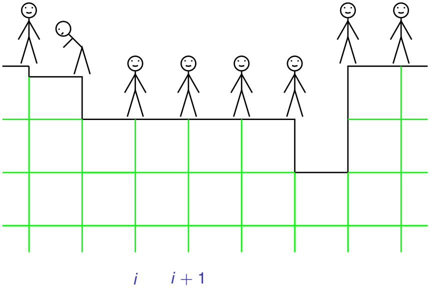

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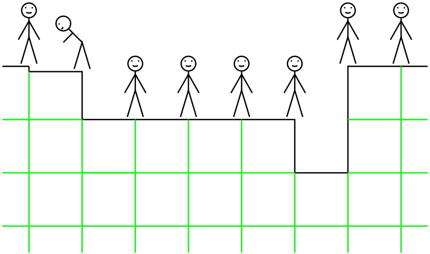

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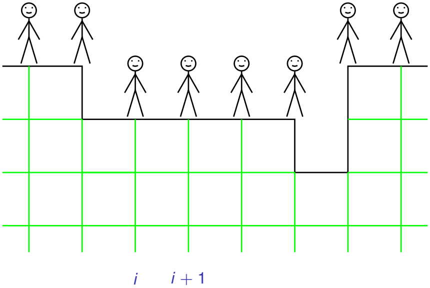

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Extremal translation-invariant stationary distributions are still product, and rather explicit in terms of $r(\cdot)$.

A special case: $r(\omega_i) = \mathrm{e}^{\beta\omega_i}$: $\omega_i \sim$ discrete Gaussian$(\frac{\theta}{\beta},\ \frac{1}{\sqrt{\beta}})$.

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$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \to \begin{pmatrix}\omega_i - 1\\ \omega_{i+1}+1\end{pmatrix}$$ with rate $r(\omega_i, \omega_{i+1})$, where

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- they satisfy some regularity conditions to make sure the dynamics exists.

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The *density* $\varrho := \mathbf{E}(\omega)$ and the *hydrodynamic flux* $H := \mathbf{E}$[growth rate] both depend on a parameter of the stationary distribution.

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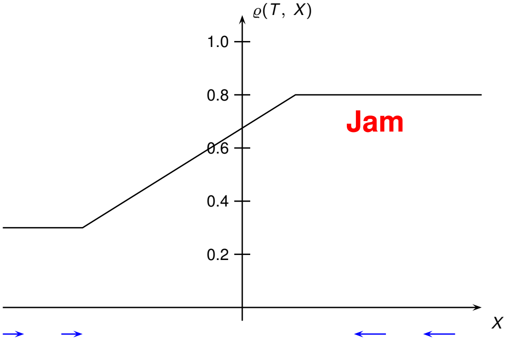
- The *characteristics* is a path $X(T)$ where $\varrho(T, X(T))$ is constant. $\dot{X}(T) = H'(\varrho)$ is the *characteristic speed*.

Shock



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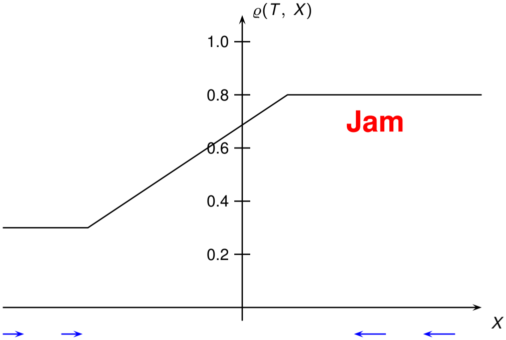

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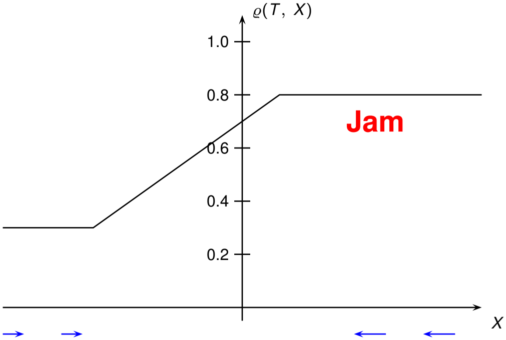

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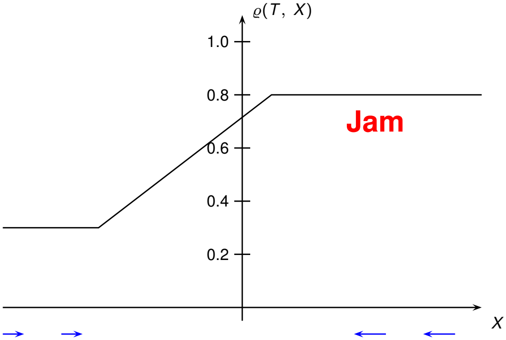

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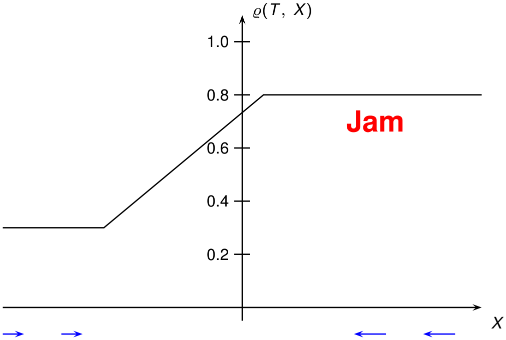

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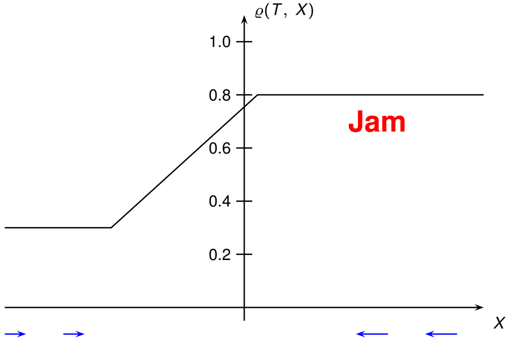

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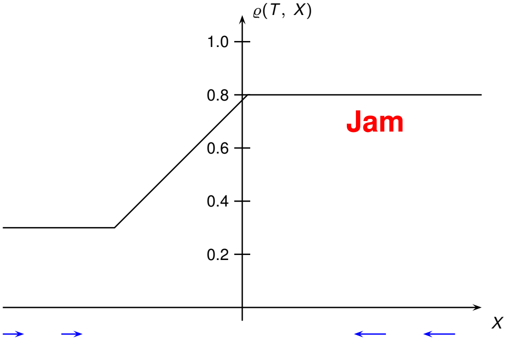

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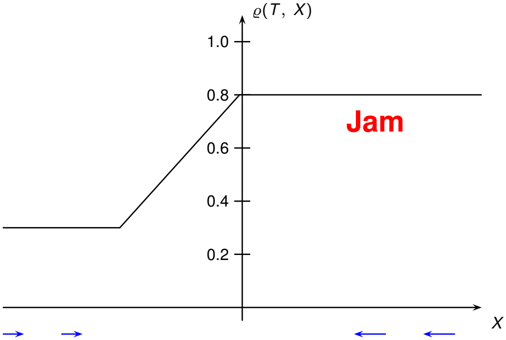

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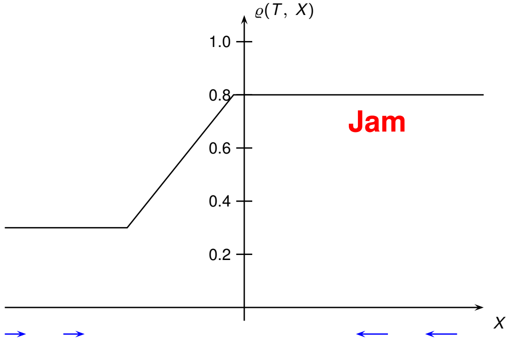

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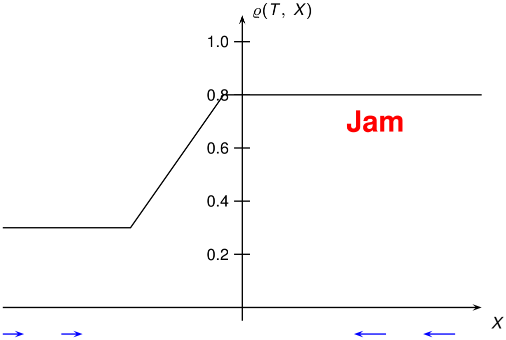

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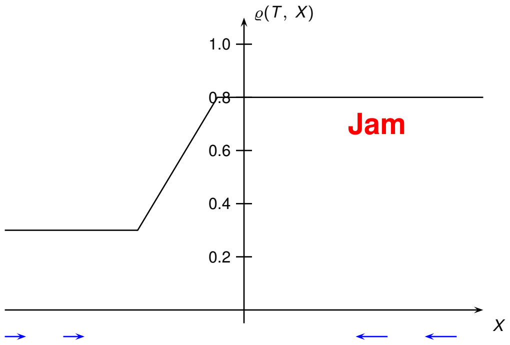

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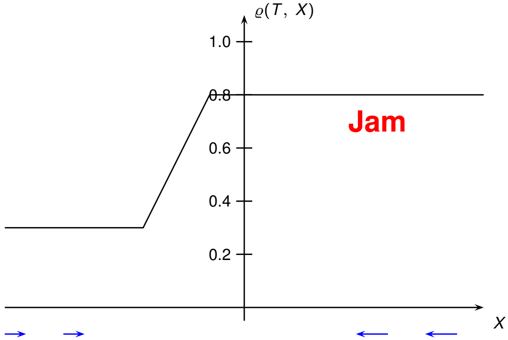

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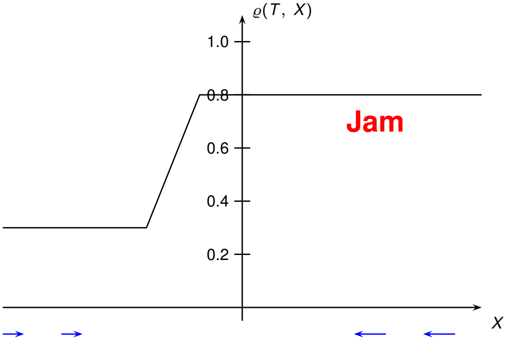

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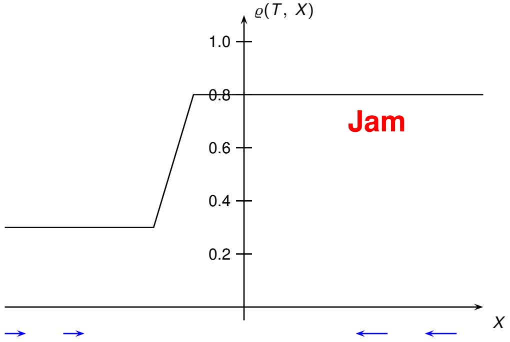

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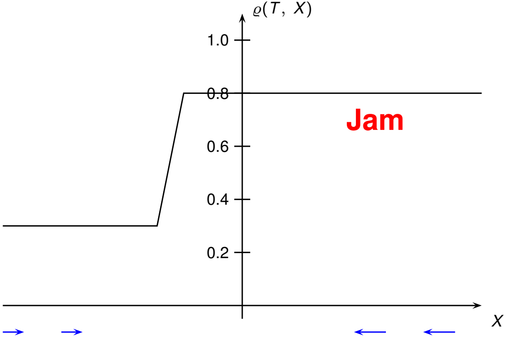

