Electric network for non-reversible Markov chains

Joint work with Áron Folly

Márton Balázs

University of Bristol

Random walks on Random Networks @ BMC 2016
University of Bristol, 24th March 2016.

Reversible chains and resistor networks (classical)

- Given a Markov Chain,

$$\mathbf{P}_x\{\tau_a < \tau_b\} = u_x$$

with $u_a = 1$, $u_b = 0$, and conductances $\sim$ transition probabilities.

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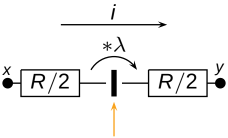
- Effective resistances are monotone functions of the edge resistances.
- Can be used for lots of things (e.g., recurrence-transience, commute times).
- Only works for reversible Markov chains.

The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

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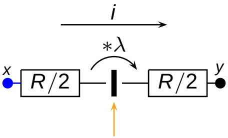

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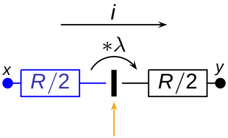

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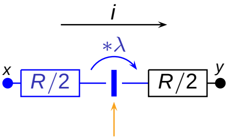

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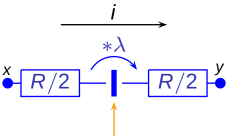

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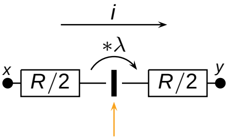

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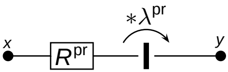
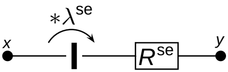


$$(u_x - i \cdot R^{pr}) \cdot \lambda^{pr} = u_y$$



$$u_x \cdot \lambda^{se} - R^{se} \cdot i = u_y$$

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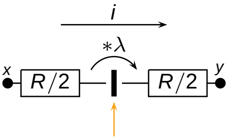

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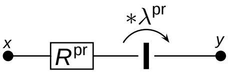

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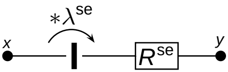

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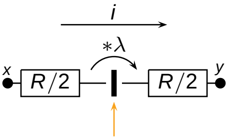

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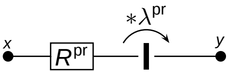

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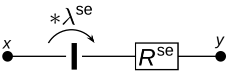
$$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$$



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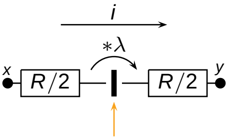

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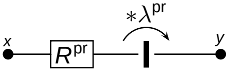

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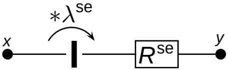

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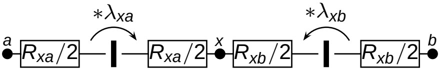

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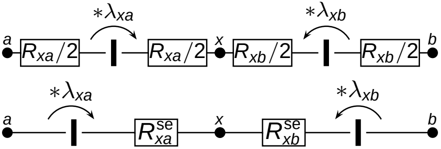

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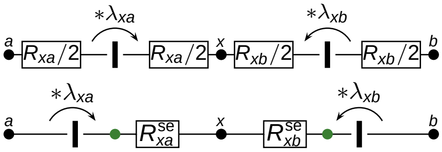

$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y$$

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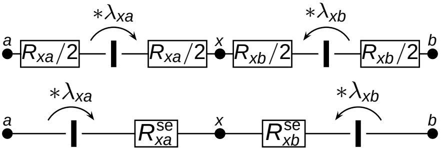

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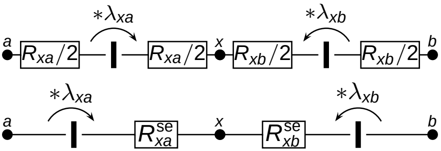

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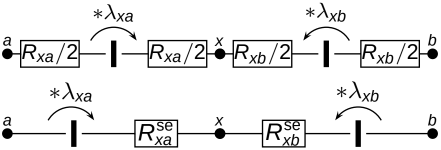

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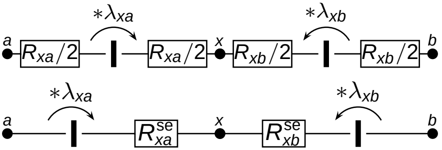

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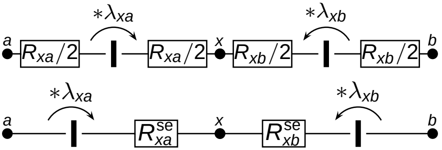

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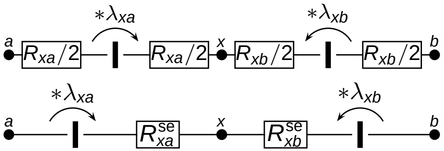

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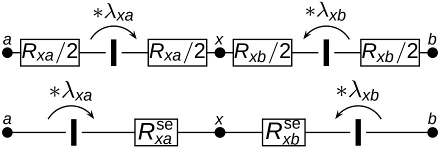

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From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

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"Markovian" property

$$u_x = \sum_z P_{xz} u_z ; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

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$$\mu_x P_{xy} \cdot \mu_y P_{yx} = D_{xy}^2;$$

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Reversed chain: Replace P_{xy} by $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$.

