Road layout in the KPZ class

Growing out of a project that started with
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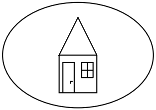
Aquib Molla

Márton Balázs

University of Bristol

Fluctuations for inhomogeneous last passage times
Sussex, September 2023.

A naive Poisson model



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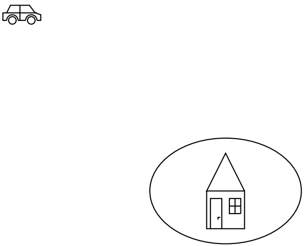

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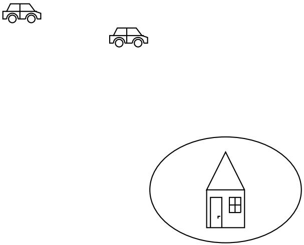

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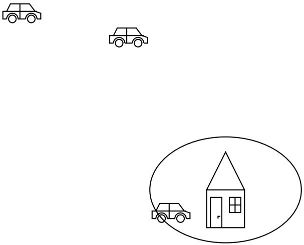

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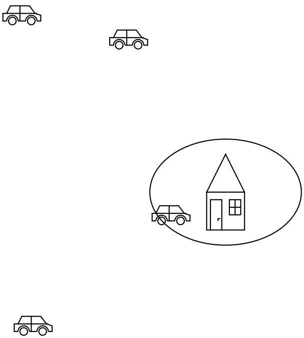

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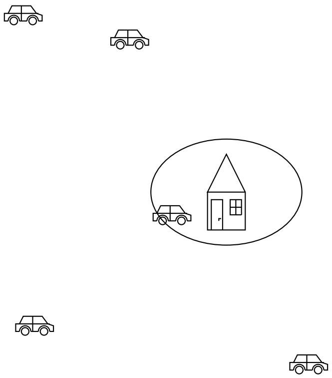

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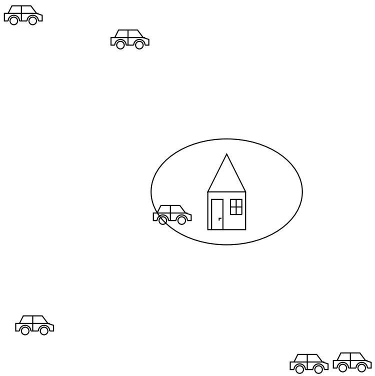

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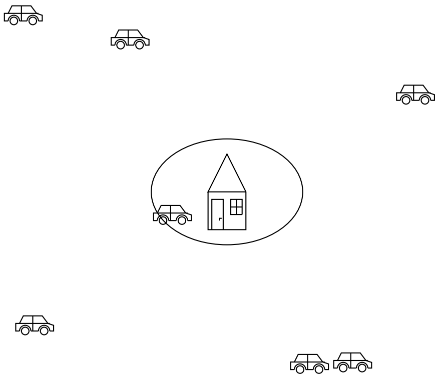

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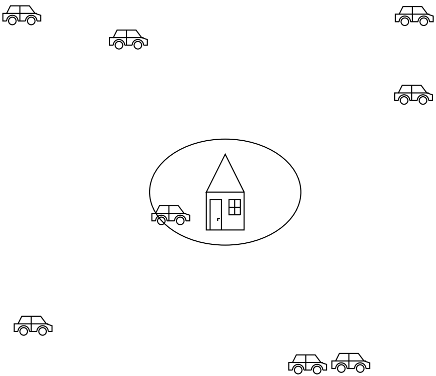

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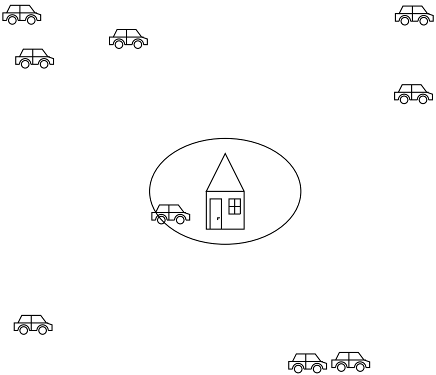

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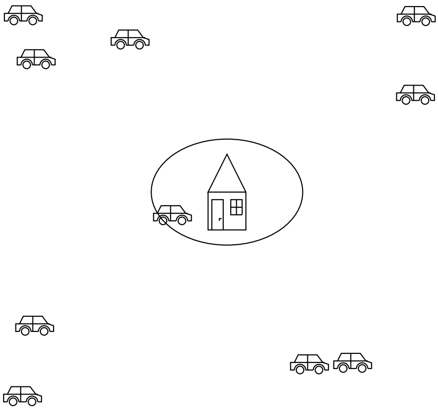