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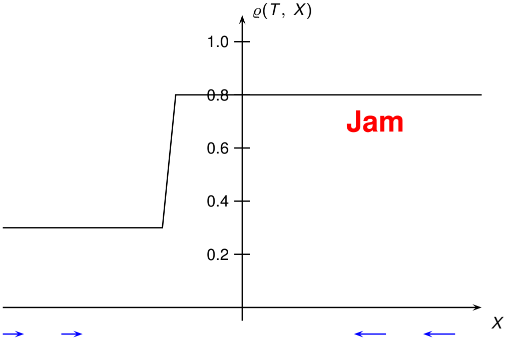

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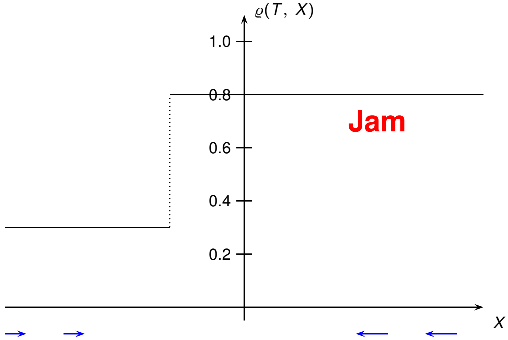

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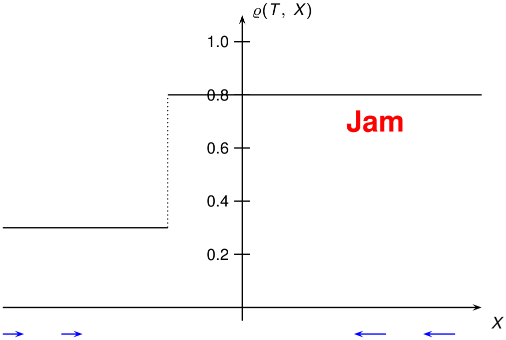

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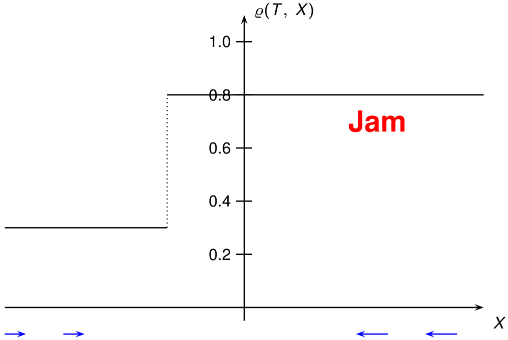

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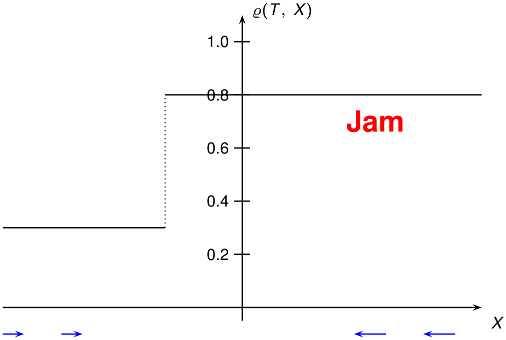

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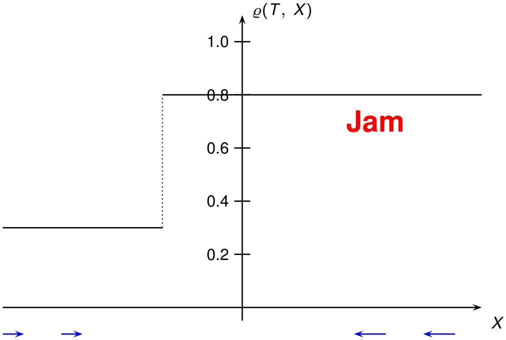

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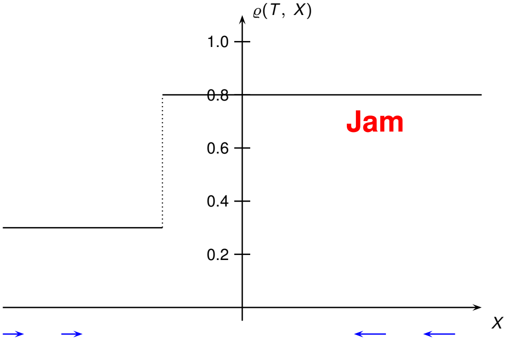

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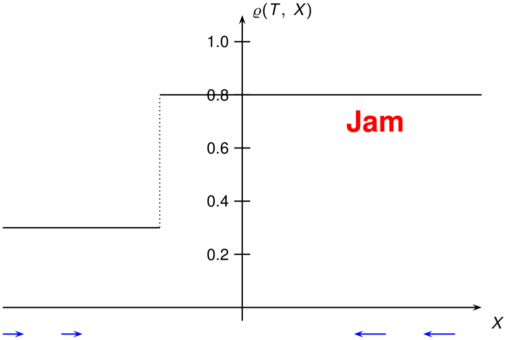

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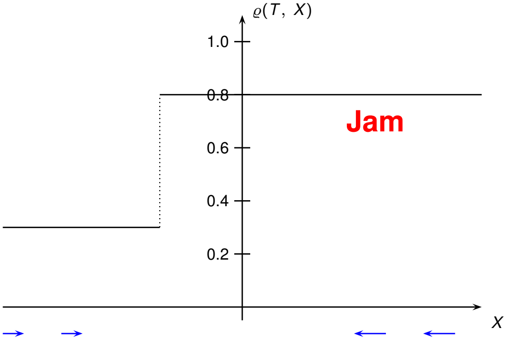
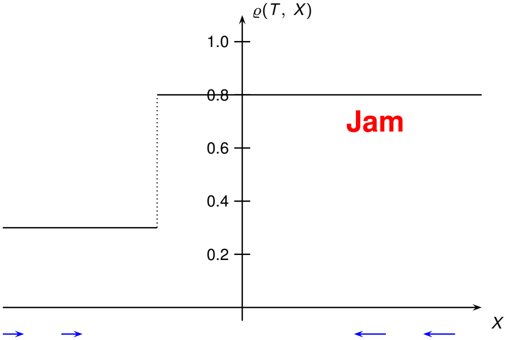


$\dot{X}(T) = H'(\varrho) \searrow$ (H concave)

Shock



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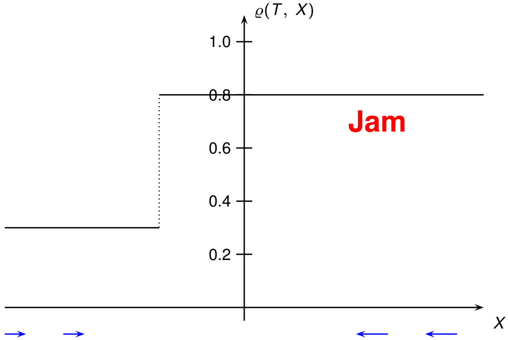
Shock



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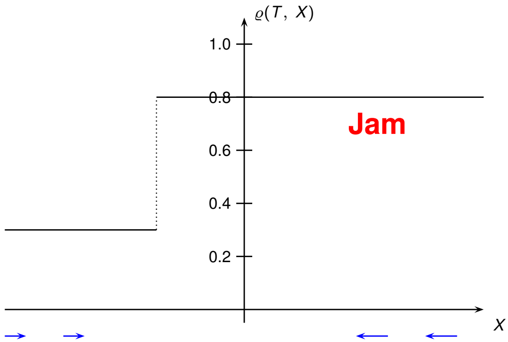

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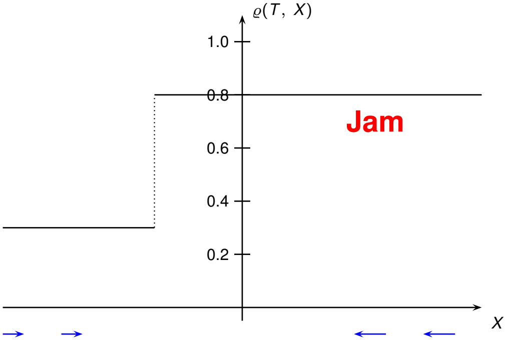

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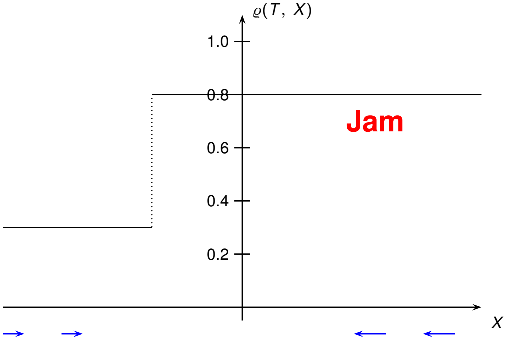
Shock



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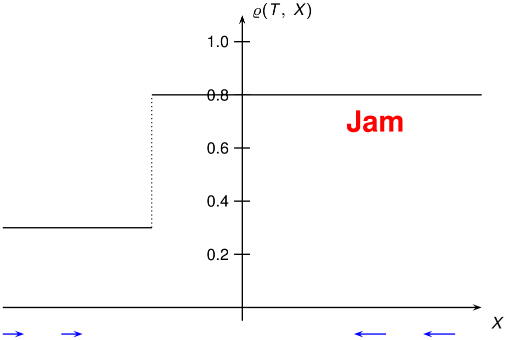

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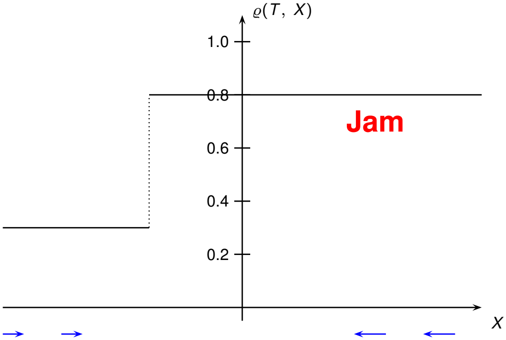

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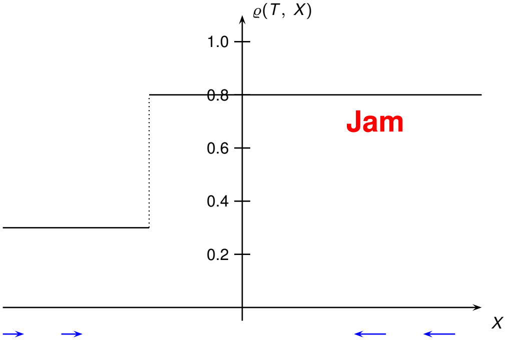

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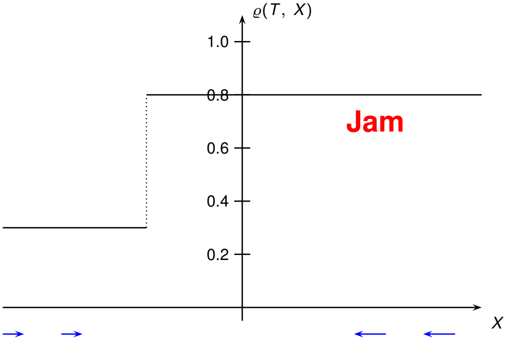

