Nonexistence of bi-infinite geodesics in exponential last passage percolation - a probabilistic way

Joint with
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Large Scale Stochastic Dynamics
Oberwolfach, 17 September, 2019.

Last passage percolation

Geodesics

The result

Tools

New boundary

Crossing

Stationarity

Proof

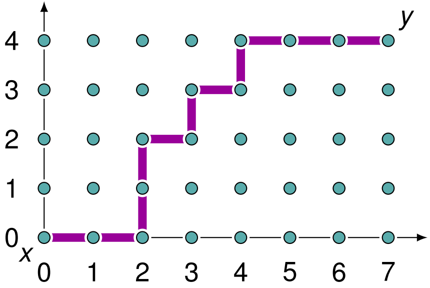
When it’s too flat

No sharp turns please

The diagonal case

Last passage percolation

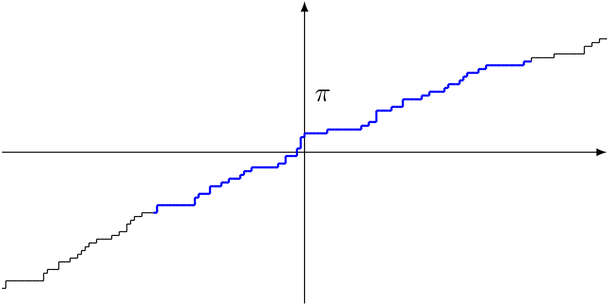
- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{x,y}$ from x to y is the a.s. unique heaviest up-right from x to y.
- $G_{x,y} = \sum_{z \in \pi_{x,y}} \omega_z$ is its weight.



Surface growth, TASEP, queuing…

Bi-infinite geodesics

A bi-infinite up-right path is a *bi-infinite geodesic*, if any of its segments is itself a geodesic between the two endpoints.



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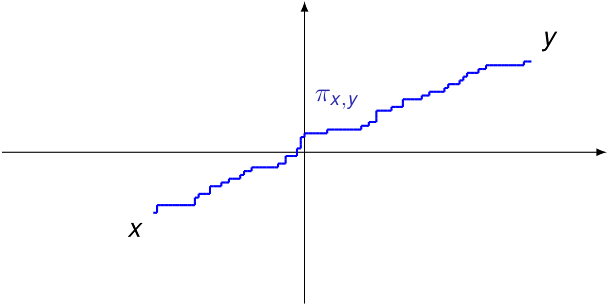

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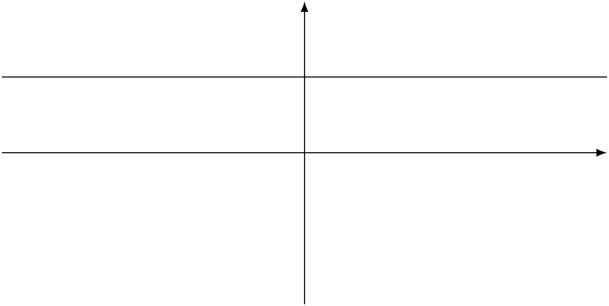

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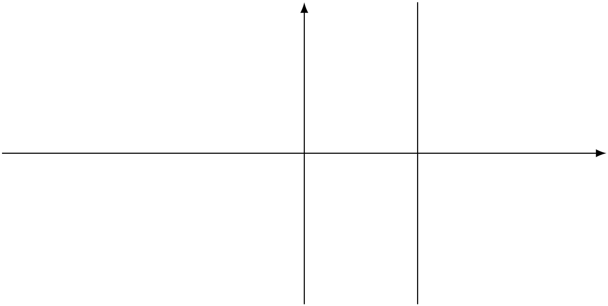

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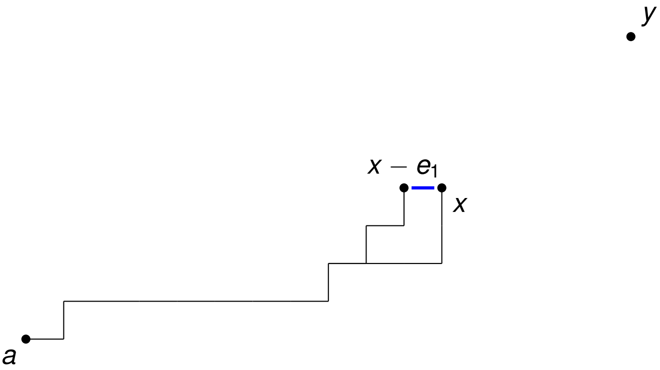
- We only need a bit of random walks, queuing, couplings.

