# Jacobi triple product via the exclusion process

Joint with
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Large Scale Stochastic Dynamics
Oberwolfach, 18 September, 2019.

# Jacobi triple product

## Theorem

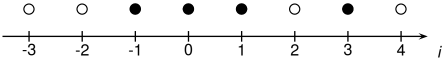
*Let* $|x| < 1$ *and* $y \neq 0$ *be complex numbers. Then*

$$\prod_{i=1}^{\infty}(1 - x^{2i})\left(1 + \frac{x^{2i-1}}{y^2}\right)(1 + x^{2i-1}y^2) = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

Mostly appears in number theory and combinatorics of partitions.

It also follows from ASEP and ZRP (for real $x$, $y$ only).

# Asymmetric simple exclusion

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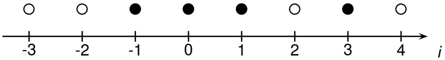


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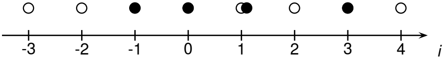


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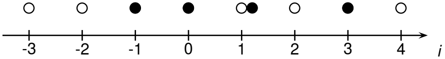


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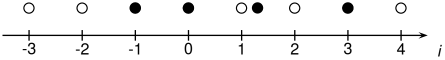


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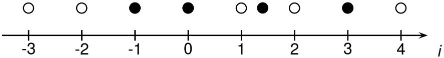


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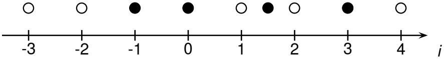


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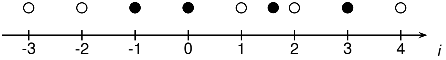


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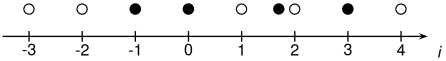


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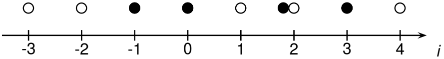


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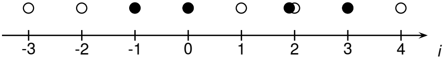


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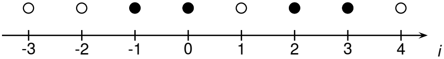


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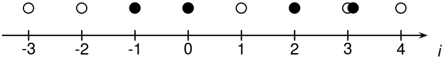


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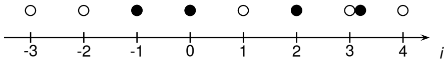


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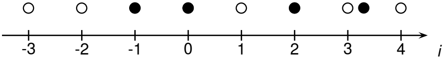


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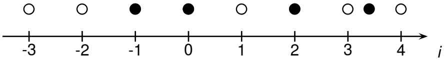


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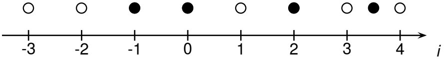


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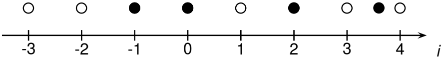


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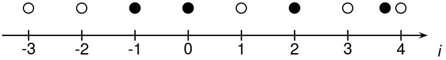


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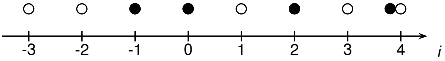


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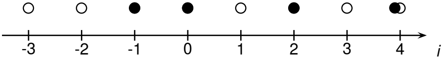


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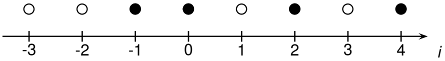


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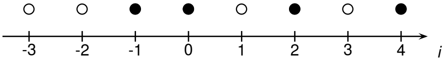


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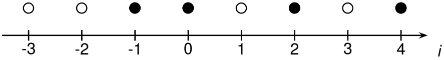


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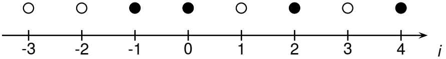


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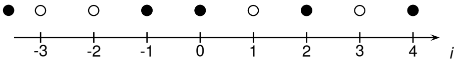


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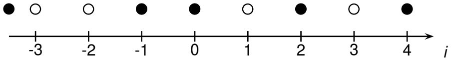


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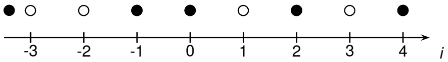


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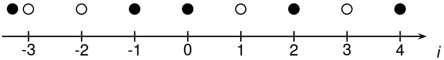


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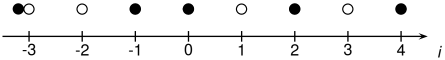


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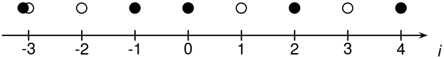


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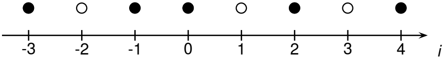


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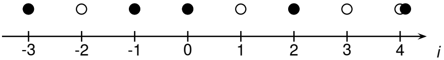


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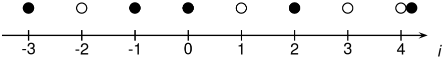


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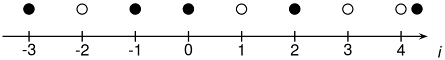


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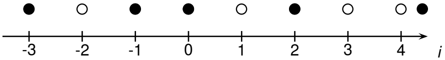


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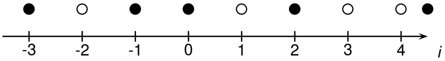


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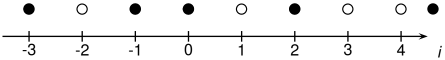


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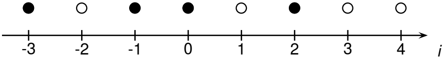


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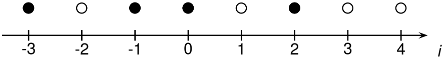


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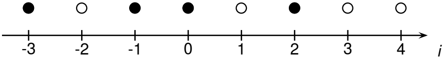


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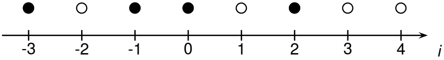


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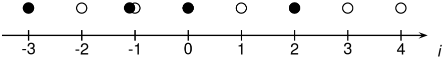


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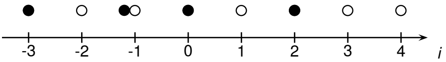


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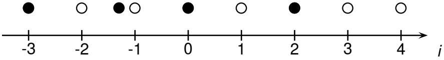


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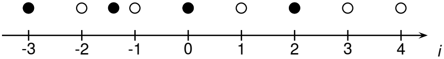