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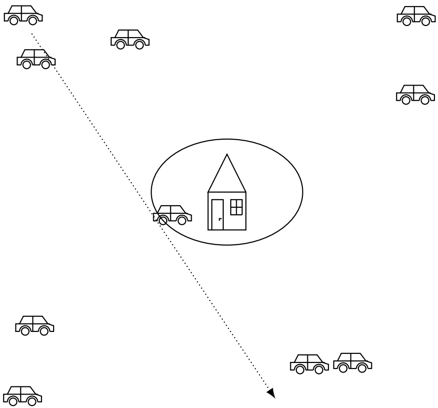

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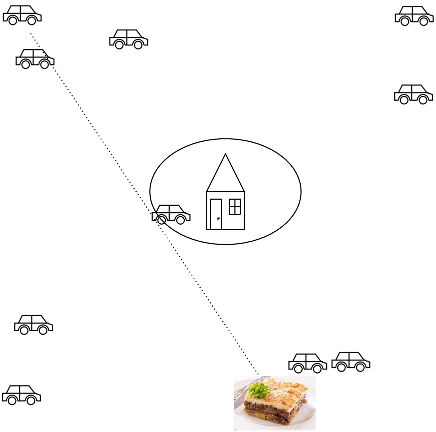

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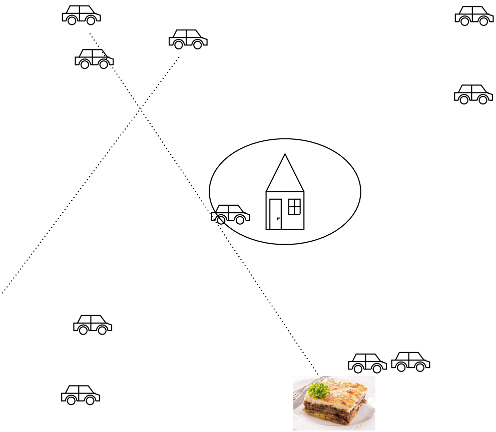

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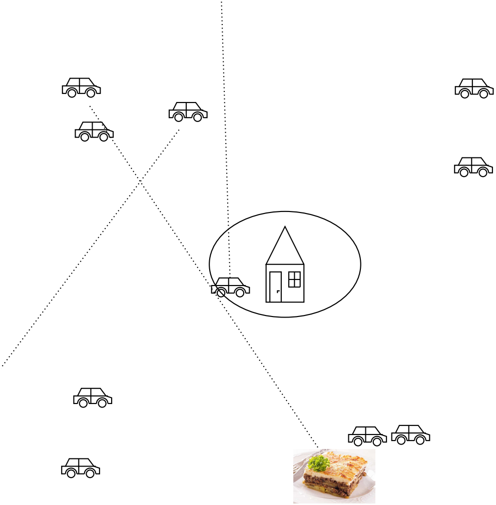

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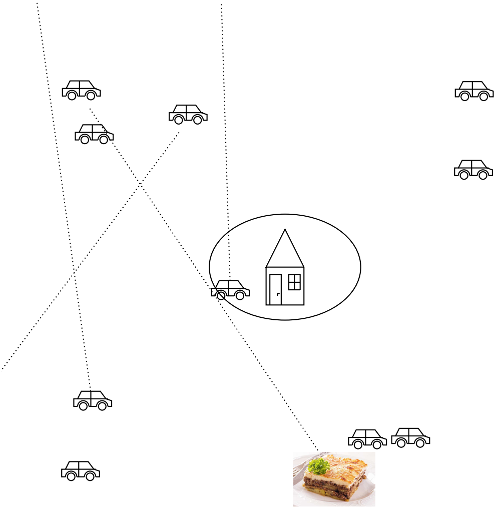

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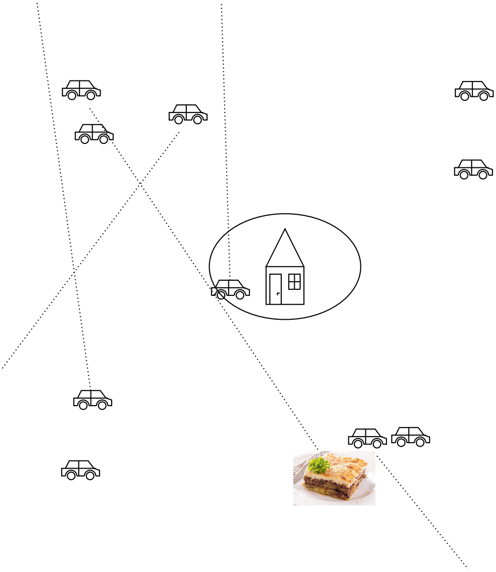