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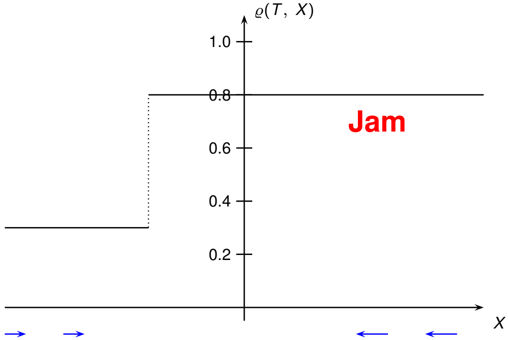

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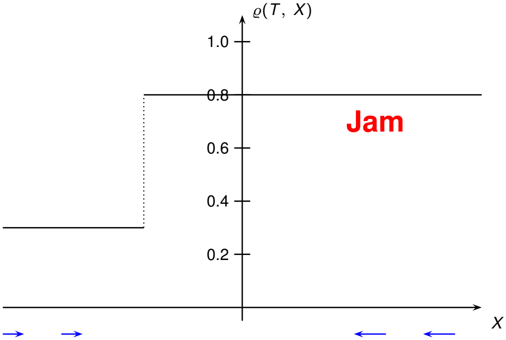

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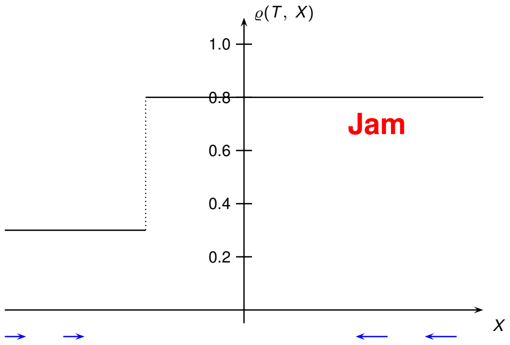

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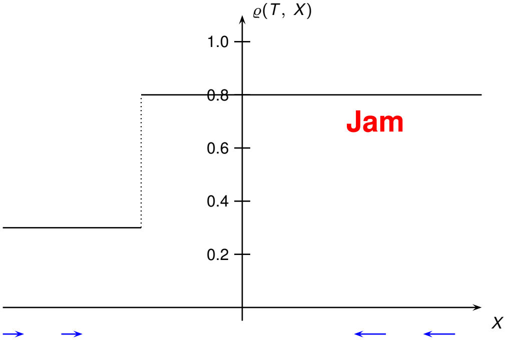

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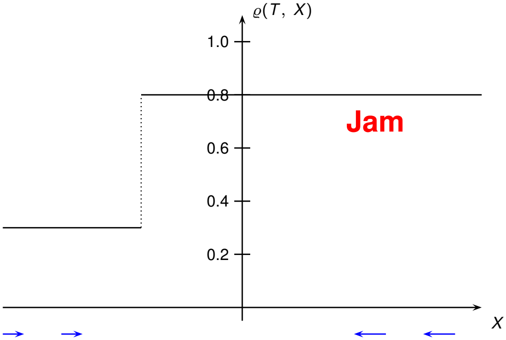

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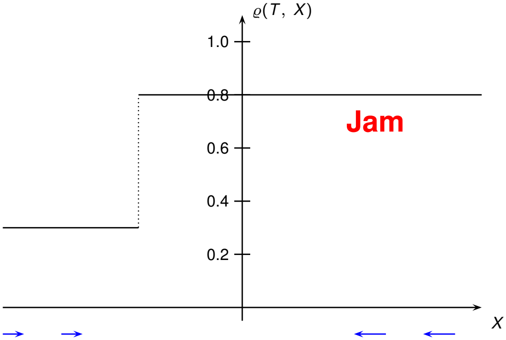

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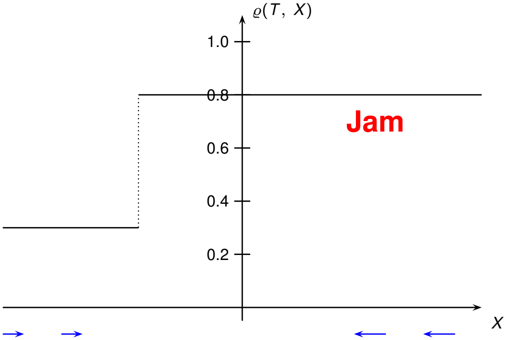

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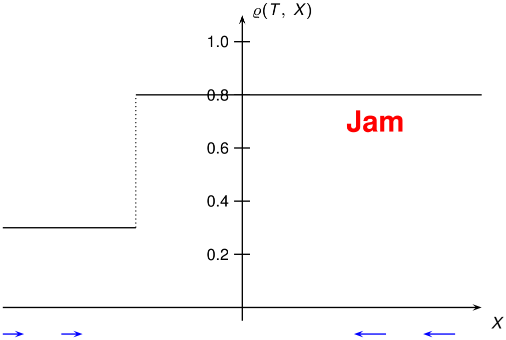

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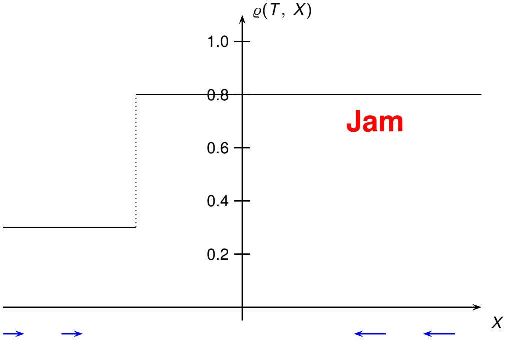

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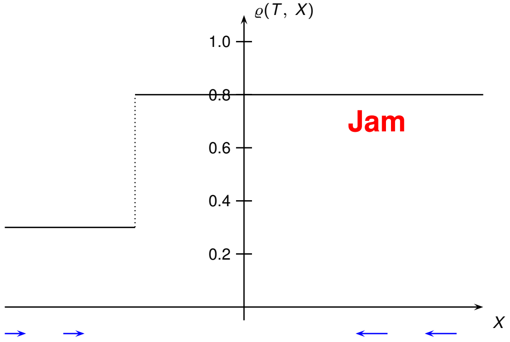

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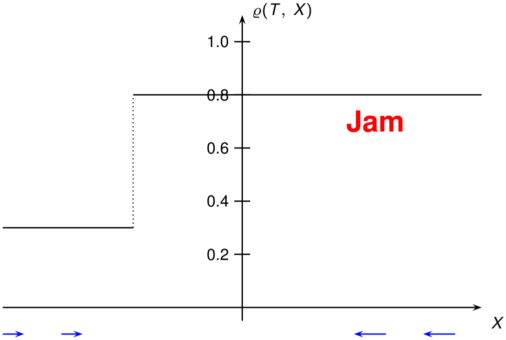

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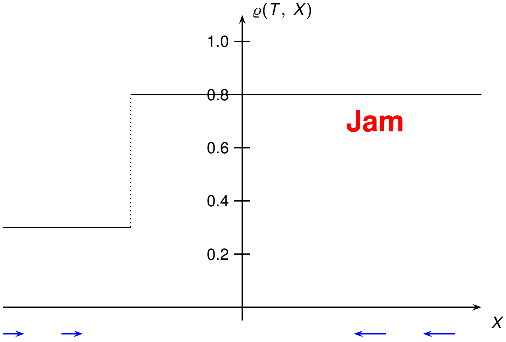

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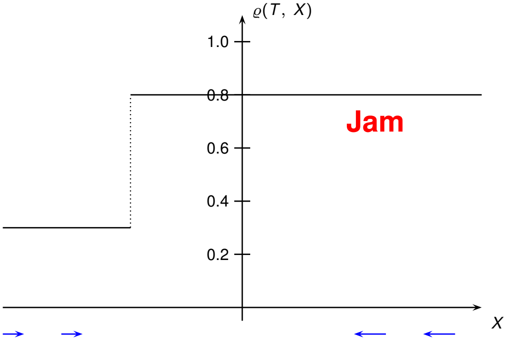
Shock



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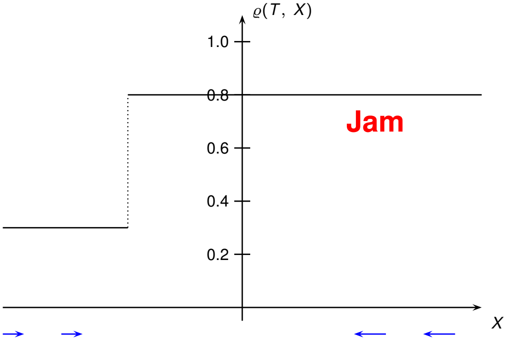

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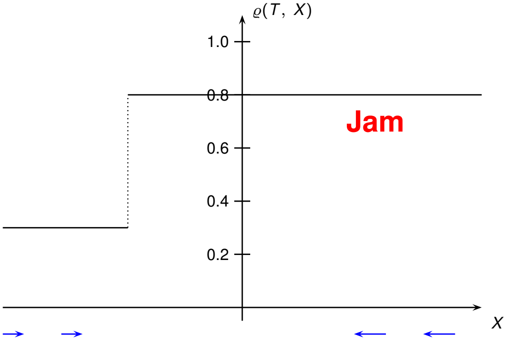
Shock



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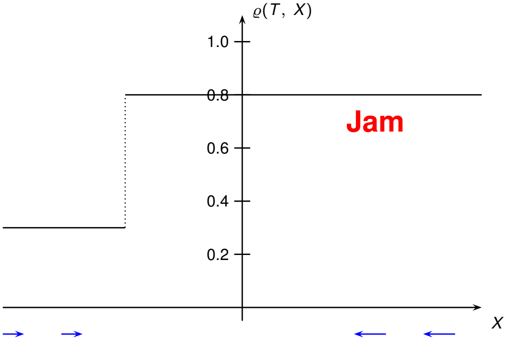

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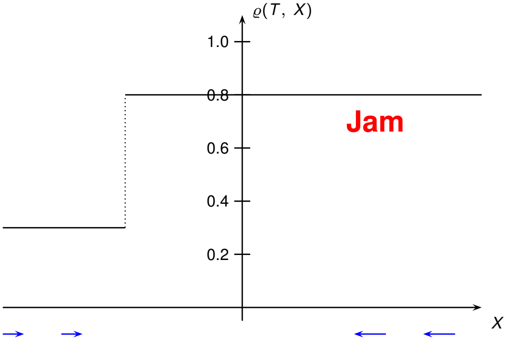

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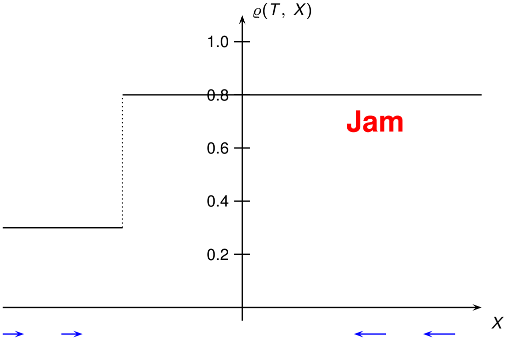