1. Increments as new boundary



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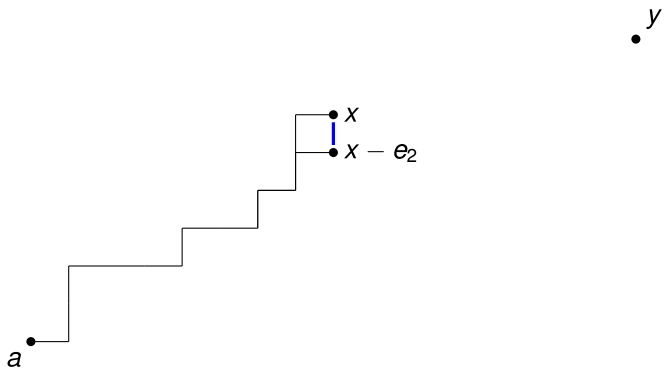

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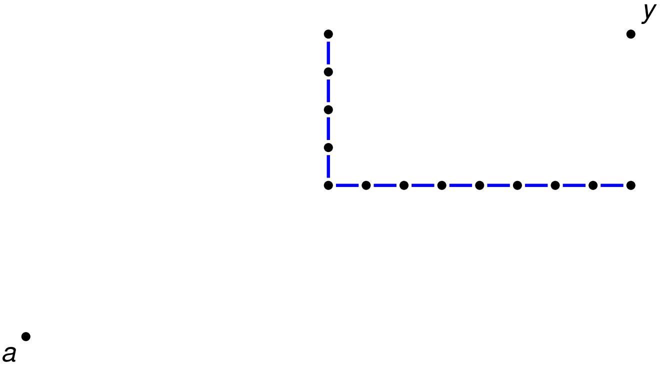

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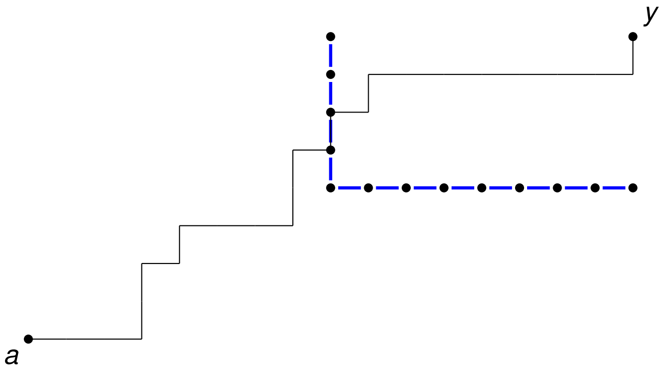

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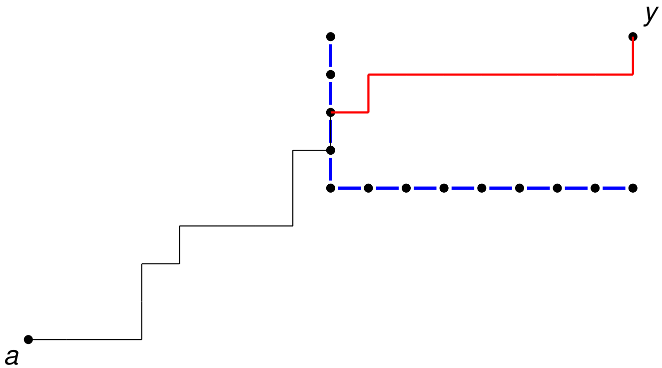