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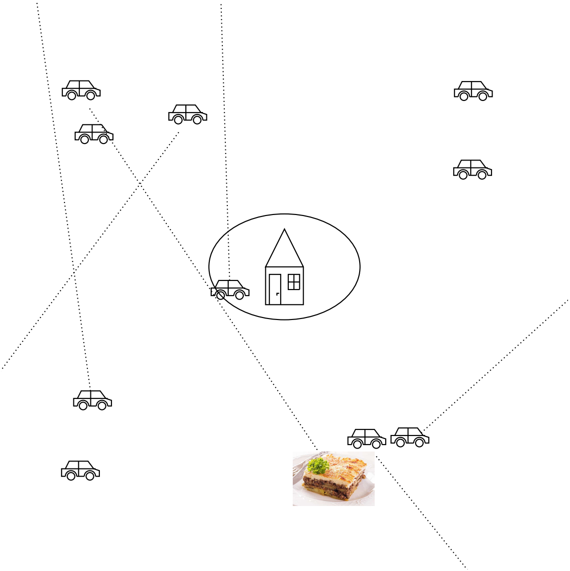

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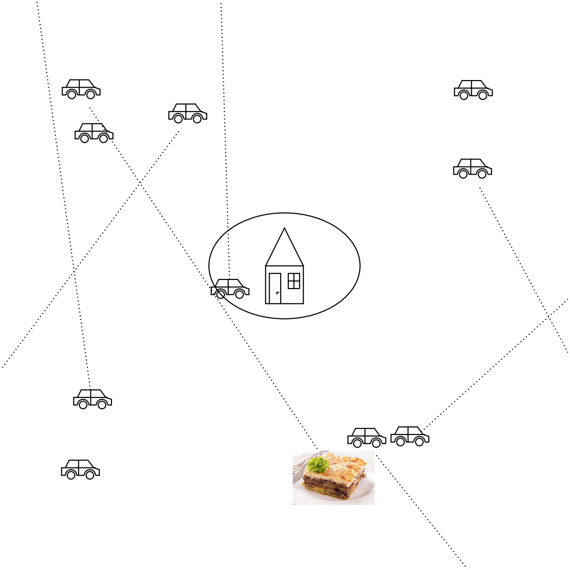

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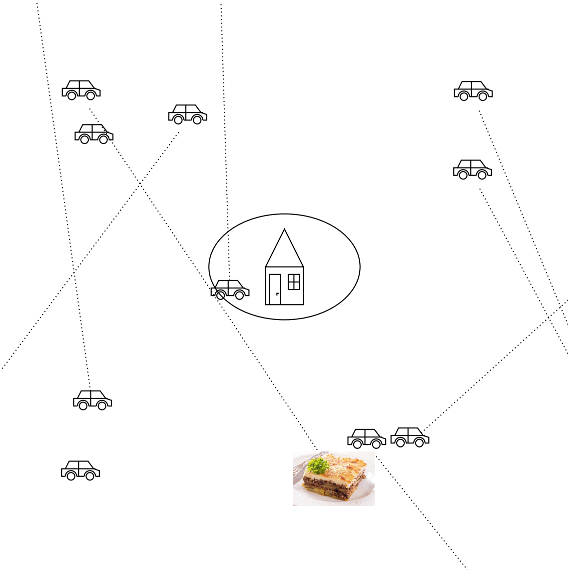

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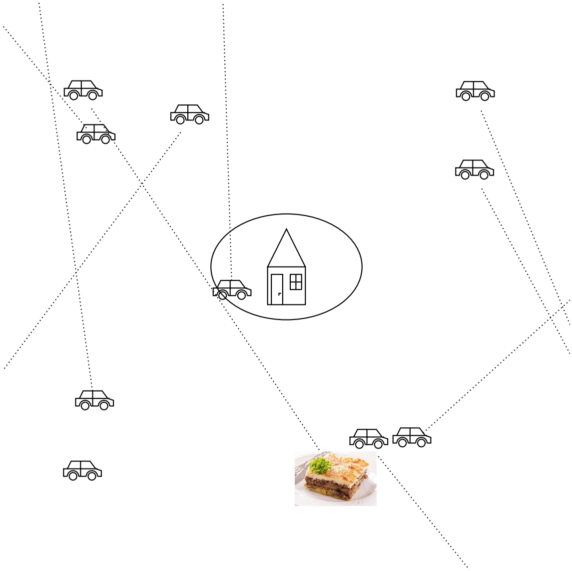