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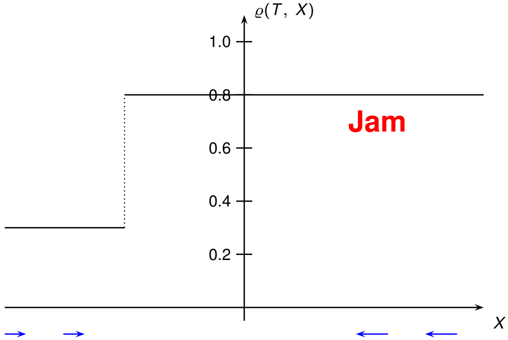

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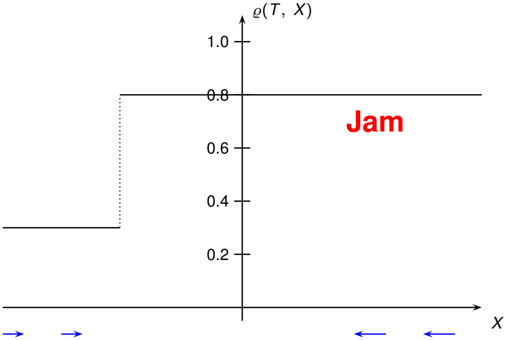

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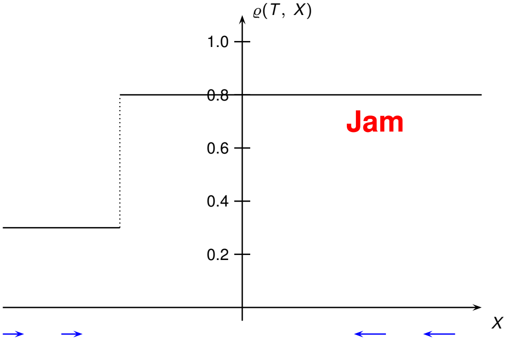

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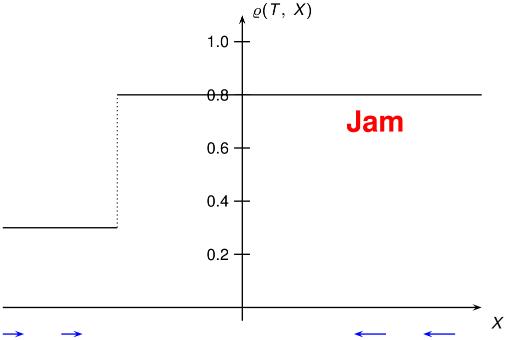

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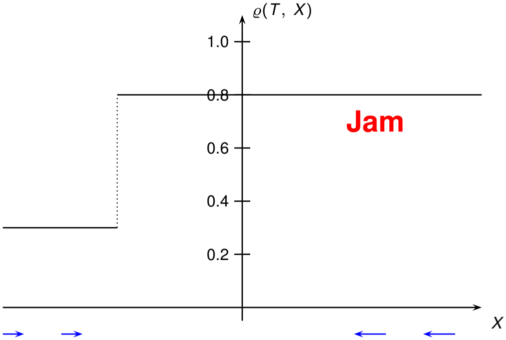

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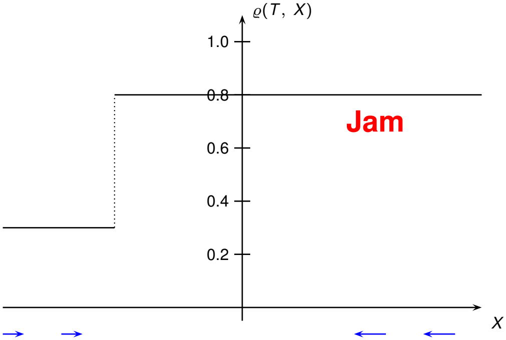

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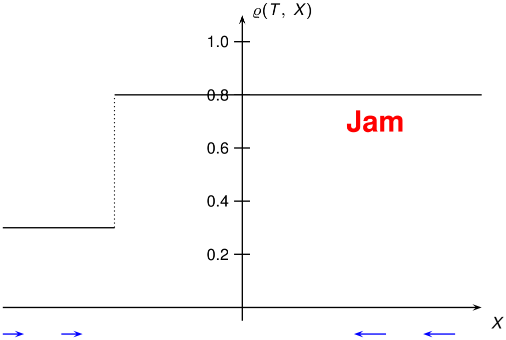

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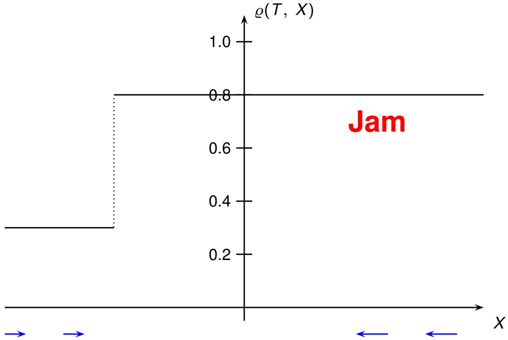

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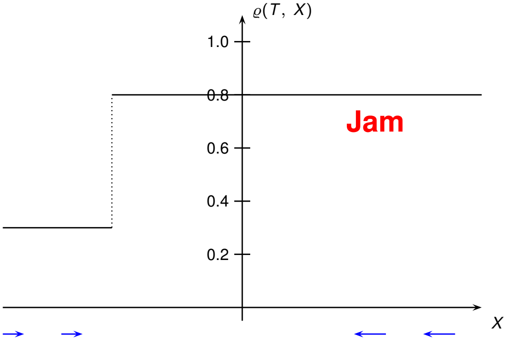

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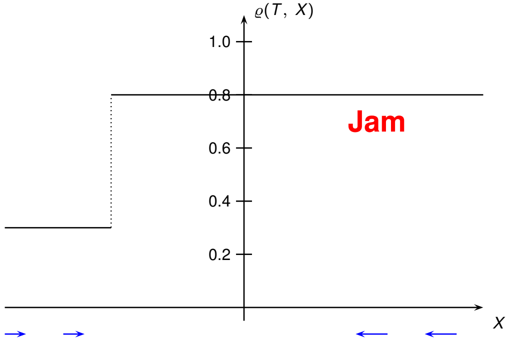
Shock



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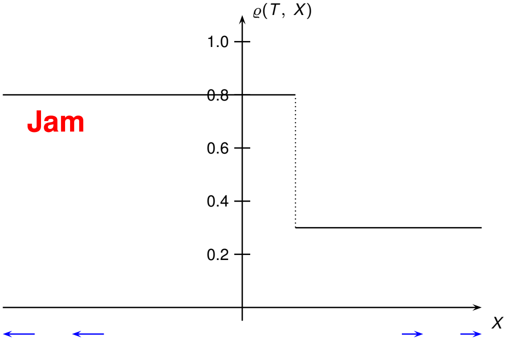
Rarefaction fan



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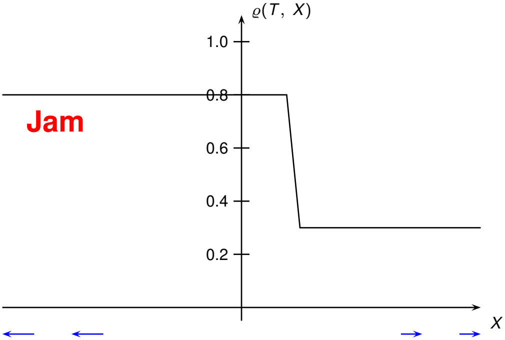

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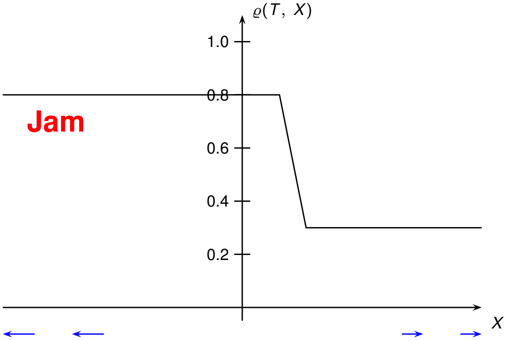

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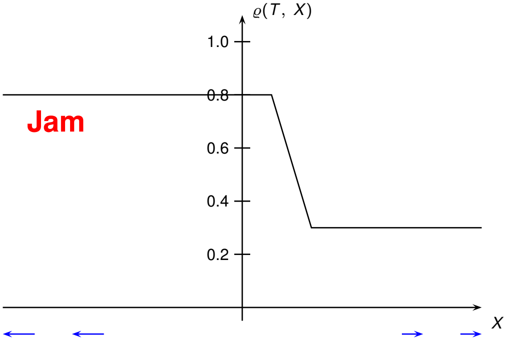

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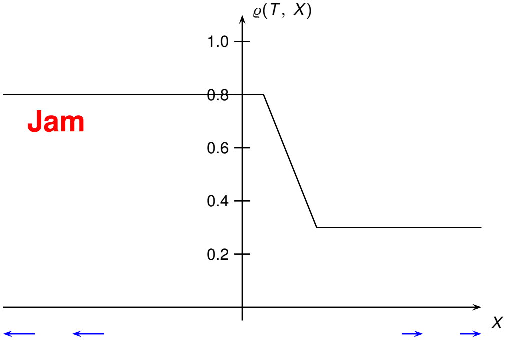

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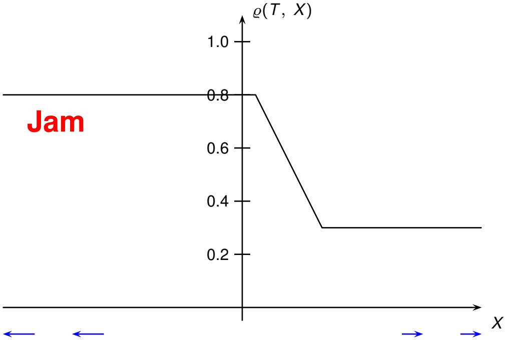

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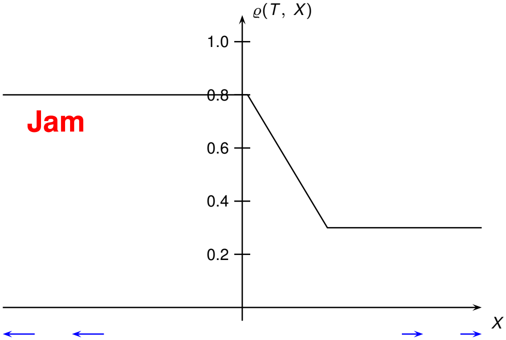

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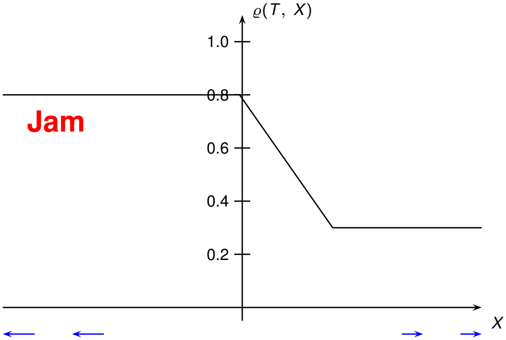

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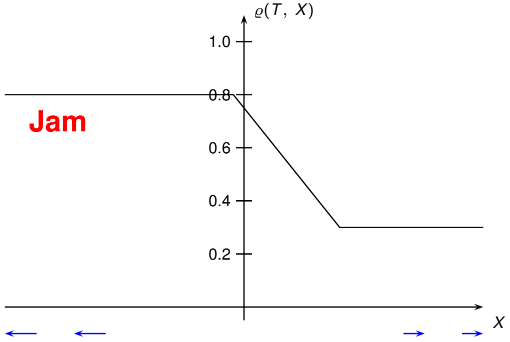