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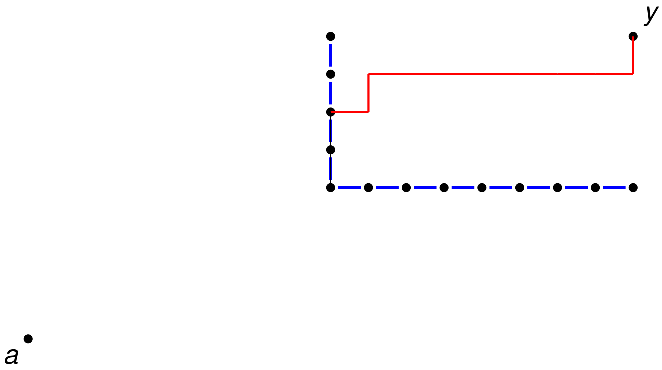

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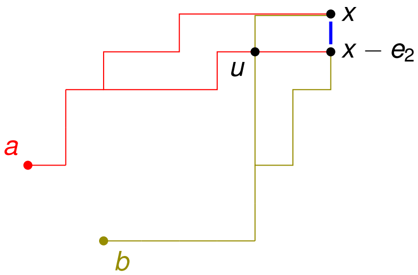
$\rightsquigarrow$ Act as boundary weights for a smaller, embedded model.

2. Crossing lemma

Let a be North-West of b.



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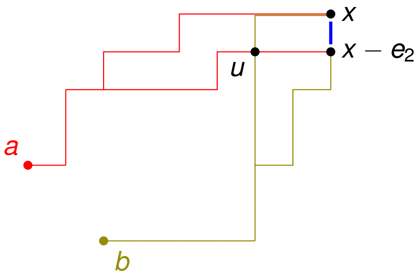

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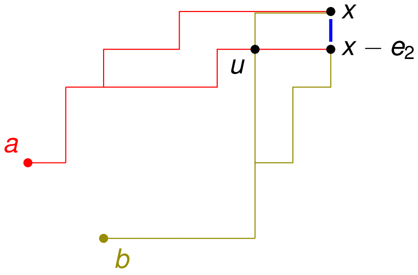

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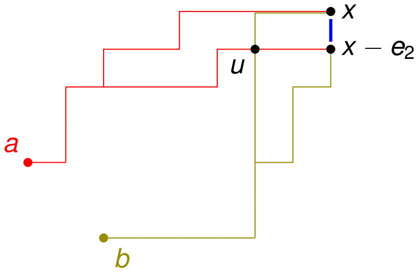

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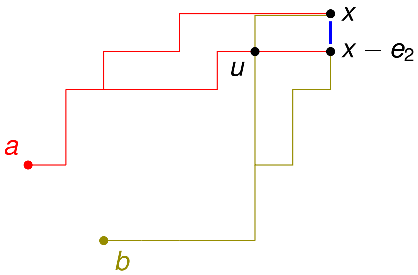

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Similarly, $I_x^{(a)} \leq I_x^{(b)}$.

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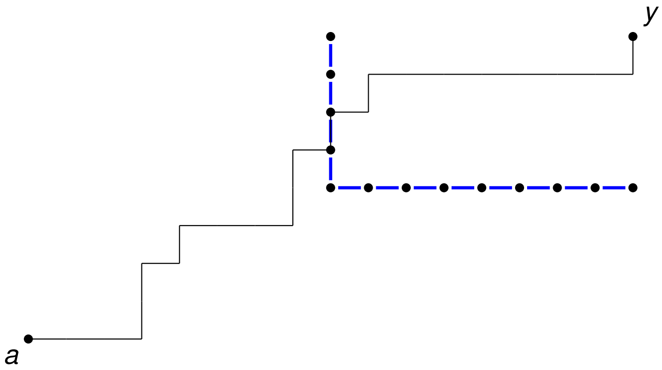

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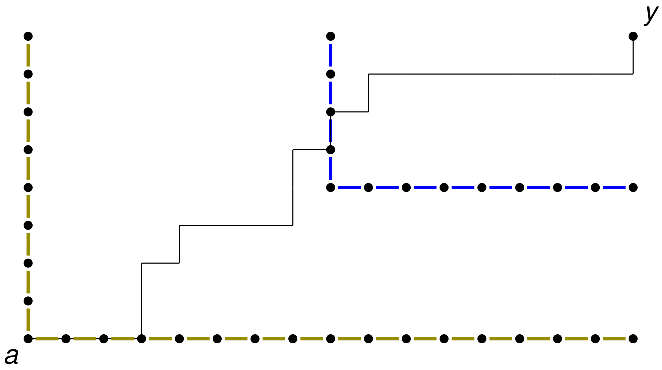
Replace the boundary to | $\sim \mathrm{Exp}(\varrho)$, — $\sim \mathrm{Exp}(1-\varrho)$ independent.