$\rightsquigarrow$ D_{xy} stays, λ_{xy} reverses to λ_{yx}.

$$\gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz}\gamma_{zx} = \sum_z D_{xz}\gamma_{xz} \quad D_{xy} = 2\gamma_{xy}C_{xy}/(\lambda_{xy}+1)$$

$$P_{xy} = D_{xy}\gamma_{xy}/D_x$$

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

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$\rightsquigarrow$ In particular, i_a is proportional to $u_a - u_b$. We have effective resistance.

What works

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The analogy with the current in the reversed network!

Theorem (Chandra, Raghavan, Ruzzo, Smolensky and Tiwari '96 for reversible)

Commute time $= R_{eff} \cdot$ *all conductances.*

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For all sets A, B, capacity$\sim$escape probability.

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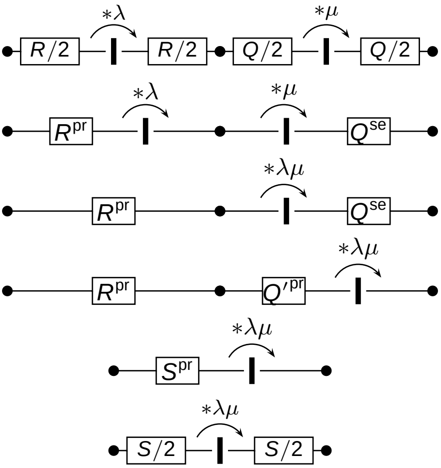
This is non-physical!

In particular, symmetrising the chain ($P_{xy} \to \frac{P_{xy}+\hat{P}_{xy}}{2}$) cannot increase escape probabilities:

- symmetrising leaves C_{xy} unchanged;
- the above sum is minimised by the symmetric voltages, not $\{u_x\}$ (Classical Dirichlet principle).

The electric network

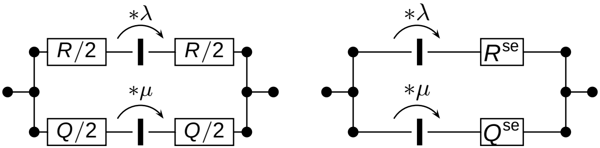
Series:



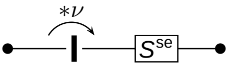
$$S = R\frac{(\lambda+1)\mu}{\lambda\mu+1} + Q\frac{\mu+1}{\lambda\mu+1}.$$

The electric network

Parallel:



Compare this with



$$S = \frac{RQ}{R+Q}$$

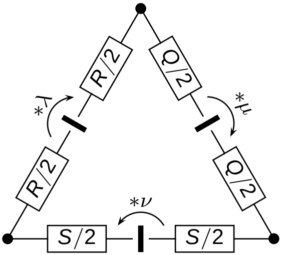
$$\nu = \frac{Q\lambda(\mu+1) + R\mu(\lambda+1)}{Q(\mu+1)+R(\lambda+1)}.$$

The electric network

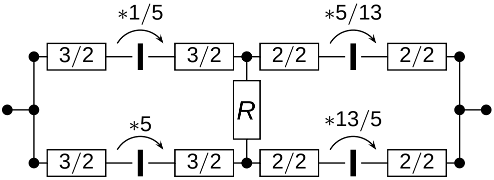
Star-Delta:

Star to Delta works,

Delta to Star only works if Delta does not produce a circular current by itself ($\lambda\mu\nu = 1$).



Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}.$$

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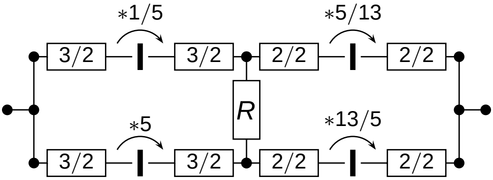

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☹

Dirichlet principle

Classical case:

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$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$
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Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

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Thank you.

Kolmogorov's criterion

Theorem (Kolmogorov's criterion)

A Markov chain is reversible if and only if for every closed cycle $x_0, x_1, x_2, \ldots, x_n = x_0$ in Ω we have

$$P_{x_0 x_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1} x_0} = P_{x_0 x_{n-1}} \cdot P_{x_{n-1} x_{n-2}} \cdots P_{x_1 x_0}.$$

In particular, any Markov chain on a finite connected tree G is necessarily reversible.

Kolmogorov's criterion

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}$$

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$$\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1.$$

- Total multiplication factor along any loop is one.

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Repeat for trees:

- There are no loops.

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