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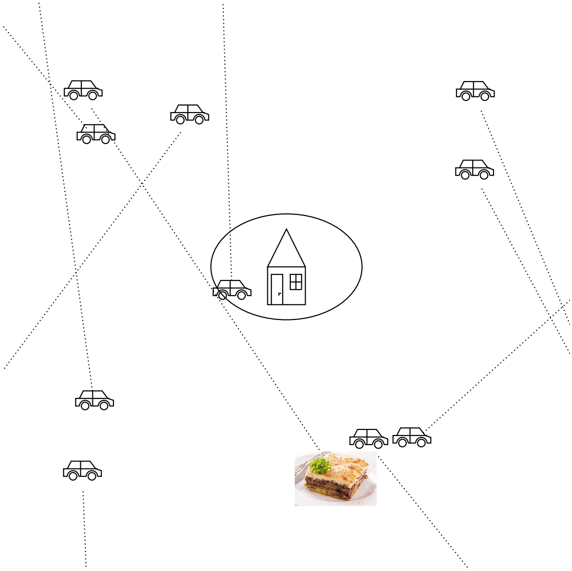

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- Unfortunately $D \gg r\ldots$

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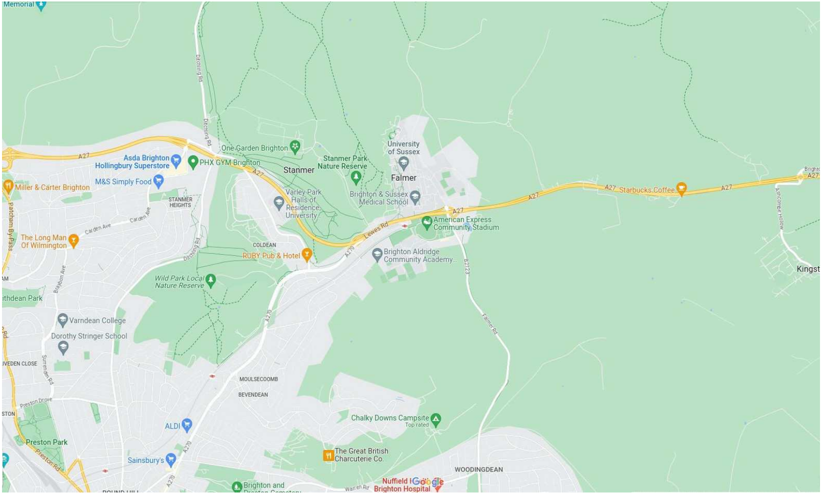

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- People who first beat a path; horse drawn carriages; road builders try to minimise obstacles. Gradients, built-up objects, etc.
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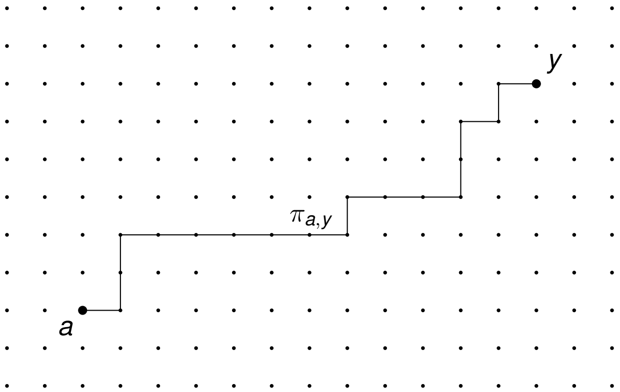
- More tools in Exponential last passage percolation (LPP). Should behave similarly to FPP.

Last passage percolation

- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{a,y}$ from a to y is the a.s. unique heaviest up-right path from a to y. Its weight is $G_{a,y}$.