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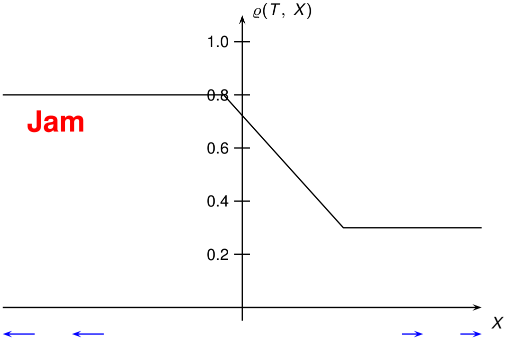

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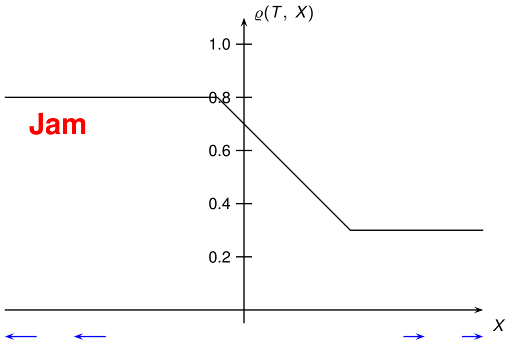

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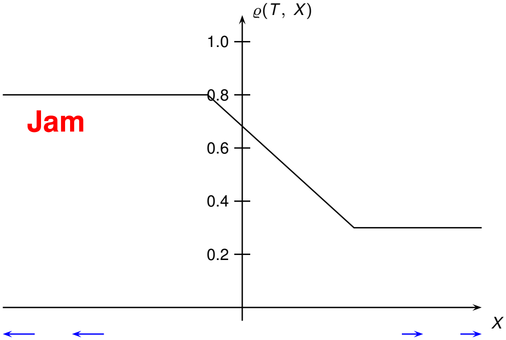

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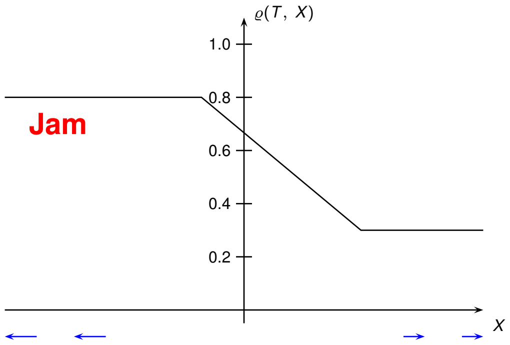

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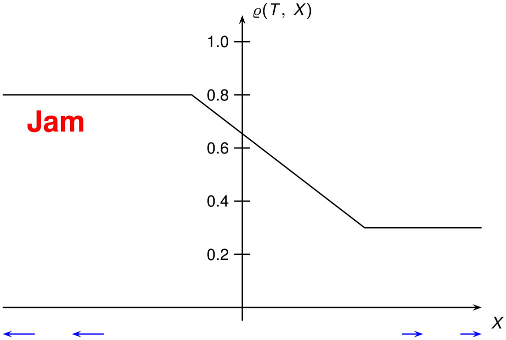

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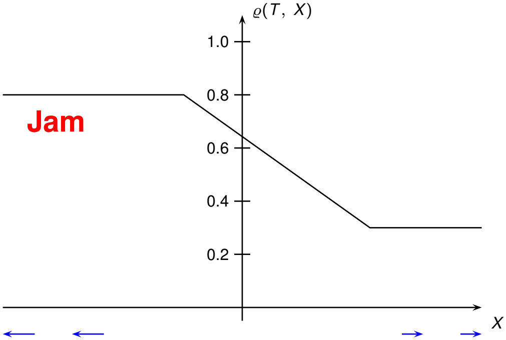

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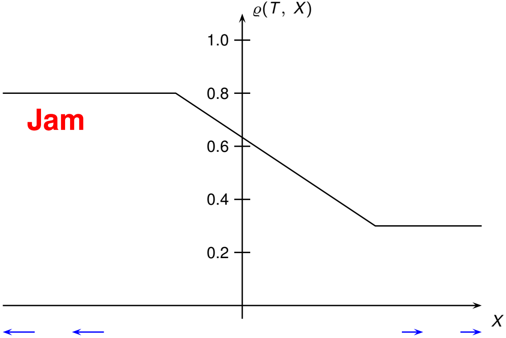

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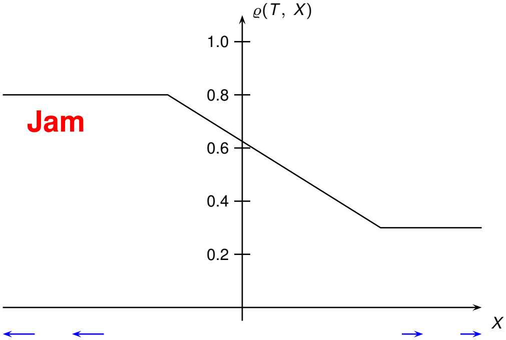

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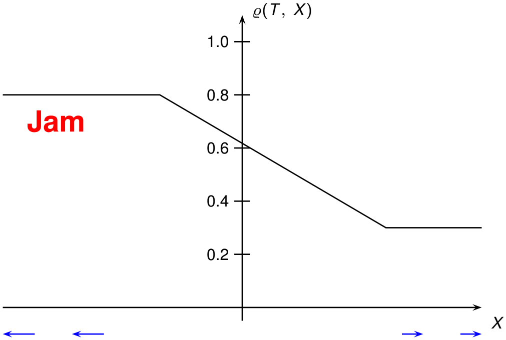

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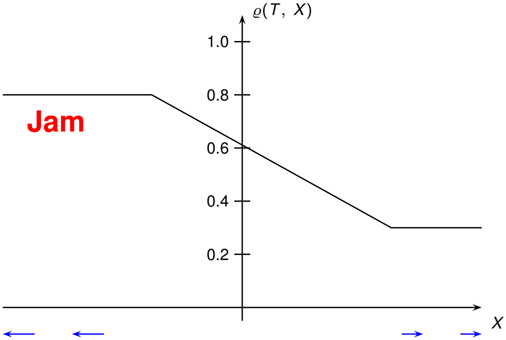
Rarefaction fan



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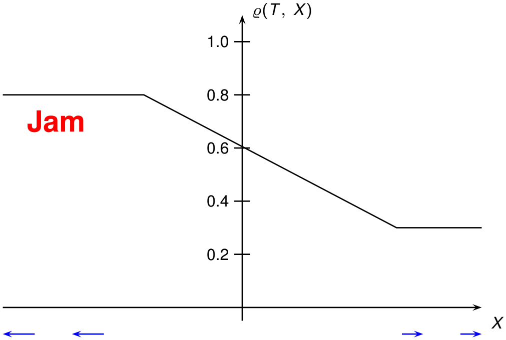

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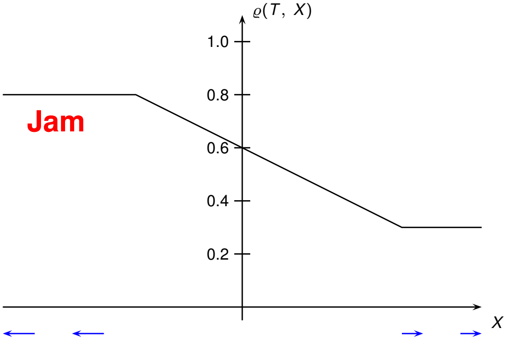

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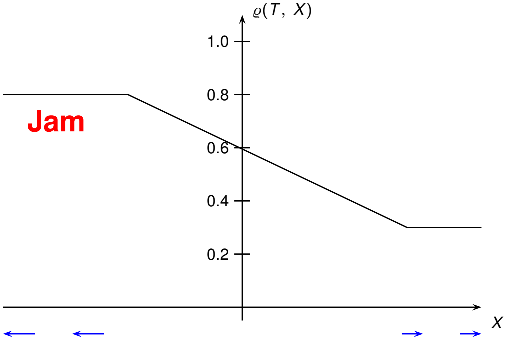

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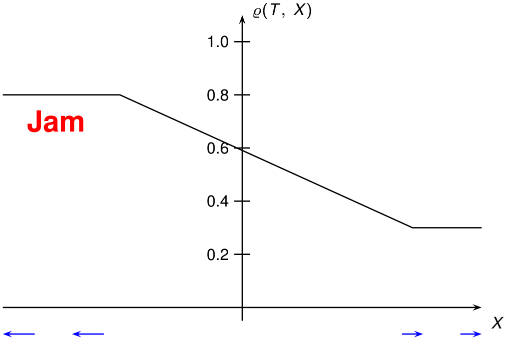
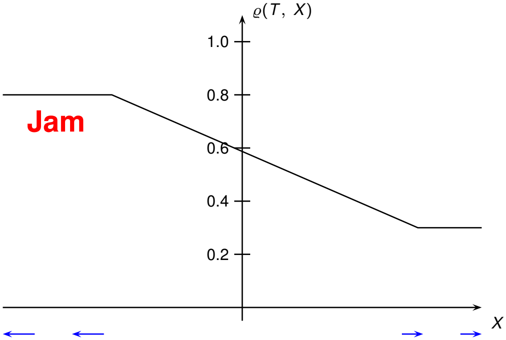


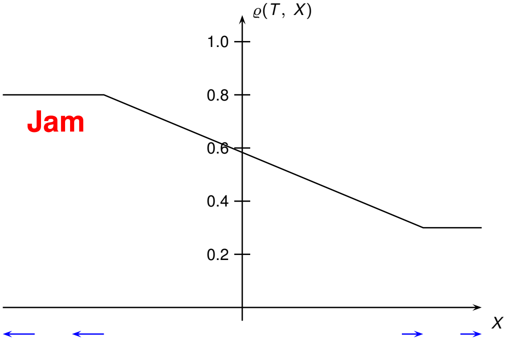
$\dot{X}(T) = H'(\varrho) \searrow$ (H concave)

Rarefaction fan



$$\dot{X}(T) = H'(\varrho) \searrow \qquad \text{(H concave)}$$

Rarefaction fan



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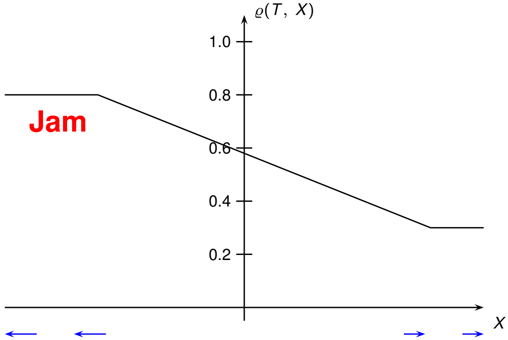
Rarefaction fan



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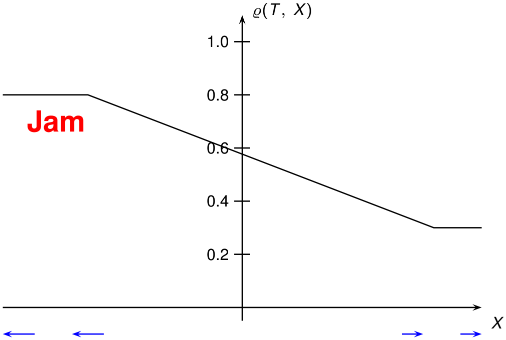