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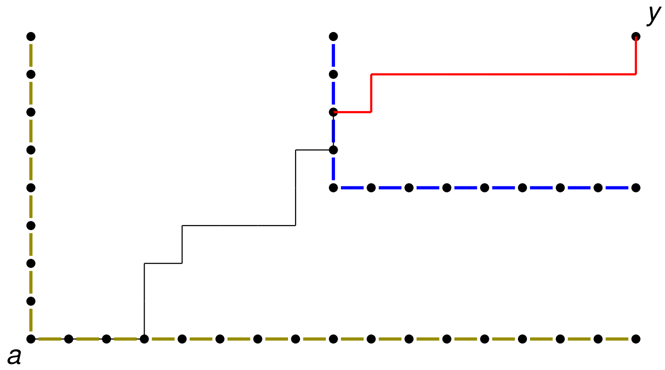

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The embedded model has the same structure.

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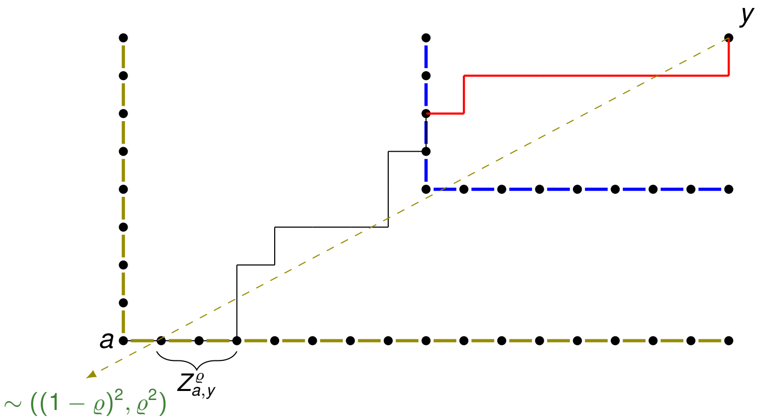
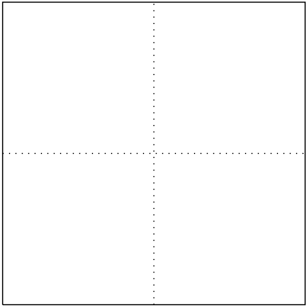


B., Cator, Seppäläinen '06: $\mathbb{P}\{|Z^{\varrho}_{a,y}| \geq \ell\} \leq \text{box}^2/\ell^3$, *good directional control*.

Proof

Take larger and larger boxes and show that geodesics avoid more and more the origin when crossing from one side to the other (Newman '95).



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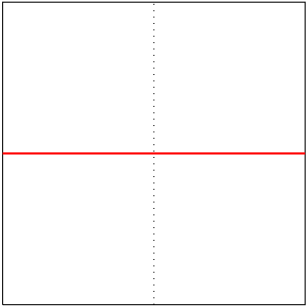

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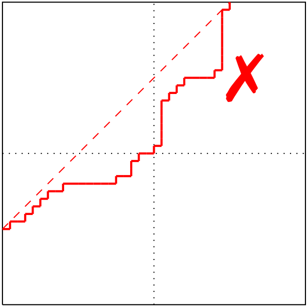

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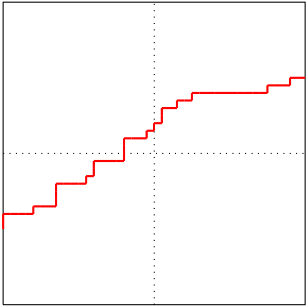
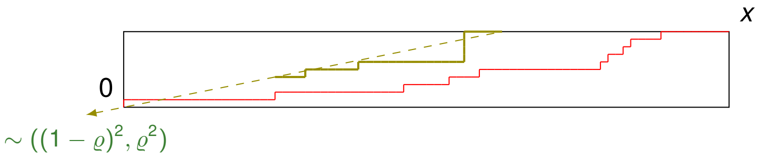
2. Otherwise, geodesics don't like to turn too much.
3. We are left with roughly diagonal ones, show that they fluctuate too much.