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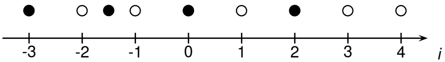


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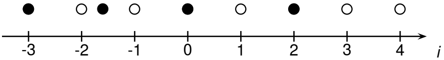


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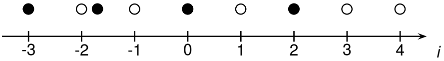


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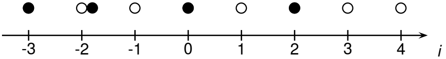


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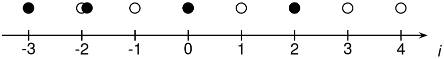


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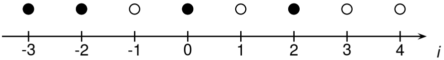


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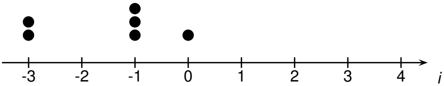


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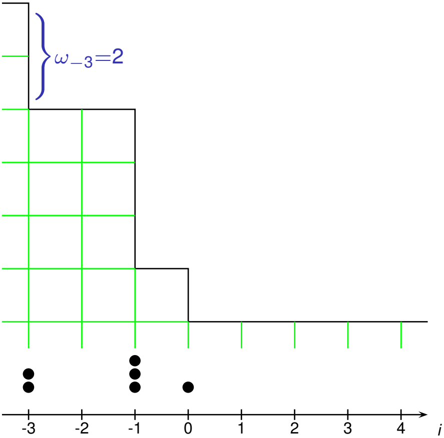


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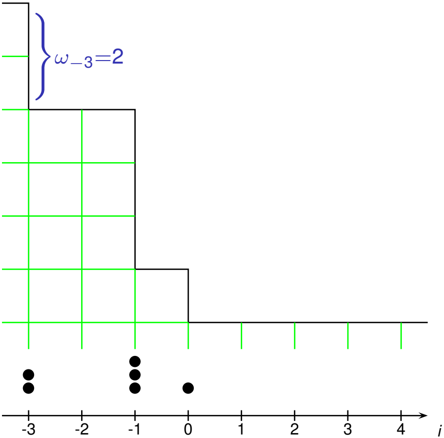


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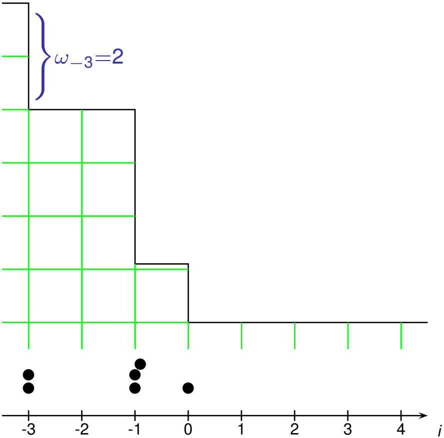


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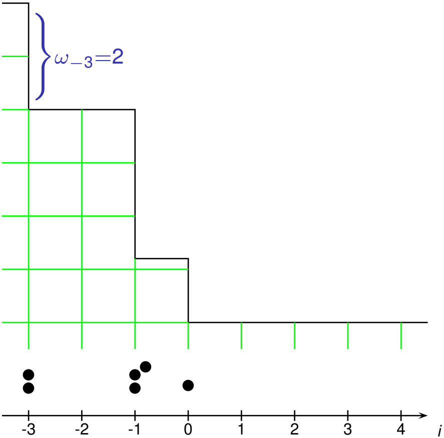


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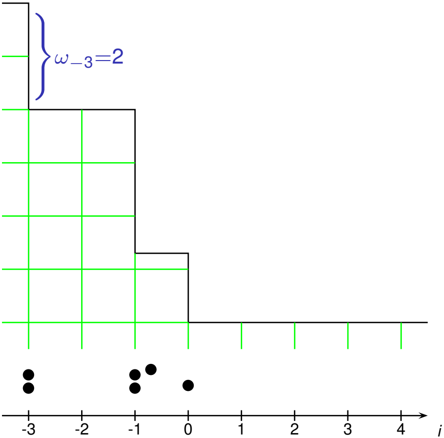


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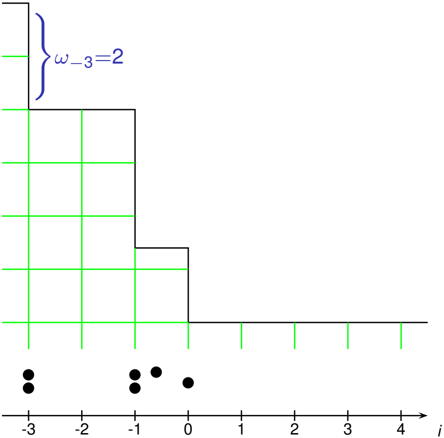


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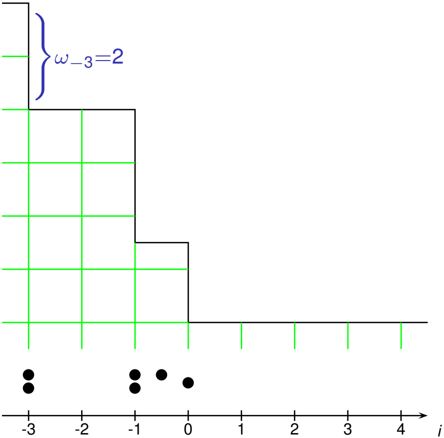


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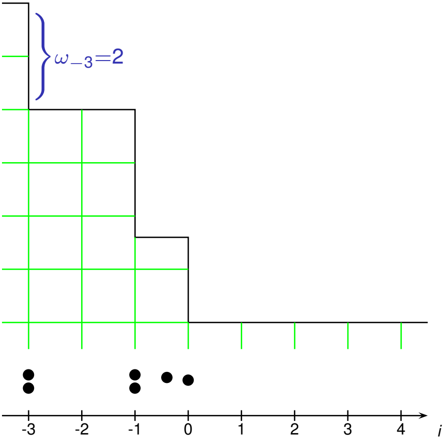


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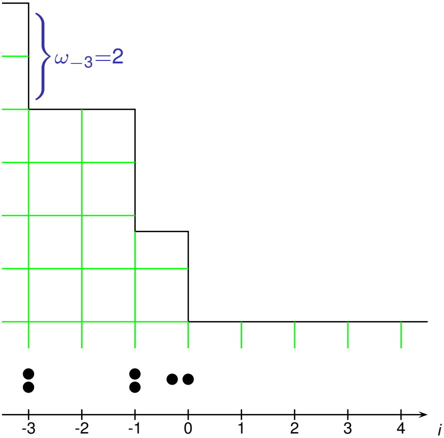