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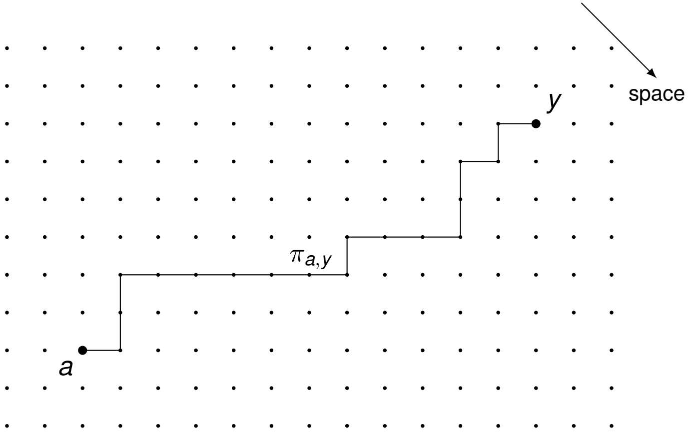

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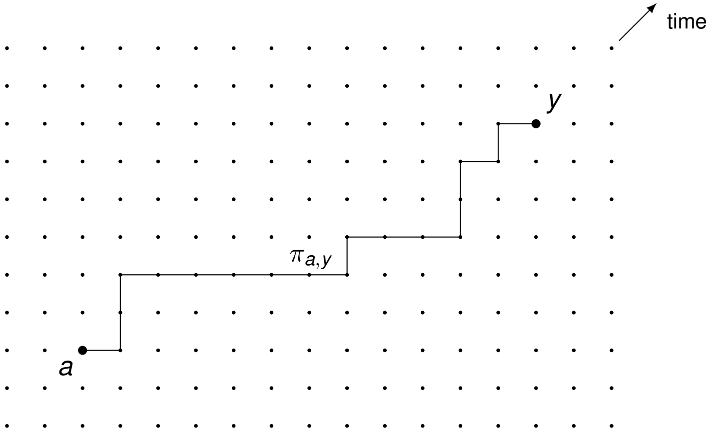

Last passage percolation

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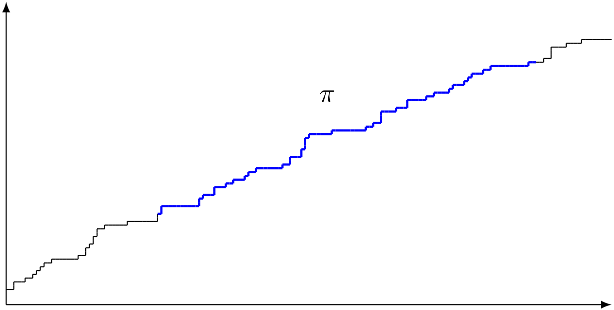

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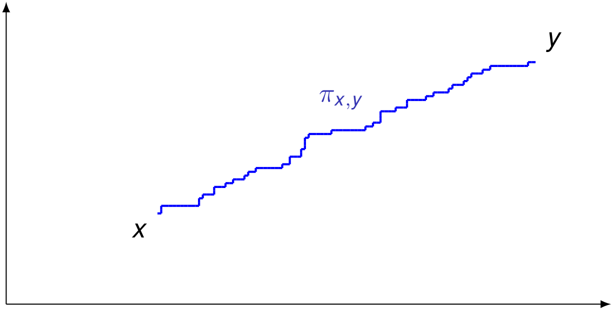

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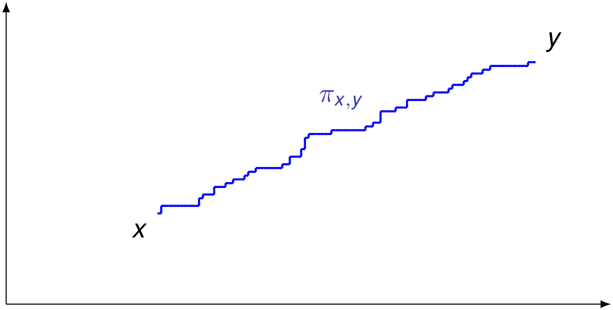
For any fixed direction this a.s. exists and is unique. [Newman (et al) '96]; [Ferrari, Pimentel '05]; [Coupier '11]; [Janjigian, Rassoul-Agha, Seppäläinen '23]



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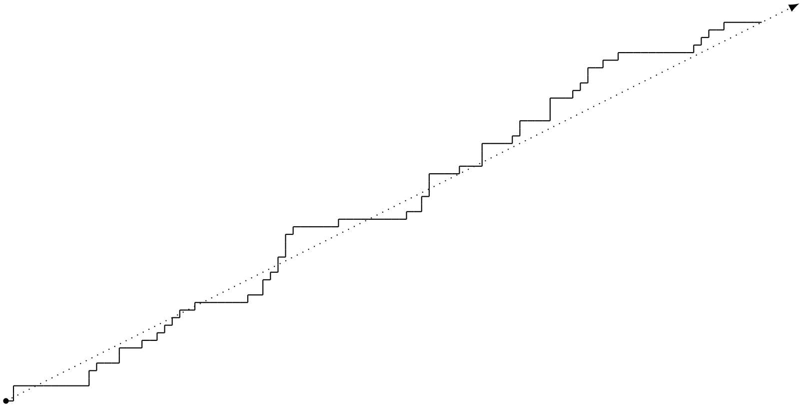
- Draw the a.s. semi-infinite geodesic for each point to its chosen direction. Many of these will coalesce when the angles are close. That's our road map with traffic data on it. A road segment is *busy* when many geodesics use that edge.