1. When it's too flat

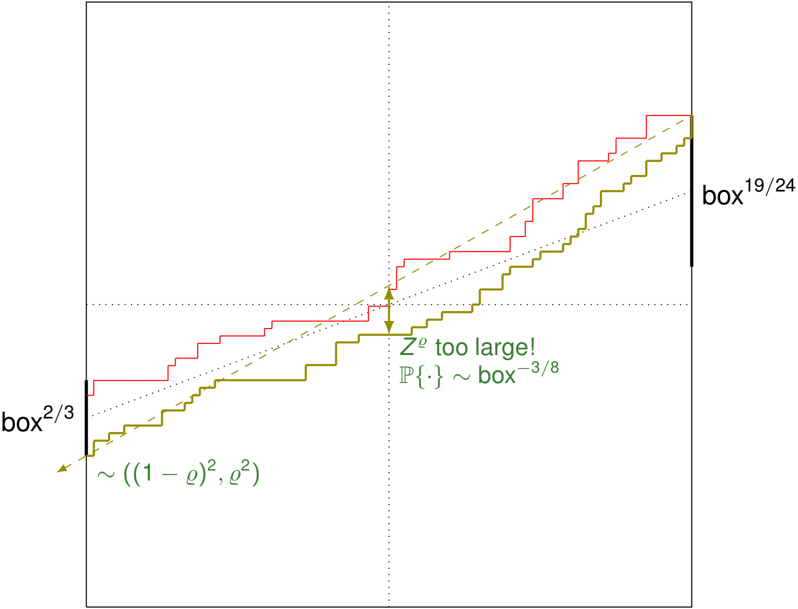


Take ϱ small, but not too small compared to x, so that with large probability the green stationary path exits on the left of x (use the shape function here).

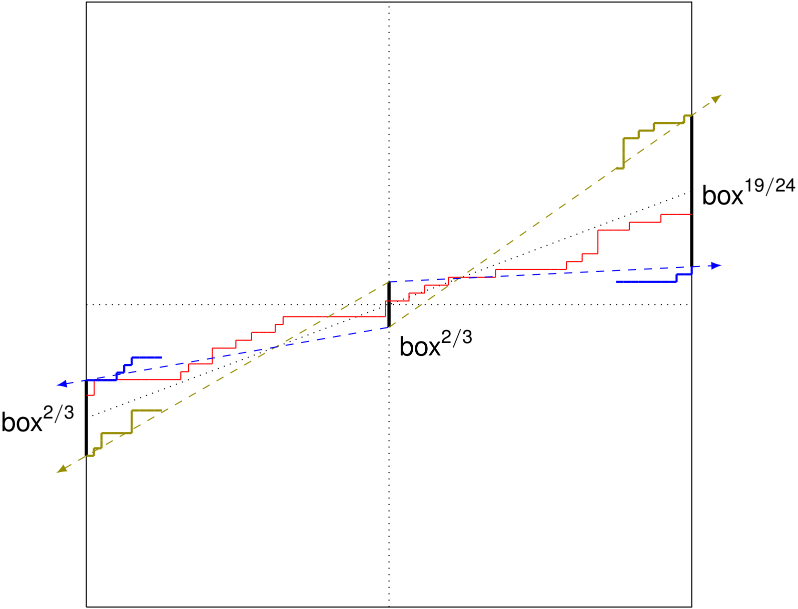
$$G_{0,x} - G_{e_2,x} = \hat{J}_{e_2} \geq \hat{J}^{\varrho}_{e_1} \sim \mathrm{Exp}(\varrho),$$

and can take $\varrho \to 0$ as the box flattens with $x \to \infty$. So, it's never worth leaving from e_2 compared from 0.