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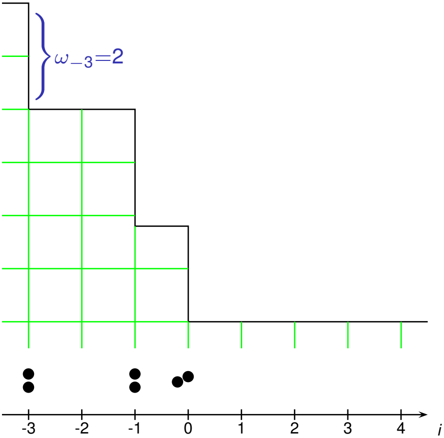


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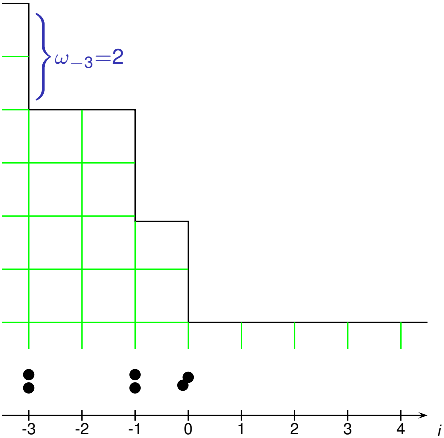


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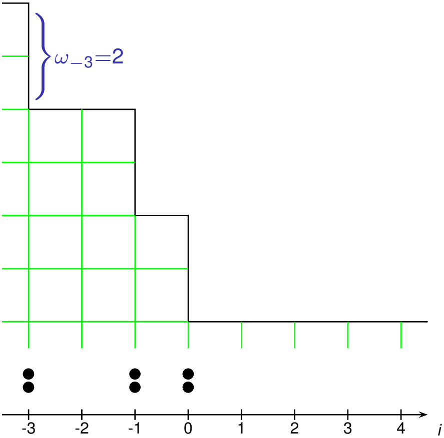


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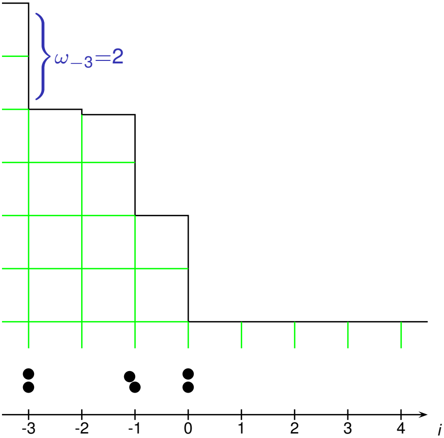


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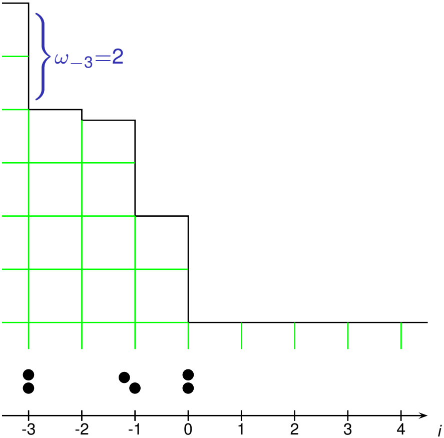


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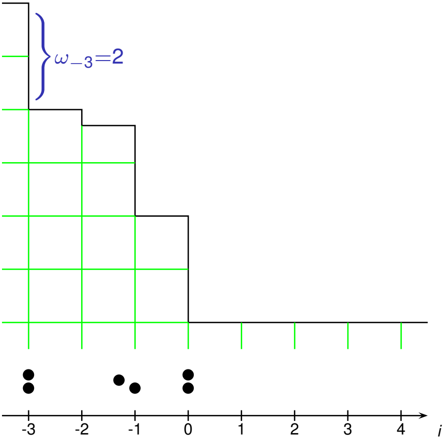


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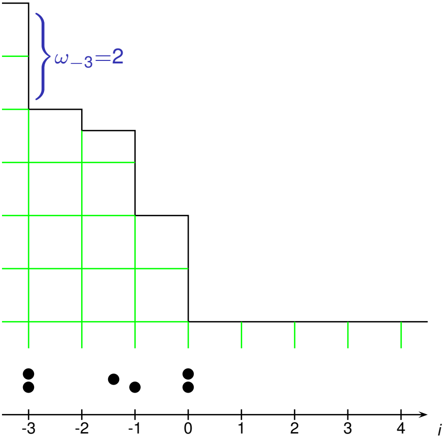


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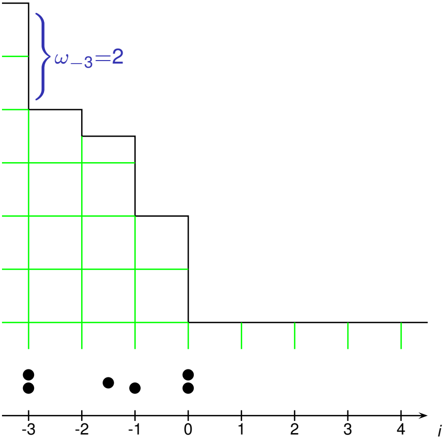


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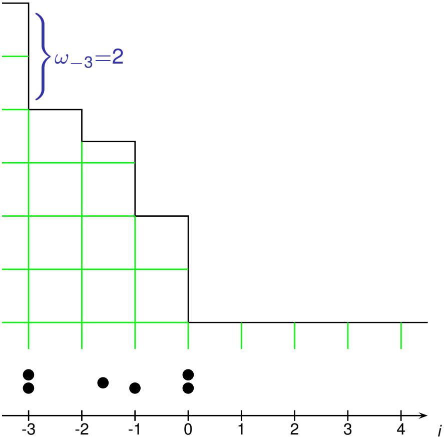


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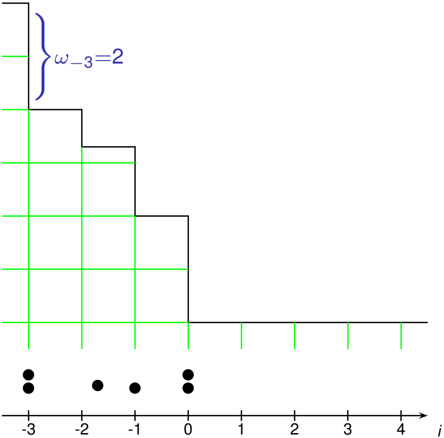