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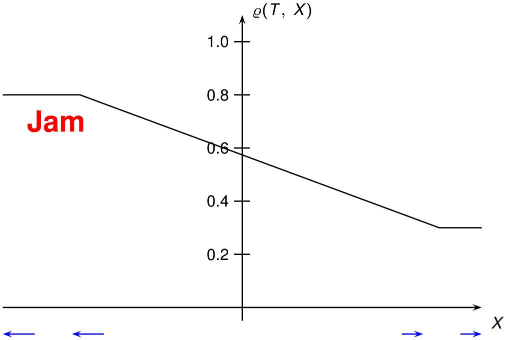

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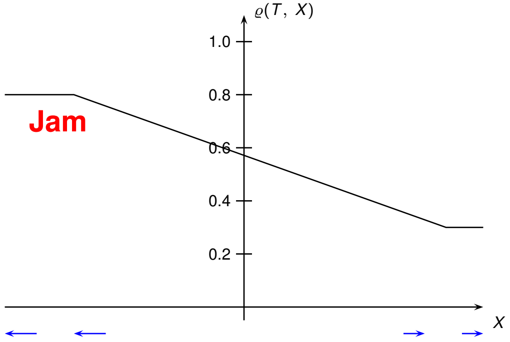

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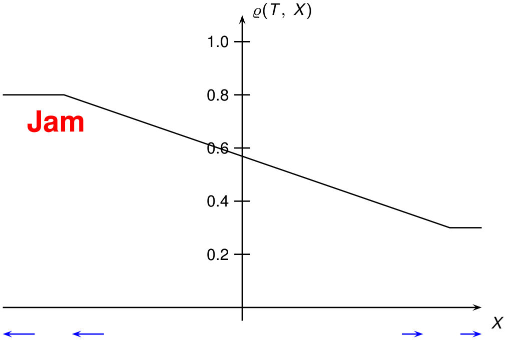

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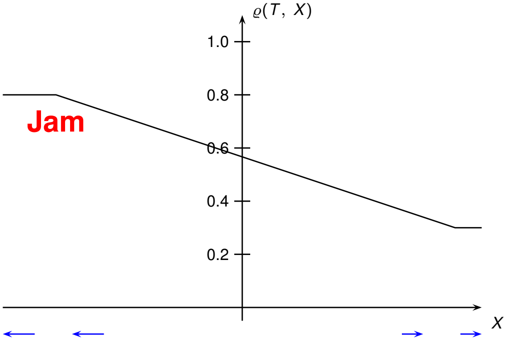
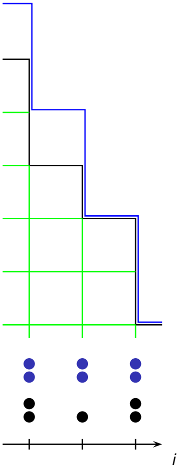


$$\dot{X}(T) = H'(\varrho) \searrow \qquad (H \text{ concave})$$

The second class particle

States ω and η only differ at one site.



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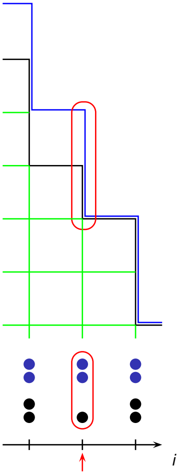

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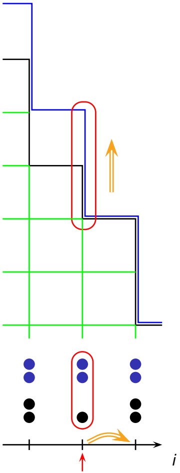

Growth on the right:
rate≤rate

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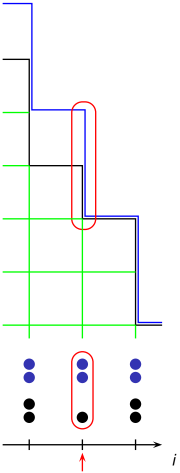

Growth on the right:
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with rate:

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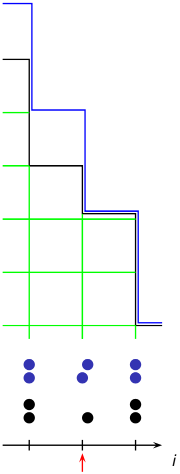

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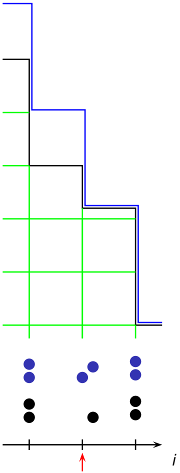

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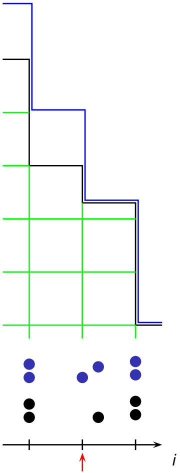

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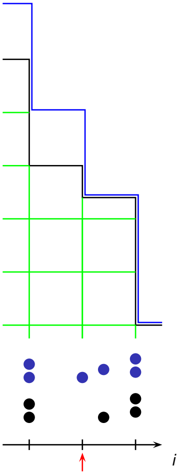

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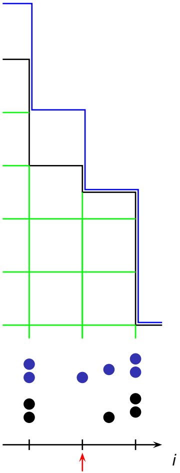

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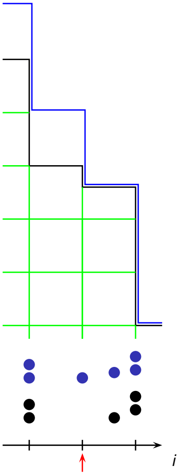

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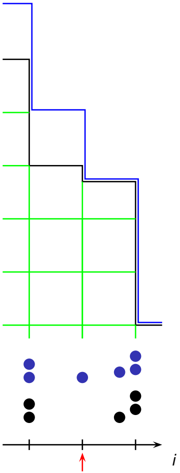

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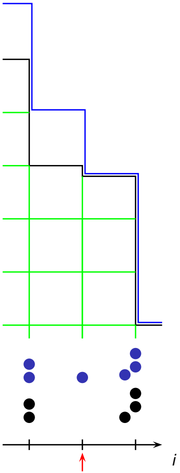

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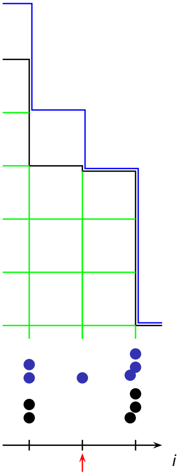

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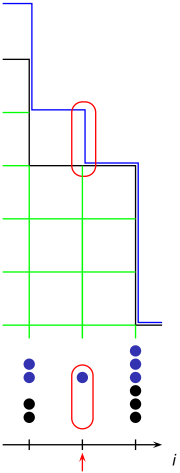

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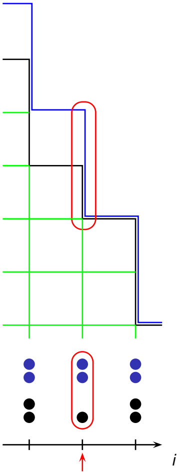

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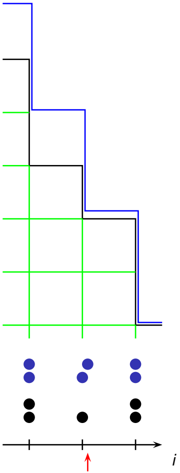

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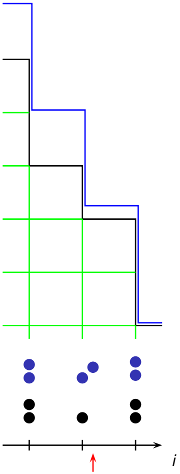

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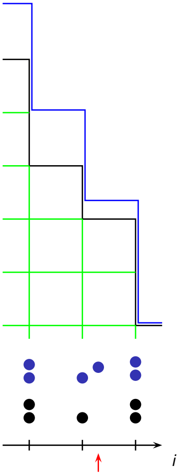

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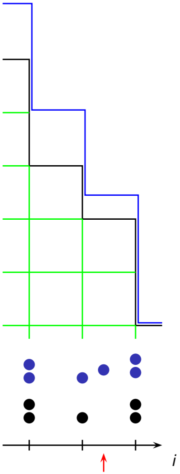

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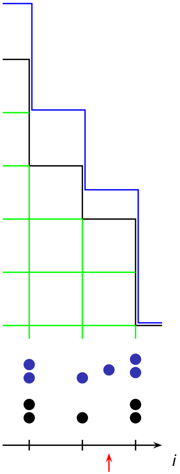

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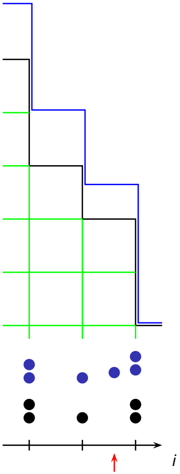

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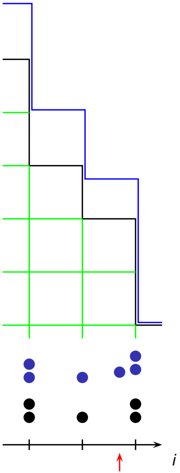

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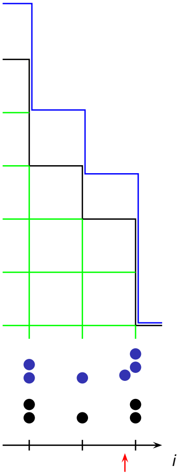

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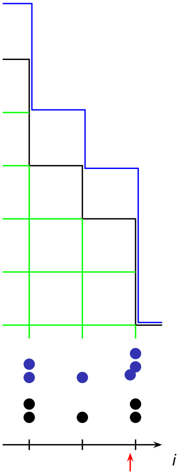

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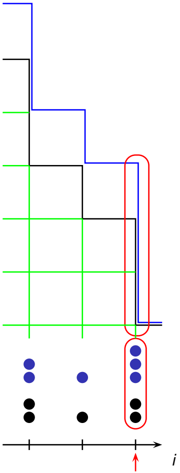

Growth on the right:
rate≤rate
with rate-rate:

The second class particle

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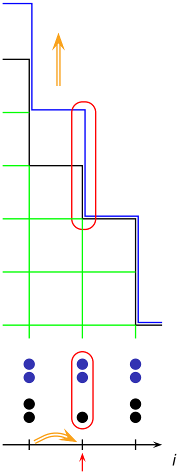
Growth on the left:

rate$\geq$rate



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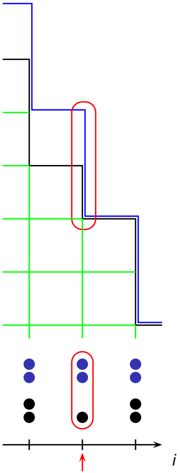