2. No sharp turns please



3. The diagonal case: the attack of the geodesics



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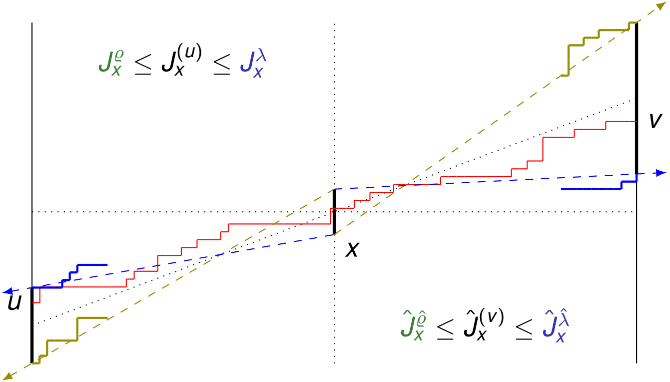

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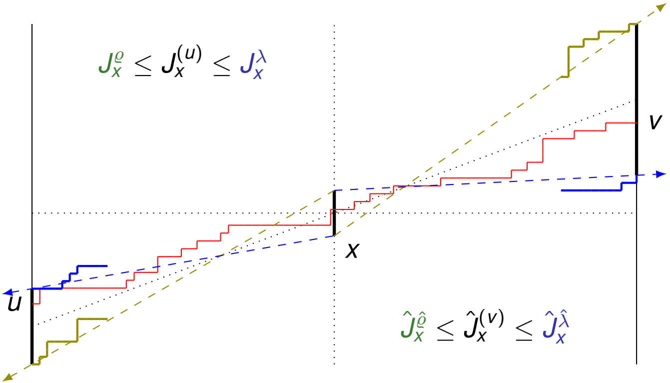

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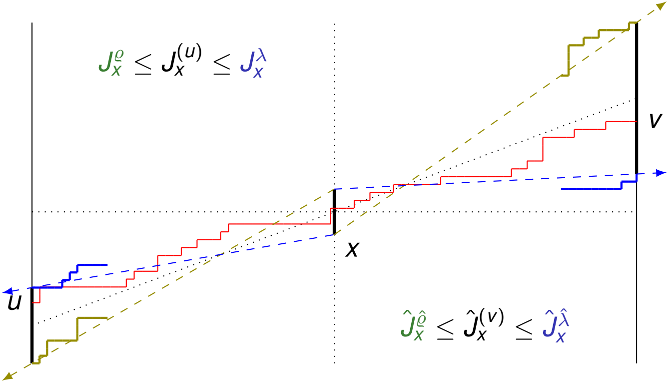

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- The bounds $J_j^{\varrho} - \hat{J}_j^{\hat{\lambda}} \leq J_j^{(u)} - \hat{J}_j^{(v)} \leq J_j^{\lambda} - \hat{J}_j^{\hat{\varrho}}$ are independent and nicely distributed.

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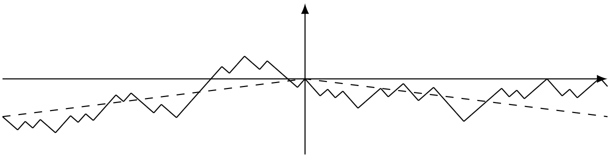
The problem boils down to whether a simple random walk minus drift reaches its maximum at 0. The answer is an asymptotic *no*, the drift is beaten by the fluctuations.



$\mathbb{P}\{\cdot\} \sim \text{box}^{-2/5}$.

So, the counting

- Intervals on the left are of size $\sim \text{box}^{2/3}$.
- Have $\text{box}/\text{box}^{2/3} \sim \text{box}^{1/3}$ many of these.

$\rightsquigarrow$ Union bound:

$$\begin{aligned}\mathbb{P}\{\text{any geodesic crosses } 0\} &\sim \text{box}^{1/3} \cdot \left(\text{box}^{-3/8} + \text{box}^{-2/5}\right) \\ &= \text{box}^{-1/24} \to 0.\end{aligned}$$

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These sharper, probabilistic estimates open up the way to further understanding of geodesics, with rather intuitive arguments.

Thank you.