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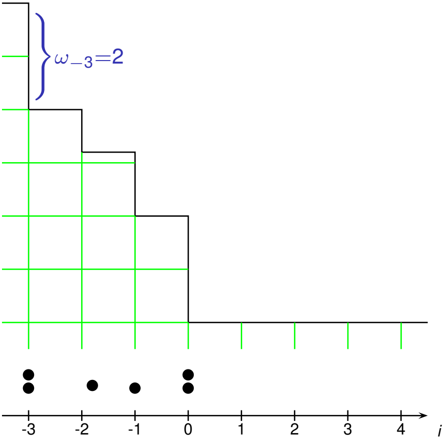


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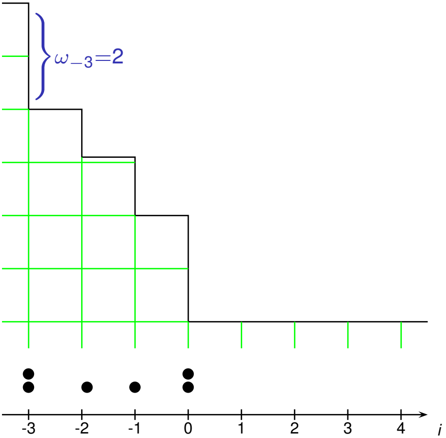


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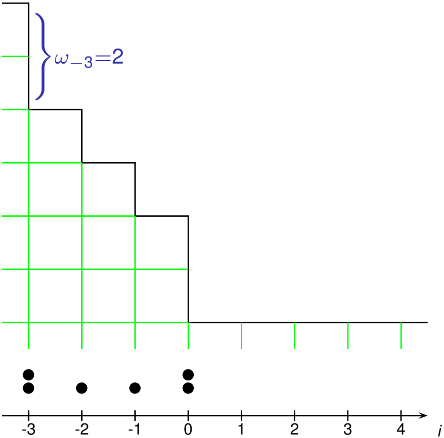


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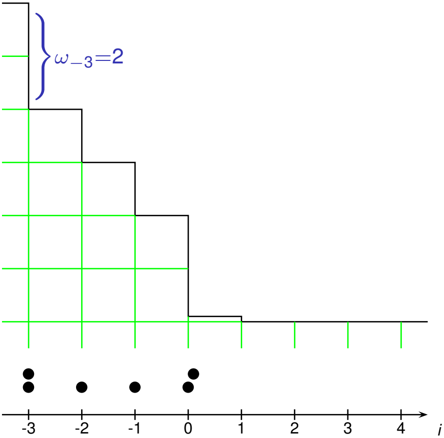


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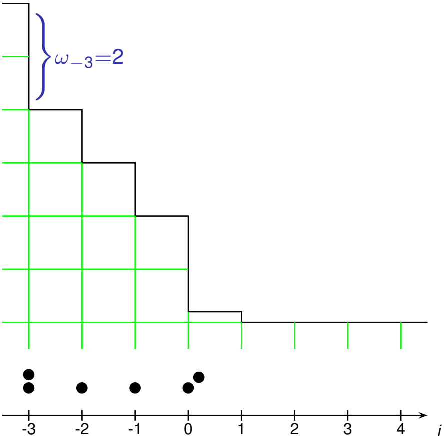


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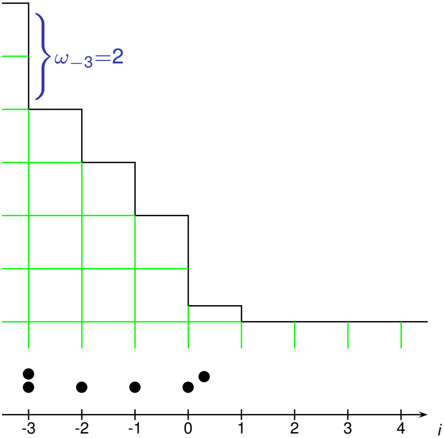


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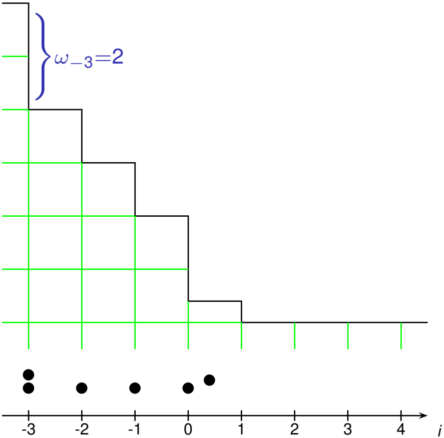


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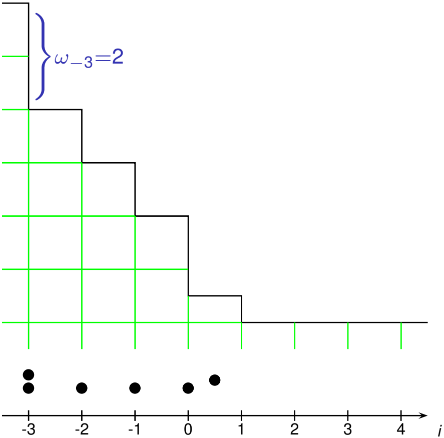


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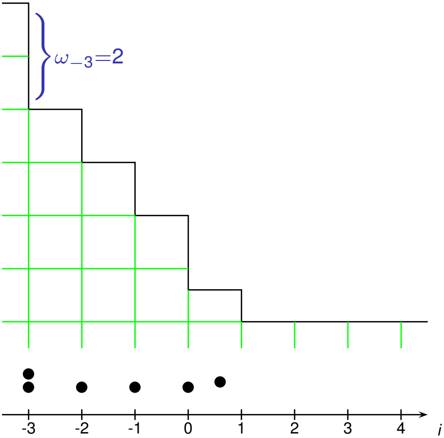


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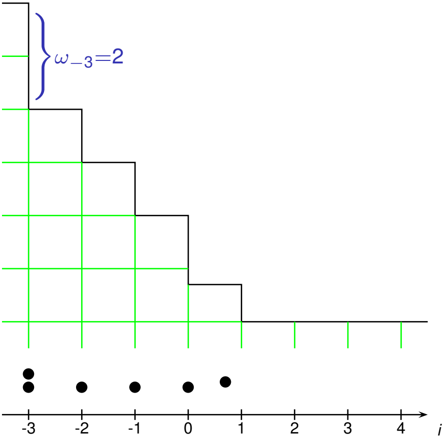