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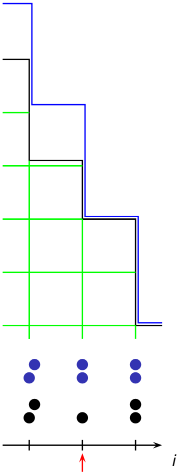

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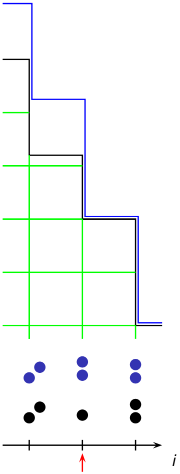

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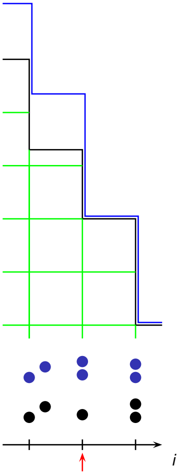

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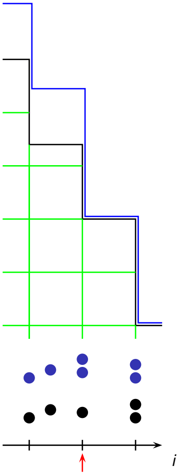

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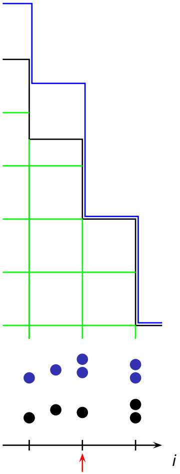

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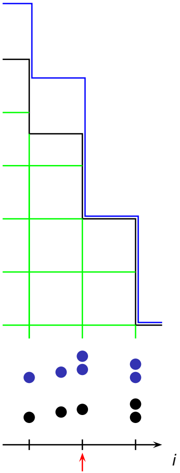

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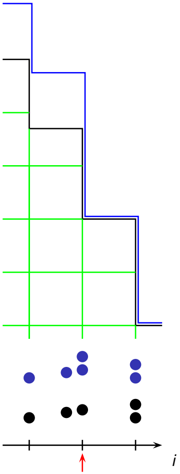

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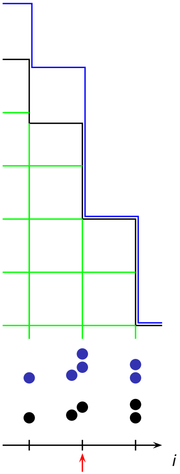

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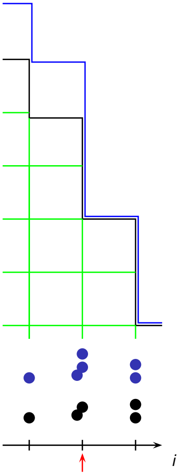

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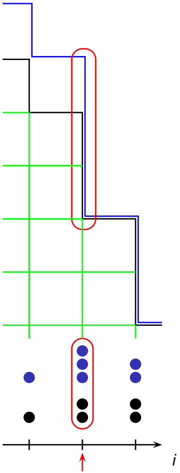

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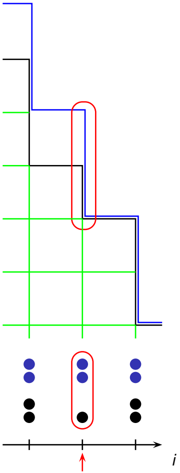

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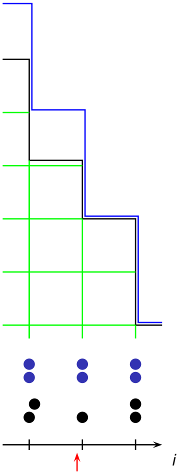

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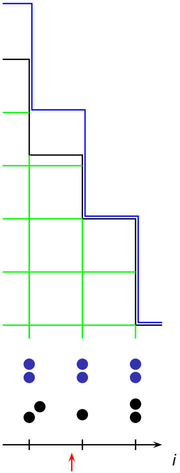

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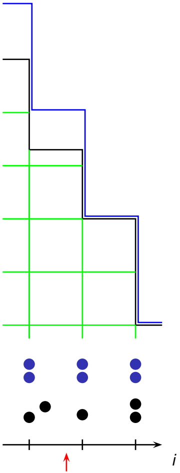

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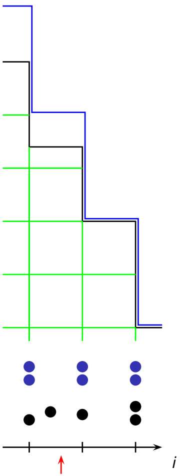

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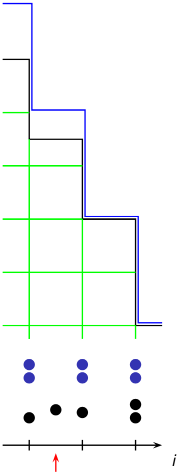

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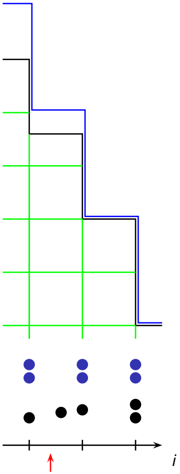

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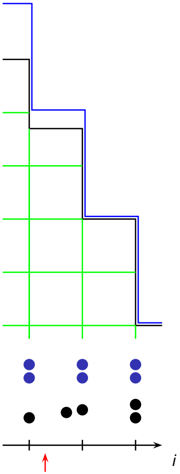

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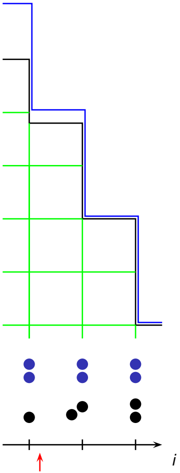

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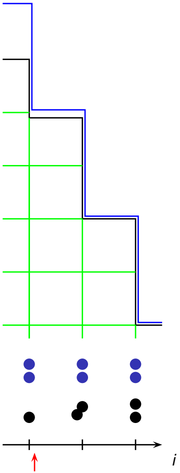

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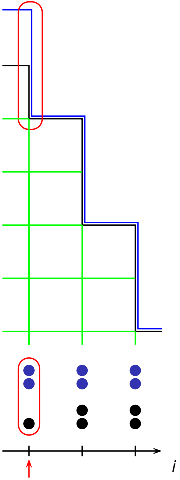

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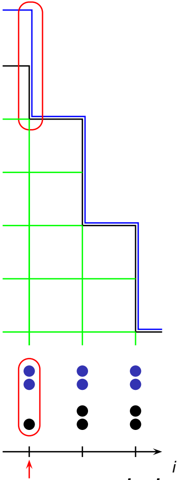
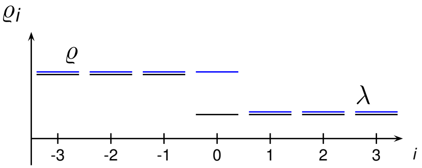


A single discrepancy ↑, the *second class particle*, is conserved.
Its position at time t is $Q(t)$.

Ferrari-Kipnis '95 for TASEP

Blue TASEP ω:
Bernoulli(ϱ) for sites $\{\dots, -2, -1, 0\}$,
Bernoulli(λ) for sites $\{1, 2, 3, \dots\}$.

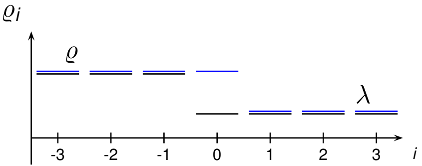
Black TASEP η:
Bernoulli(ϱ) for sites $\{\dots, -3, -2, -1\}$,
Bernoulli(λ) for sites $\{0, 1, 2, \dots\}$.



$h_i(t)$, $g_i(t)$ are the respective numbers of particles jumping over the edge $(i, i+1)$ by time t $(i > 0)$.

Ferrari-Kipnis '95 for TASEP, Part 1

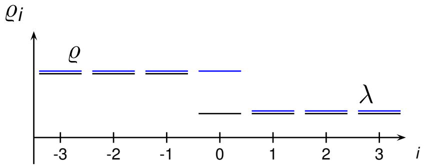
First realization:



Ferrari-Kipnis '95 for TASEP, Part 1

First realization:

- $\omega_i(0) = \eta_i(0) \sim$ Bernoulli(ϱ) for $i < 0$



Ferrari-Kipnis '95 for TASEP, Part 1

First realization:

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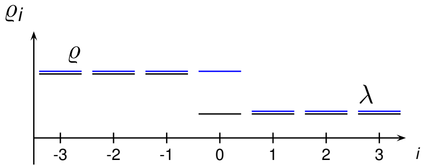

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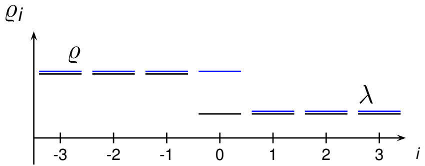

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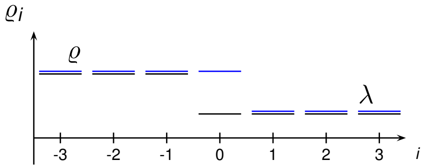

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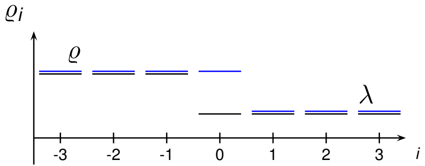
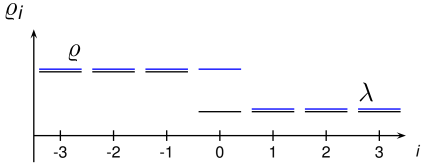


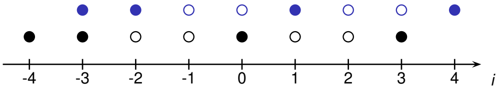
$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = (\varrho - \lambda) \cdot \mathbf{P}\{Q(t) > i\}.$$

Ferrari-Kipnis '95 for TASEP, Part 2

Second realization:

$$\omega_i(t) \equiv \eta_{i-1}(t) \qquad \forall i,\ \forall t.$$





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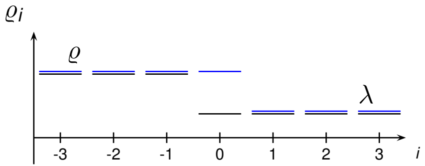

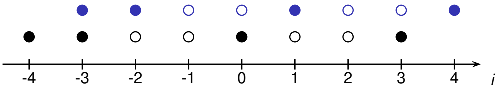

$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = \mathbf{E}(\eta_i(t) - \eta_i(0)) = \mathbf{E}\eta_i(t) - \mathbf{E}\eta_i(0).$$

Ferrari-Kipnis '95 for TASEP

Thus,

$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = (\varrho - \lambda) \cdot \mathbf{P}\{Q(t) > i\},$$
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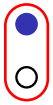
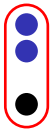
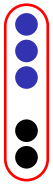
Combine with hydrodynamics to conclude

$$\frac{Q(t)}{t} \Rightarrow \begin{cases} \text{shock velocity} & \text{in a shock,} \\ \mathrm{U}\big(H'(\varrho), H'(\lambda)\big) & \text{in a rarefaction wave.} \end{cases}$$

Let's generalise

Other models have more than 0 or 1 particles per site. How do we start the second class particle?

- Shall we do

 ? Or  ? Or  ? Or . . .