Our model



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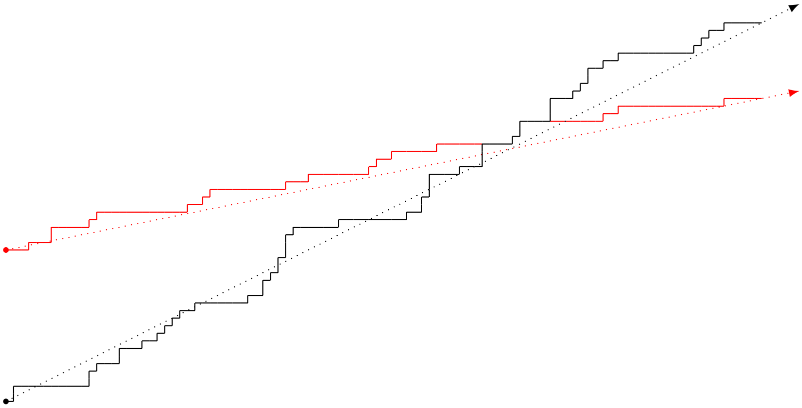

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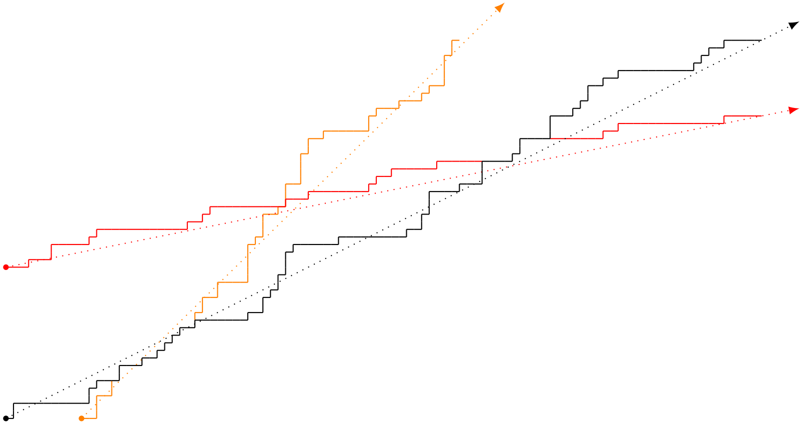

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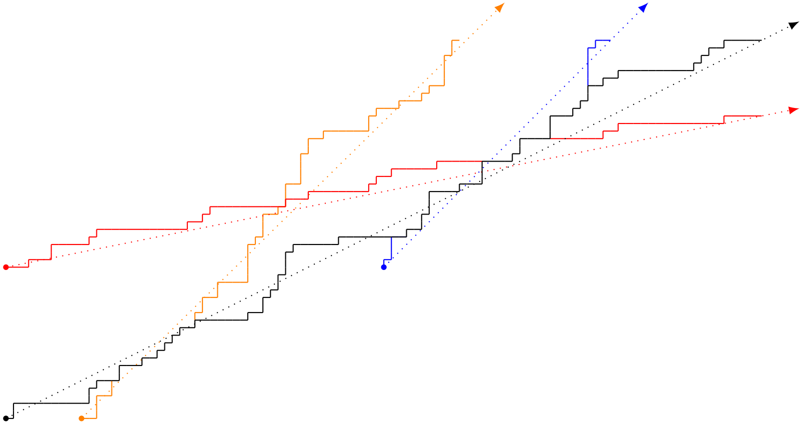