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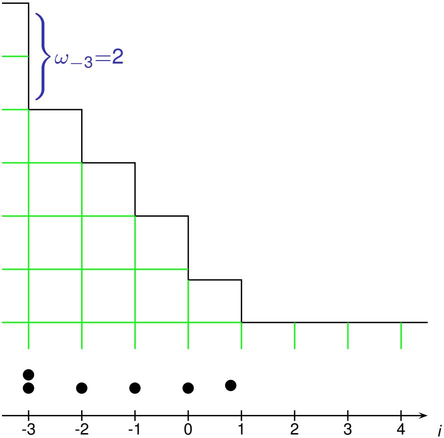


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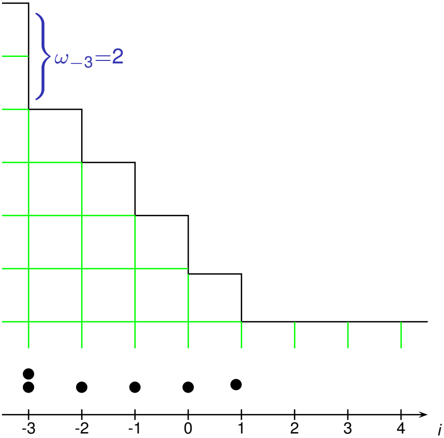


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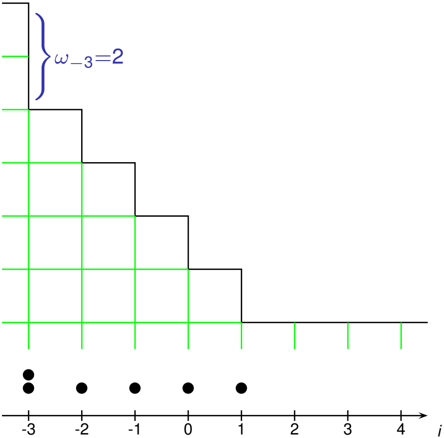


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# The asymmetric zero range process

We need $r$ non-decreasing and assume, as before, $q = 1 - p < p$.

Examples:

- 'Classical' ZRP: $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$.
- Independent walkers: $r(\omega_i) = \omega_i$.

# Product blocking measures

Can we have a reversible stationary distribution in product form:

$$\underline{\mu}(\underline{\omega}) = \bigotimes_i \mu_i(\omega_i);$$

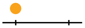
$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}\big(\underline{\omega} \to \underline{\omega}^{i\curvearrowright i+1}\big) = \underline{\mu}\big(\underline{\omega}^{i\curvearrowright i+1}\big) \cdot \text{rate}\big(\underline{\omega}^{i\curvearrowright i+1} \to \underline{\omega}\big) \quad ?$$

Here

$$\underline{\omega}^{i\curvearrowright i+1} = \underline{\omega} - \underline{\delta}_i + \underline{\delta}_{i+1}.$$

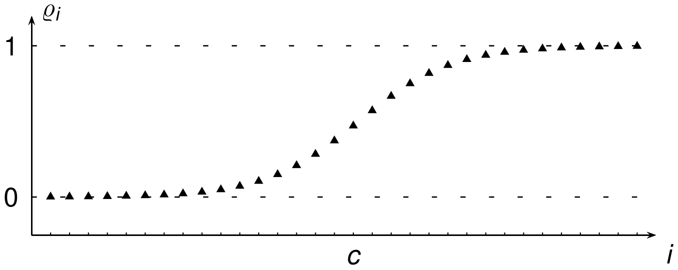
# Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}\big(\underline{\eta} \to \underline{\eta}^{i\curvearrowright i+1}\big) = \underline{\mu}\big(\underline{\eta}^{i\curvearrowright i+1}\big) \cdot \text{rate}\big(\underline{\eta}^{i\curvearrowright i+1} \to \underline{\eta}\big)$$

ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$;  $\underline{\eta}$

$$\varrho_i(1-\varrho_{i+1}) \cdot p = (1-\varrho_i)\varrho_{i+1} \cdot q$$

Solution: $$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1+(\frac{p}{q})^{i-c}} = \frac{1}{(\frac{q}{p})^{i-c}+1}$$



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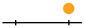
ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$;  $\underline{\eta}^{i\curvearrowright i+1}$

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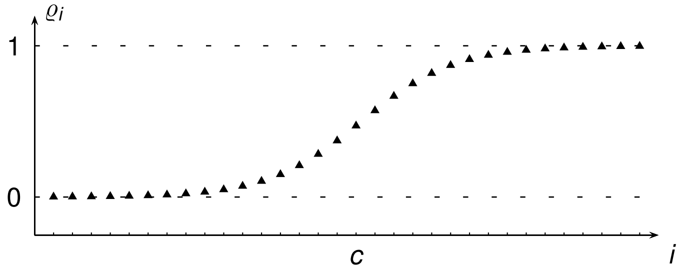


# Asymmetric zero range process

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}\big(\underline{\omega} \to \underline{\omega}^{i \curvearrowright i+1}\big) = \underline{\mu}(\underline{\omega}^{i \curvearrowright i+1}) \cdot \text{rate}\big(\underline{\omega}^{i \curvearrowright i+1} \to \underline{\omega}\big) \quad ?$$

AZRP:

$$\mu_i(\omega_i)\mu_{i+1}(\omega_{i+1}) \cdot p\mathbf{1}\{\omega_i > 0\} = \mu_i(\omega_i - 1)\mu_{i+1}(\omega_{i+1} + 1) \cdot q$$

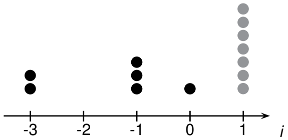
Solution: $\mu_i \sim \text{Geometric}\Big(1 - \Big(\dfrac{p}{q}\Big)^{i-\text{const}}\Big).$

# State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\Big(1 - \Big(\frac{p}{q}\Big)^{i-\text{const}}\Big).$$

$\rightsquigarrow$ we have a problem: cannot do this for all $i$! We'll pick const $= 1$ and have a *right boundary* instead.