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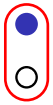
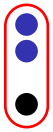
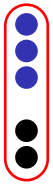
 ? Or  ? Or  ? Or . . .

- Recall for TASEP we increased λ to ϱ by adding or not adding a 2nd class particle.
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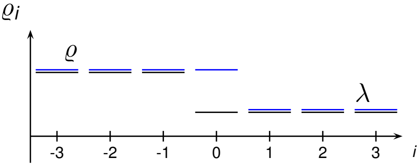

Let's generalise: problems with coupling

Fix $\lambda < \varrho \leq \lambda + 1$. Is there a joint distribution of (ω_0, η_0) such that

- the first marginal is $\omega_0 \sim$ stati. μ^ϱ;
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- *Of course for Bernoulli (TASEP).*

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- *No for Poisson (indep. walkers with $r(\omega_i) = \omega_i$).*
- *Yes for discrete Gaussian (bricklayers with $r(\omega_i) = \mathrm{e}^{\beta\omega_i}$).*

Let's generalise

Keep calm and couple anyway.

Find a coupling measure ν with

- first marginal $\omega_0 \sim$ stati. μ^{ϱ};
- second marginal $\eta_0 \sim$ stati. μ^{λ};
- zero weight whenever $\omega_0 \notin \{\eta_0, \eta_0 + 1\}$.

Not many choices:

$$\begin{aligned} \nu(x, x) &= \mu^{\varrho}\{-\infty \ldots x\} - \mu^{\lambda}\{-\infty \ldots x - 1\}, \\ \nu(x+1, x) &= \mu^{\lambda}\{-\infty \ldots x\} - \mu^{\varrho}\{-\infty \ldots x\}, \\ \nu &= \text{zero elsewhere.} \end{aligned}$$

Let's generalise

$$\nu(x,\,x) = \mu^{\varrho}\{-\infty\dots x\} - \mu^{\lambda}\{-\infty\dots x-1\},$$

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- Bad news: $\nu(x,\,x)$ can be negative (e.g., Geom., Poi).

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Let's generalise

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- Good news: *Who cares?* No 2nd class particle there.
- Good news: $\nu(x+1,\,x)\geq 0$ (attractivity).

We can still use the *signed measure* ν formally, as we only care about $\nu(x+1,\,x)$. Scale this up to get the initial distribution at the site of the second class particle:

$$\mu(\omega_0,\,\eta_0)=\mu(\eta_0+1,\,\eta_0)=\frac{\nu(\eta_0+1,\,\eta_0)}{\sum_x\nu(x+1,\,x)}=\frac{\nu(\eta_0+1,\,\eta_0)}{\varrho-\lambda}.$$

Let's generalise

$$\mu(\omega_0,\,\eta_0)=\frac{\nu(\eta_0+1,\,\eta_0)}{\varrho-\lambda}$$

- is a proper probability distribution;
- actually agrees with the coupling measure ν conditioned on a 2nd class particle when ν behaves nicely (Bernoulli, discr.Gaussian);

Let's generalise

So, how exactly use this to repeat

$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = (\varrho - \lambda) \cdot \mathbf{P}\{Q(t) > i\}$$

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&\mathbf{E}h_i(t) - \mathbf{E}g_i(t) \\
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\end{aligned}$$

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Let's generalise

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&\qquad + \sum_{x} \big[\mathbf{E}(h_i(t)\,|\,\omega_0(0) = x) - \mathbf{E}(g_i(t)\,|\,\eta_0(0) = x+1)\big] \\
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\end{aligned}
$$

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&\qquad + \sum_{x} \big[\mathbf{E}(h_i(t) \,|\, \omega_0(0) = x) - \mathbf{E}(g_i(t) \,|\, \eta_0(0) = x+1)\big] \\
&\qquad \cdot \nu(x+1,\, x) \\
&\quad = \sum_{x} \big[\mathbf{E}(h_i(t) \,|\, \omega_0(0) = x) - \mathbf{E}(g_i(t) \,|\, \eta_0(0) = x+1)\big] \\
&\qquad (\varrho - \lambda) \cdot \mu(x+1,\, x) \\
&\quad = (\varrho - \lambda) \cdot \mathbf{E}^{\mu}(h_i(t) - g_i(t))
\end{aligned}
$$

Let's generalise

$$
\begin{aligned}
&\mathbf{E}h_i(t) - \mathbf{E}g_i(t)\\
&\quad = \sum_{x,\,y} \big[\mathbf{E}(h_i(t)\,|\,\omega_0(0) = x) - \mathbf{E}(g_i(t)\,|\,\eta_0(0) = y)\big] \cdot \nu(x,\, y)\\
&\quad = \sum_{x} \big[\mathbf{E}(h_i(t)\,|\,\omega_0(0) = x) - \mathbf{E}(g_i(t)\,|\,\eta_0(0) = x)\big] \cdot \nu(x,\, x)\\
&\qquad + \sum_{x} \big[\mathbf{E}(h_i(t)\,|\,\omega_0(0) = x) - \mathbf{E}(g_i(t)\,|\,\eta_0(0) = x+1)\big]\\
&\qquad \cdot \nu(x+1,\, x)\\
&\quad = \sum_{x} \big[\mathbf{E}(h_i(t)\,|\,\omega_0(0) = x) - \mathbf{E}(g_i(t)\,|\,\eta_0(0) = x+1)\big]\\
&\qquad (\varrho - \lambda) \cdot \mu(x+1,\, x)\\
&\quad = (\varrho - \lambda) \cdot \mathbf{E}^{\mu}(h_i(t) - g_i(t)) = (\varrho - \lambda) \cdot \mathbf{P}^{\mu}\{Q(t) > i\}.
\end{aligned}
$$

Let's generalise

Theorem

Starting in

$$\bigotimes_{i<0} \mu_i^{\varrho} \otimes \mu_0 \otimes \bigotimes_{i>0} \mu_i^{\lambda},$$

$$\lim_{N\to\infty} \mathbf{P}\left\{ \frac{Q(NT)}{N} > X \right\} = \frac{\varrho(X,\ T) - \lambda}{\varrho - \lambda}$$

where $\varrho(X,\ T)$ *is the entropy solution of the hydrodynamic equation with initial data*

$$\begin{array}{ll} \varrho & \text{on the left} \\ \lambda & \text{on the right.} \end{array}$$

What do we have?

$$\lim_{N\to\infty} \mathbf{P}\left\{\frac{Q(NT)}{N} > X\right\} = \frac{\varrho(X,\,T)-\lambda}{\varrho-\lambda}$$

$\rightsquigarrow$ The solution $\varrho(X,\,T)$ is the distribution of the velocity for Q.

- Shock: distribution is step function, velocity is deterministic (LLN).
- Rarefaction wave: distribution is continuous, velocity is random (e.g., Uniform for TASEP).

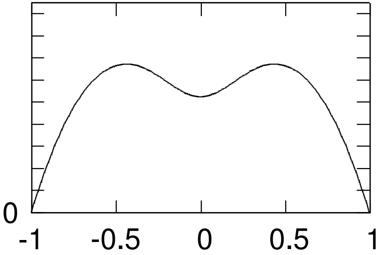
A fun model (B., A.L. Nagy, I. Tóth, B. Tóth)

$\omega_i = -1, 0, 1;$

$$
\begin{aligned}
(0, -1) &\to (-1, 0) && \text{with rate } \frac{1}{2}, \\
(1, 0) &\to (0, 1) && \text{with rate } \frac{1}{2}, \\
(1, -1) &\to (0, 0) && \text{with rate } 1, \\
(0, 0) &\to (-1, 1) && \text{with rate } c.
\end{aligned}
$$

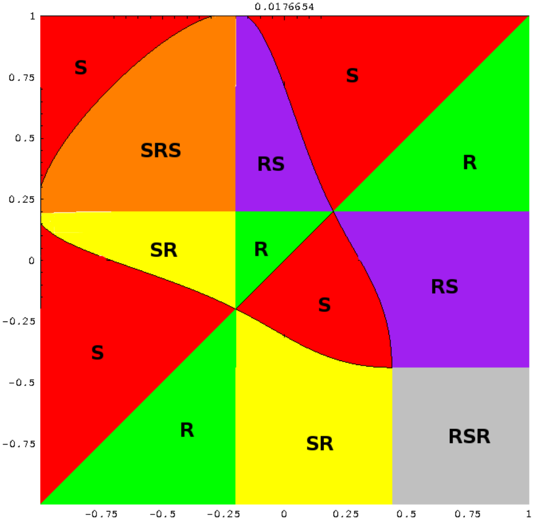
A fun model (B., A.L. Nagy, I. Tóth, B. Tóth)

Hydrodynamic flux $H(\varrho)$, for certain c:



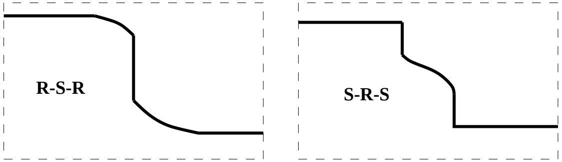
A fun model (B., A.L. Nagy, I. Tóth, B. Tóth)

Here is what can happen (**R**: rarefaction wave, **S**: Shock):



A fun model (B., A.L. Nagy, I. Tóth, B. Tóth)

Examples for $\varrho(T, X)$:



$$\lim_{N\to\infty} \mathbf{P}\left\{\frac{Q(NT)}{N} > X\right\} = \frac{\varrho(X,\ T) - \lambda}{\varrho - \lambda}$$

$\rightsquigarrow$ The solution $\varrho(X,\ T)$ is the distribution of the velocity for Q.

I haven't seen a walk with a random velocity of *mixed distribution* before.

Storytelling. . .

$$\mu(\omega_0,\, \eta_0) = \frac{\nu(\eta_0 + 1,\, \eta_0)}{\varrho - \lambda}$$

In the 1/3-fluctuations papers (B., J. Komjáthy, T. Seppäläinen) we had to start the second class particle in a $\varrho = \lambda$ flat environment. We came up with a measure $\hat{\mu}$ for this which worked nicely with our formulas. *But at that time we had no idea why.*

As it turns out: $\hat{\mu} = \lim_{\lambda \nearrow \varrho} \mu$.

Symmetric case

Everything works with partially asymmetric models (allow left jumps too).

In fact everything works for symmetric models as well. The hydrodynamic scaling is diffusive there with the limit being of heat equation type. In this case:

Symmetric case

Theorem (Symmetric version)

Starting in

$$\bigotimes_{i<0} \mu_i^{\varrho} \otimes \mu_0 \otimes \bigotimes_{i>0} \mu_i^{\lambda},$$

$$\lim_{N\to\infty} \mathbf{P}\Big\{ \frac{Q(NT)}{\sqrt{N}} > X \Big\} = \frac{\varrho(X,\, T) - \lambda}{\varrho - \lambda}$$

where $\varrho(X,\, T)$ *is the entropy solution of the hydrodynamic equation with initial data*

$$\begin{array}{ll} \varrho & \textit{on the left} \\ \lambda & \textit{on the right.} \end{array}$$

SSEP: CLT (of course. . .). Other models: interesting!

One more result

Theorem

If μ^ϱ are the stationary product marginals then, under our initial distribution, $\eta_{Q(t)}(t)$ is stationary in time.

Proof.

Repeat the argument with $\mathbf{E}\Phi\big(\eta_i(t)\big)$ instead of $\mathbf{E}g_i(t)$. $\square$

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Thank you.