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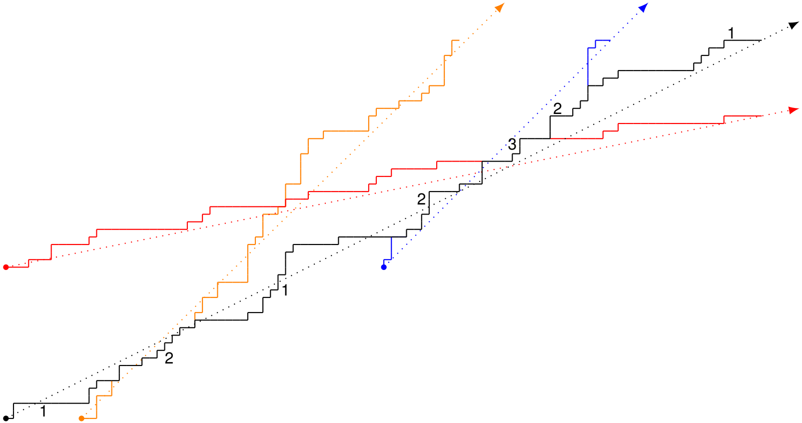

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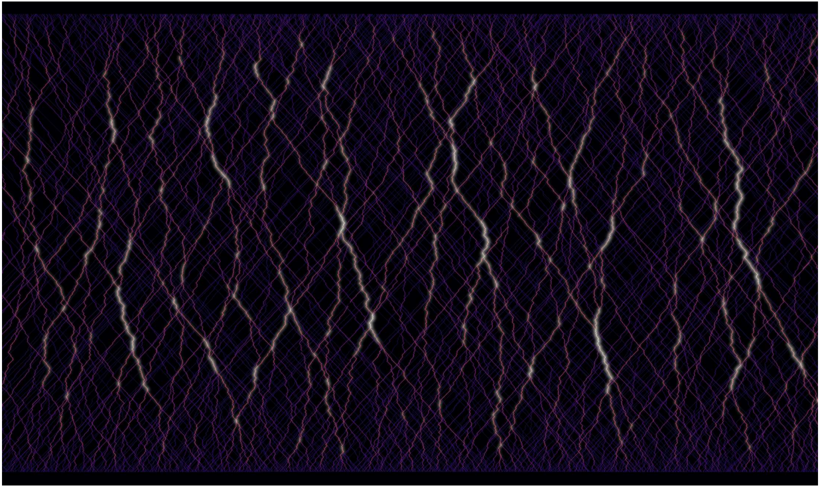

Simulation by David Harper

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- How far is the nearest busy road? I.e., within distance k in the space-direction, what is the probability to find an edge with k^{α} geodesics on it (and what is the interesting α)?
- Is this actually a good model of real road networks out there?

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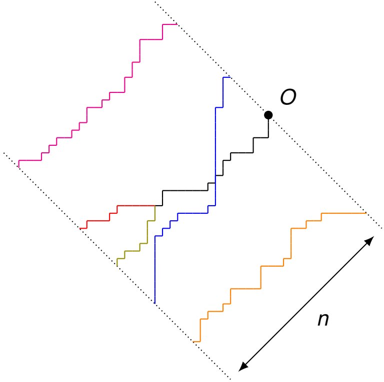
The number N_n that start from distance n and go through O is mean 1 (mass transport principle).



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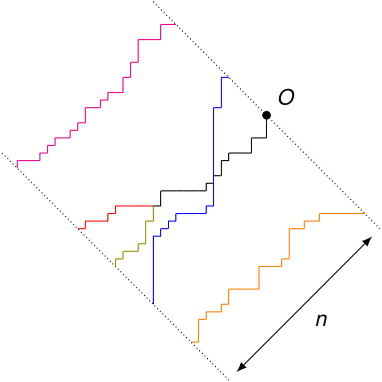

From all layers: $N = \sum_{n=1}^{\infty} N_n$ is of infinite mean.

Answers

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$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

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We don't believe the log.

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- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
- We already know they cannot come from too far.
- Then either too many cars picked the direction of the origin (independent choices), or many wandered in from a wrong direction (transversal fluctuations).
- $\rightsquigarrow$ upper bound.

How many cars go through the origin?

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With similar methods,

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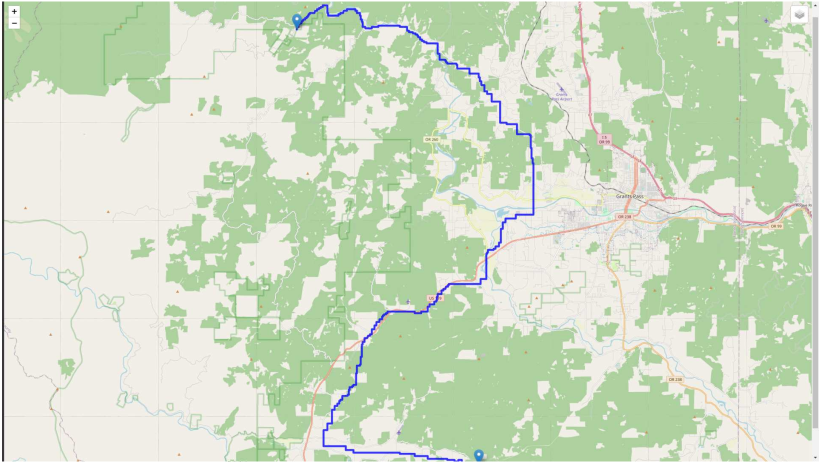
Theorem

$$\mathbb{P}\{\textit{yes, road with} \geq \textit{const} \cdot k^4 \textit{ cars within distance } k\} \geq c.$$