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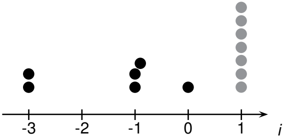

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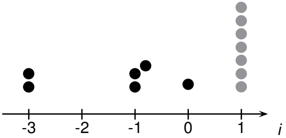

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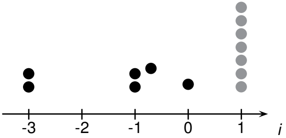

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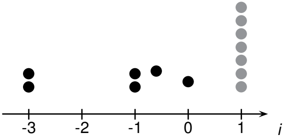

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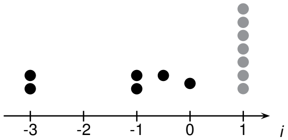

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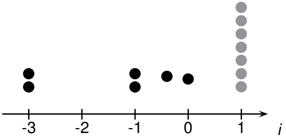

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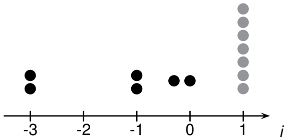

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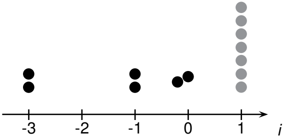

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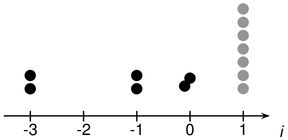

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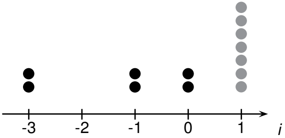

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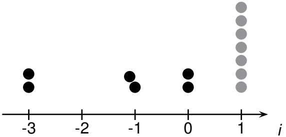

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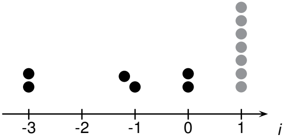

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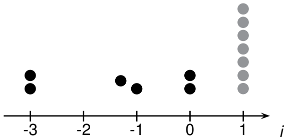

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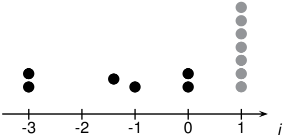

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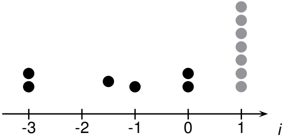

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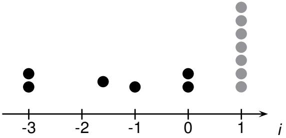

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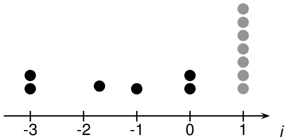

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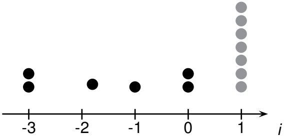

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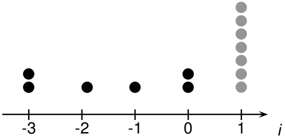

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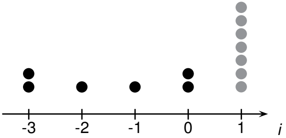

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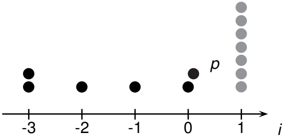

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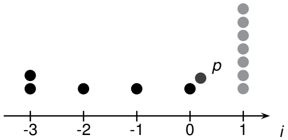

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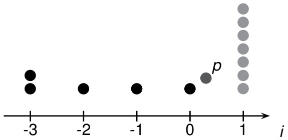

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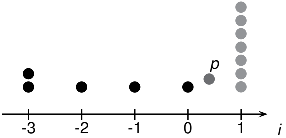

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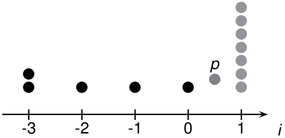

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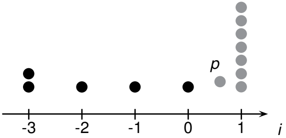

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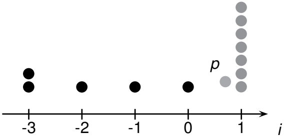

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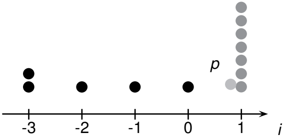

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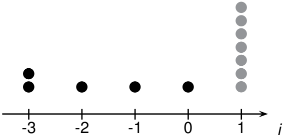

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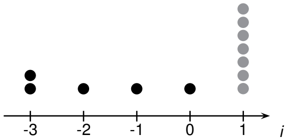

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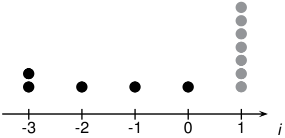

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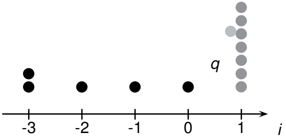

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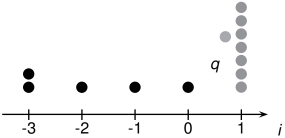

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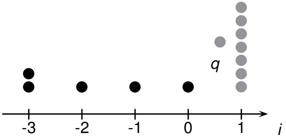

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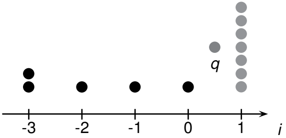

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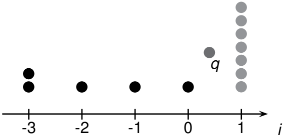

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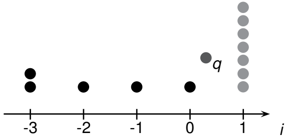

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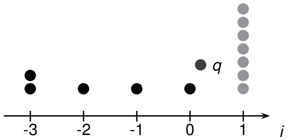

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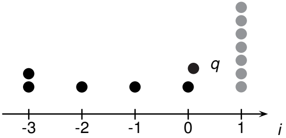

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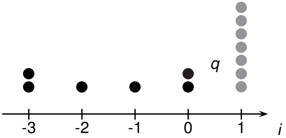
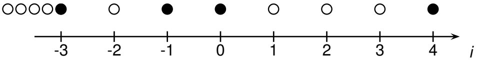



$\rightsquigarrow$ The product measure stays stationary on the half-line.

