Electric network for irreversible walks - but is it useful?

Work in progress, joint with
Áron Folly

Márton Balázs

University of Bristol

Large Scale Stochastic Dynamics
30 October, 2013.

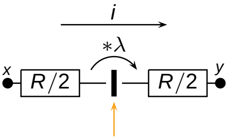
Irreversible chains and electric networks

- The part
- From network to chain
- From chain to network
- Effective resistance
- What works

The electric network

- Reducing the network
- Nonmonotonicity
- What doesn’t work

The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

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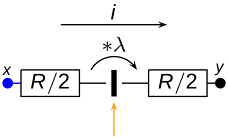

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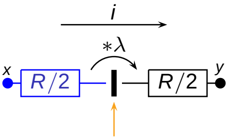

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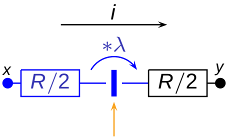

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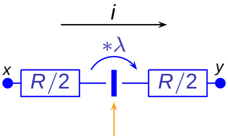

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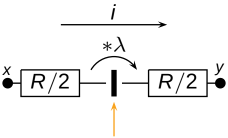

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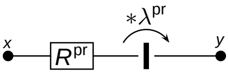
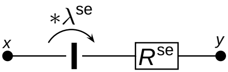


$$(u_x - i \cdot R^{\mathrm{pr}}) \cdot \lambda^{\mathrm{pr}} = u_y$$



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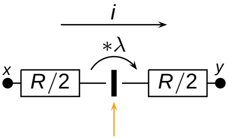

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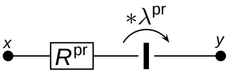

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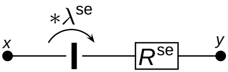

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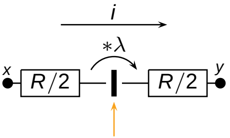

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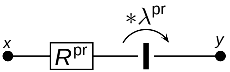

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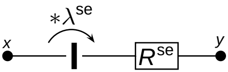

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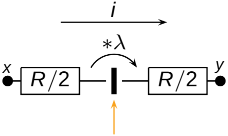

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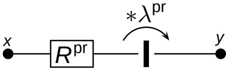

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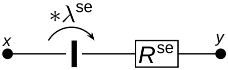

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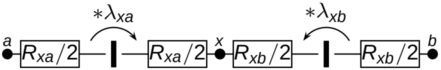

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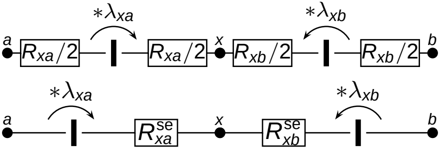

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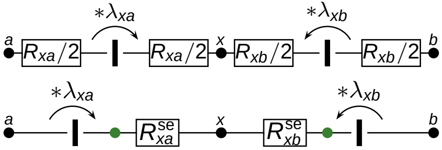

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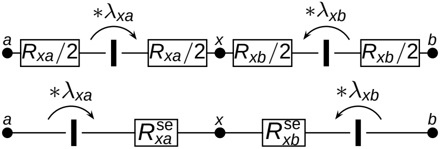

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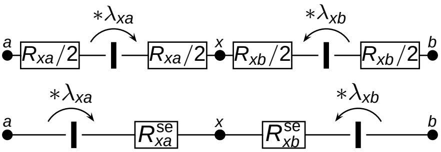

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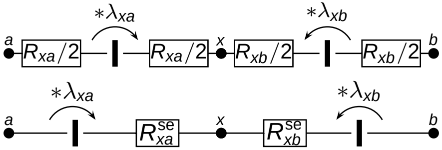

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From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

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$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

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Stationary distribtuion:

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From chain to network

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Reversed chain: Replace P_{xy} by $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$.

$\rightsquigarrow$ D_{xy} stays, λ_{xy} reverses to λ_{yx}.

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Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$.
Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

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$\rightsquigarrow$The Markov property has another solution: constant u_b potentials with zero external currents.

$\rightsquigarrow$ Take difference + use linearity: We have effective resistance.

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Modulo normalisation...

$$\mathbf{E}_a(\text{signed current } x \to y \text{ before absorbed in } b) = \hat{i}_{xy}.$$

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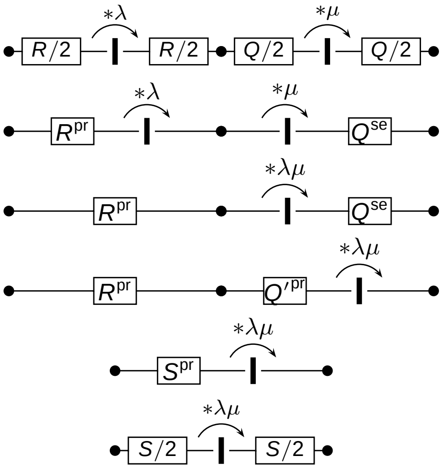
Theorem

Commute time $= R_{eff} \cdot$ *all conductances.*

The electric network

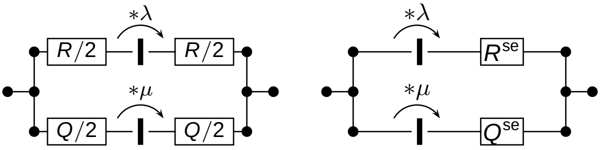
Series:



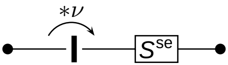
$$S = R\frac{(\lambda+1)\mu}{\lambda\mu+1} + Q\frac{\mu+1}{\lambda\mu+1}.$$

The electric network

Parallel:



Compare this with



$$S = \frac{RQ}{R+Q}$$

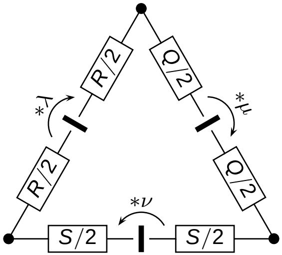
$$\nu = \frac{Q\lambda(\mu+1) + R\mu(\lambda+1)}{Q(\mu+1) + R(\lambda+1)}.$$

The electric network

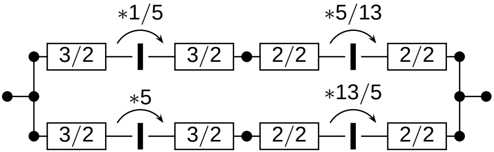
Star-Delta:

Star to Delta works,

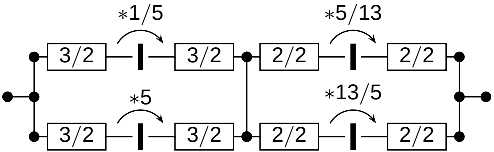
Delta to star only works if Delta does not produce a circular current by itself ($\lambda\mu\nu = 1$).



Nonmonotonicity



$$R_{\text{eff}} = \frac{27}{14}.$$



$$R_{\text{eff}} = \frac{5}{2} = \frac{35}{14}.$$

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Thank you.