Is this all any good?



Simulation by David Harper

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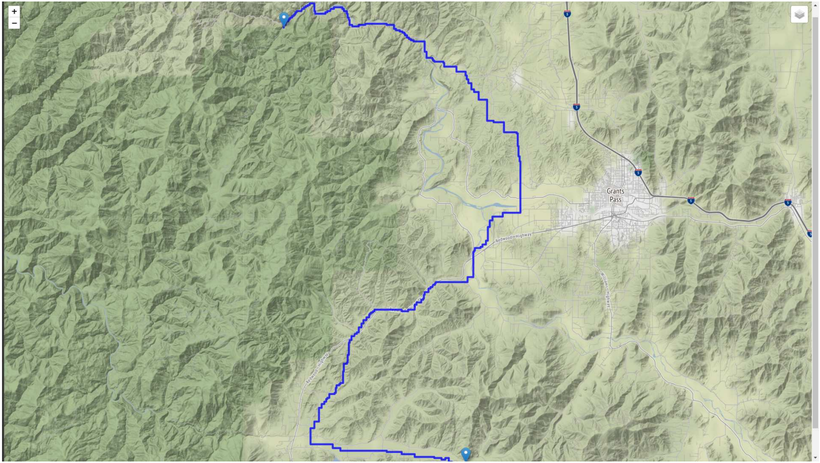

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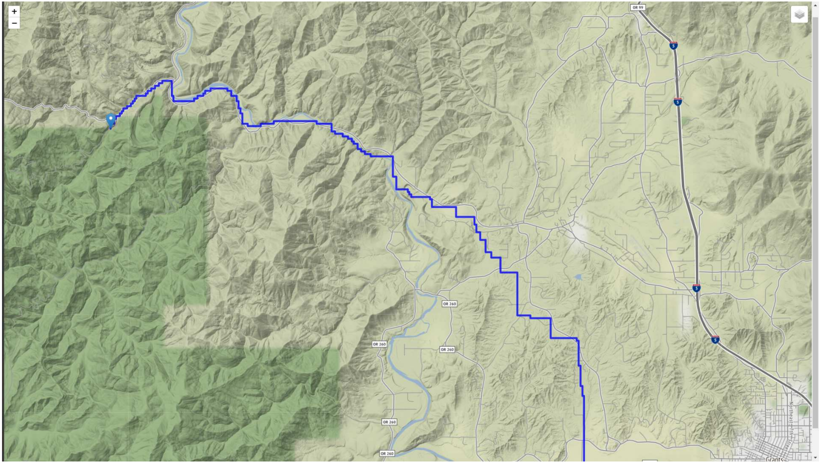

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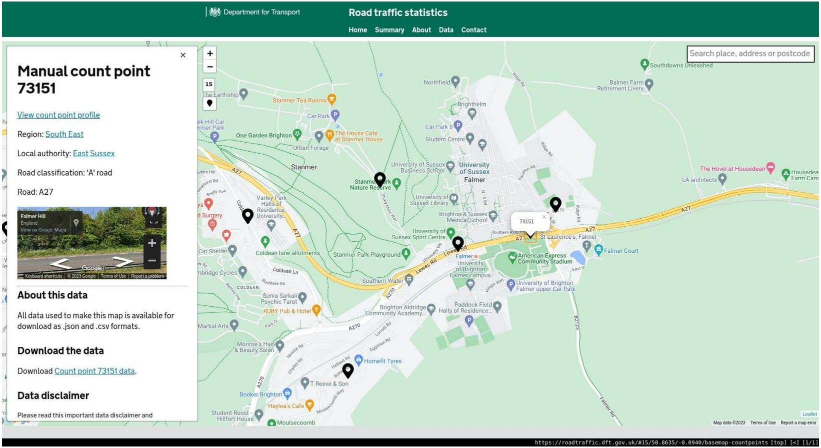

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Thank you.