# Lay down / stand up

ASEP



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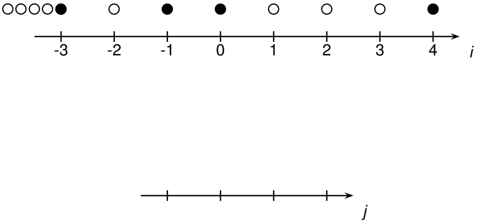

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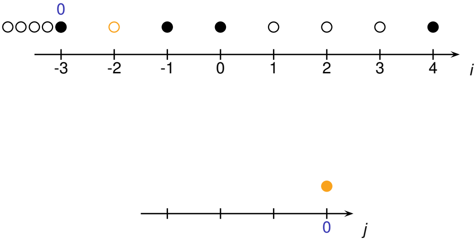

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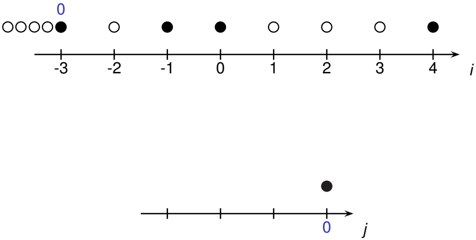

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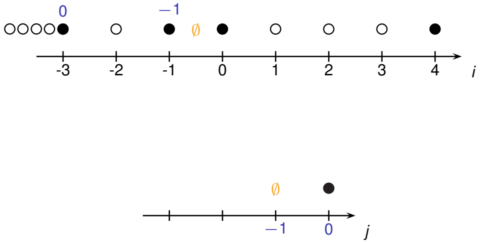
ASEP



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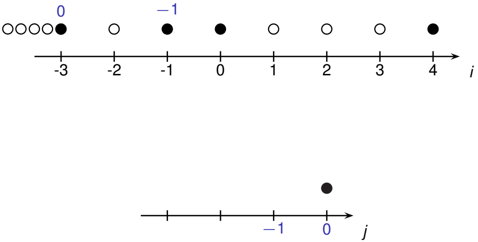

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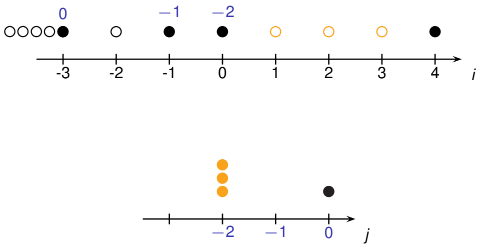

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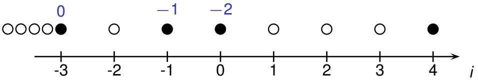


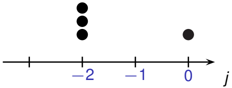

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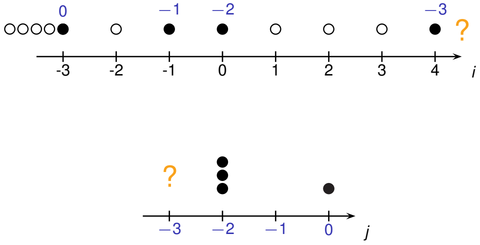

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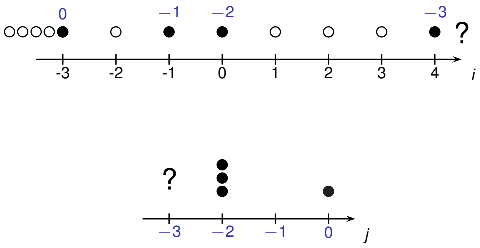

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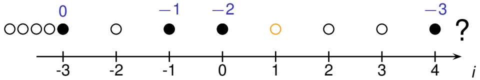


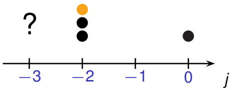

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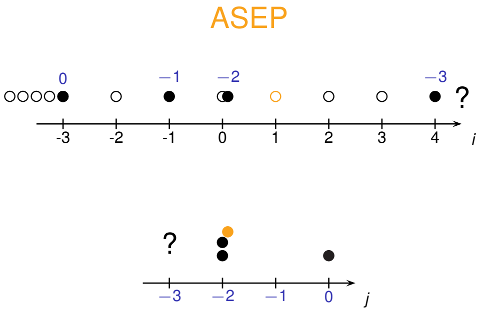

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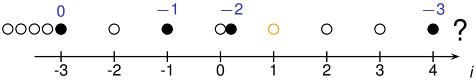
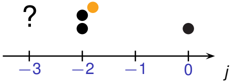
## ASEP



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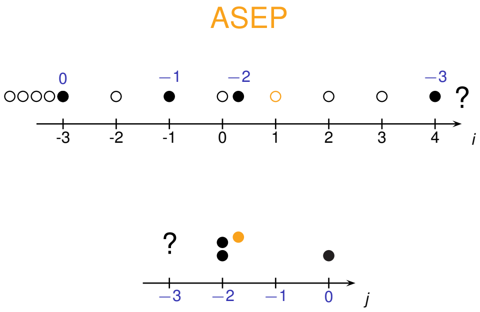

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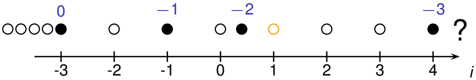


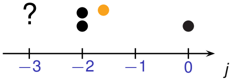

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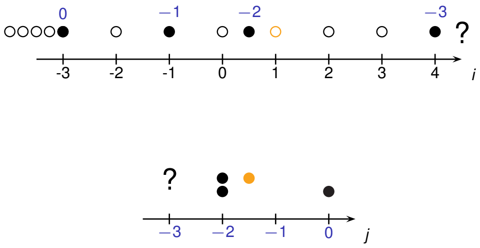

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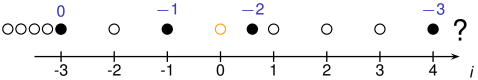


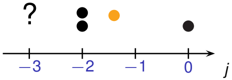

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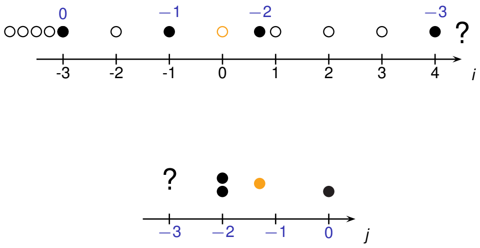

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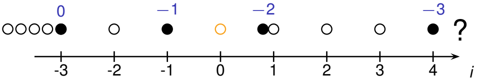


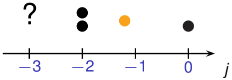

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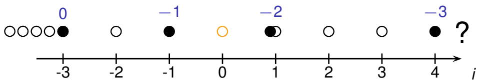


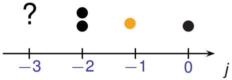

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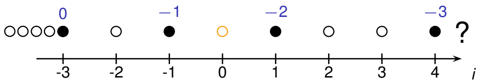
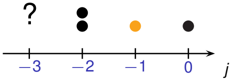
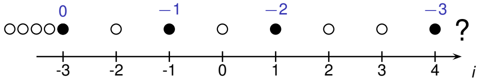


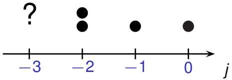



# Lay down / stand up

## ASEP




## AZRP



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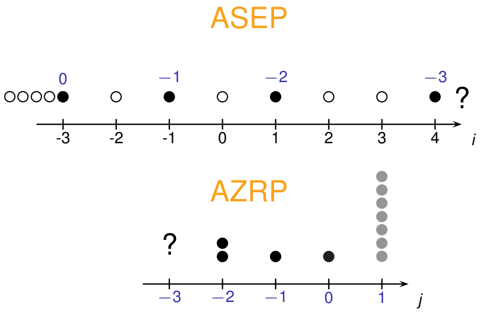

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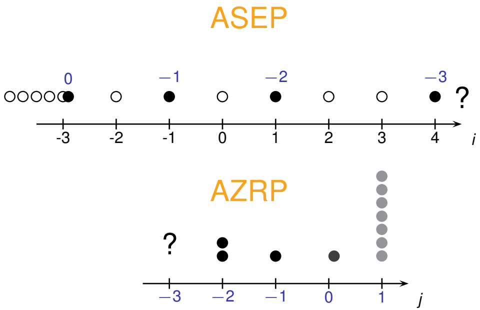

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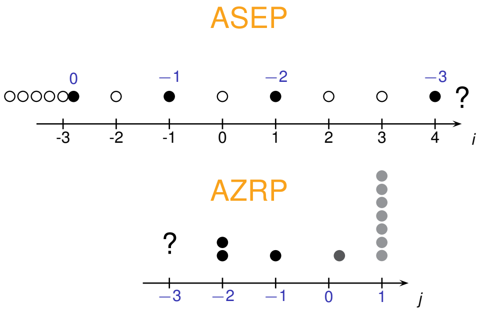

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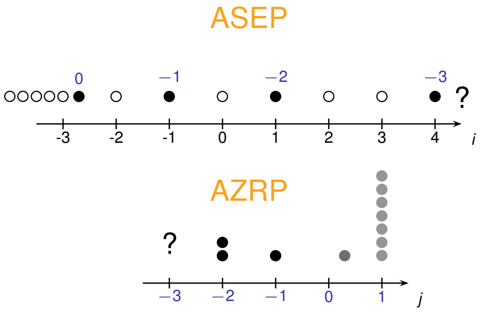

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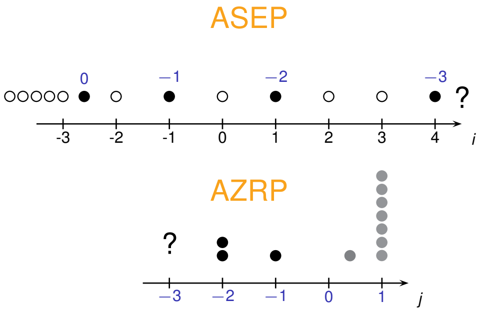

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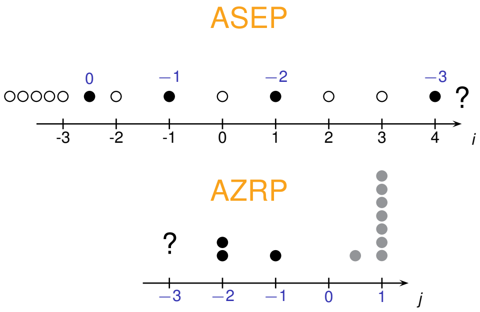

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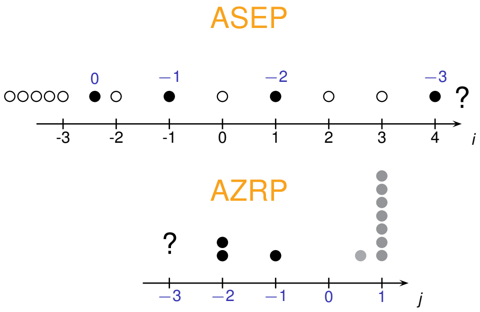

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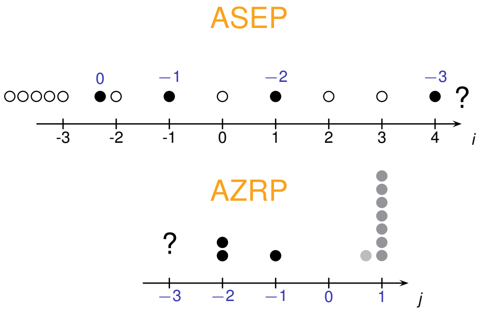

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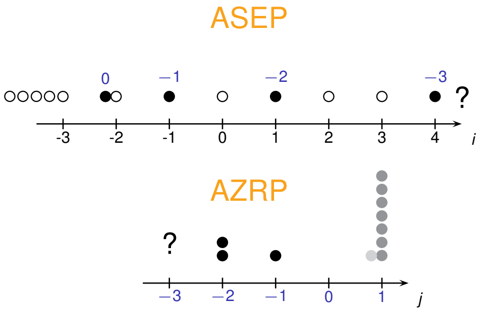

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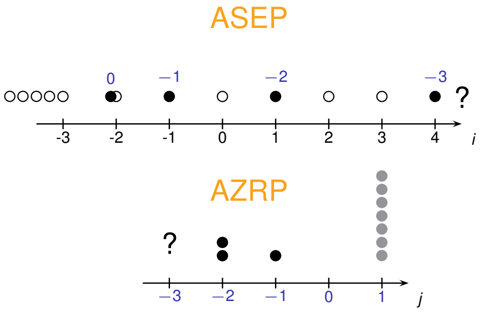

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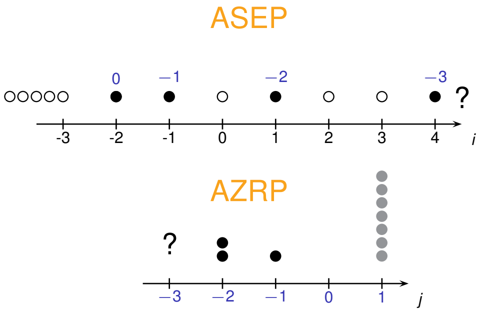


ASEP $\stackrel{T^n}{=}$ AZRP

$$\underline{\nu}^n \stackrel{T^n}{=} \prod_{i\le 0} \text{Geometric}\Big(1-\Big(\frac{p}{q}\Big)^{i-1}\Big)$$

since stationary distributions of countable irreducible Markov chains are unique.

$\rightsquigarrow$ The Jacobi Triple Product follows.

# Jacobi triple product

## Theorem

*Let* $|x| < 1$ *and* $y \neq 0$ *be complex numbers. Then*

$$\prod_{i=1}^{\infty}(1 - x^{2i})\left(1 + \frac{x^{2i-1}}{y^2}\right)(1 + x^{2i-1}y^2) = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

Thank you.