

Jacobi triple product via the exclusion process

Joint with
Ross Bowen

Márton Balázs

University of Bristol

University of Bristol:
Mathematical Physics Seminar &
Probability Seminar
14 December, 2018.

Jacobi triple product

Theorem

Let $|x| < 1$ and $y \neq 0$ be complex numbers. Then

$$\prod_{i=1}^{\infty} (1 - x^{2i}) \left(1 + \frac{x^{2i-1}}{y^2}\right) (1 + x^{2i-1}y^2) = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

Mostly appears in **number theory** and **combinatorics of partitions**.

We'll prove it using interacting particles (for real x , y only).

Models

- Asymmetric simple exclusion
- Zero range

Blocking measures

State space

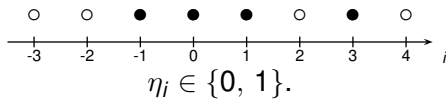
- No boundaries
- Boundaries

Lay down - stand up

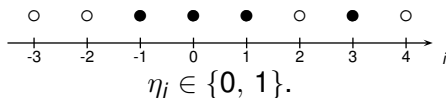
Jacobi triple product

More models

Asymmetric simple exclusion



Asymmetric simple exclusion



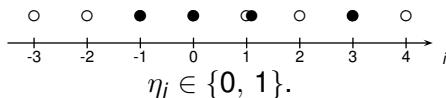
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



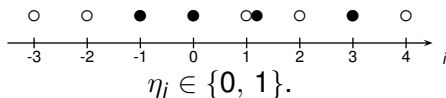
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



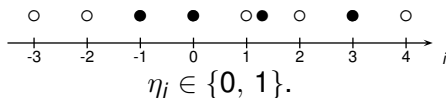
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



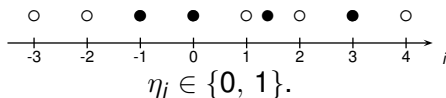
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



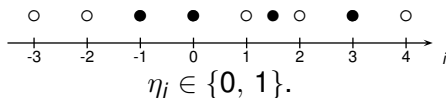
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



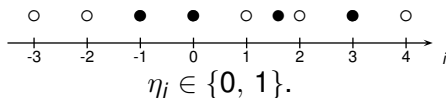
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



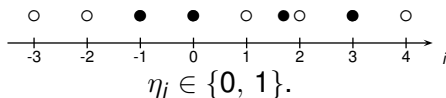
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



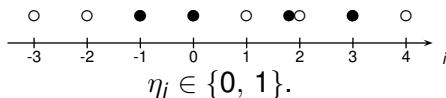
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



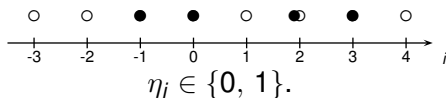
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



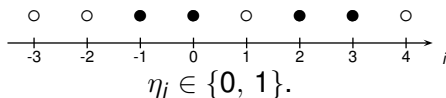
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



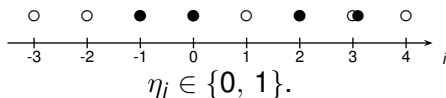
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



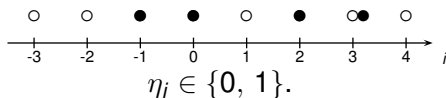
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



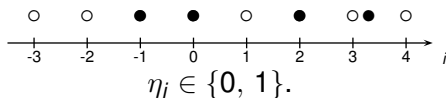
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



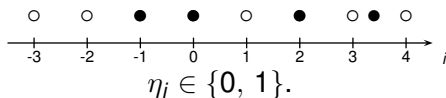
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



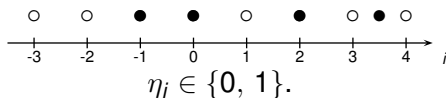
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



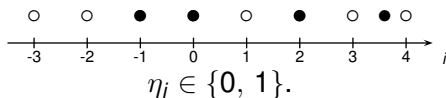
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



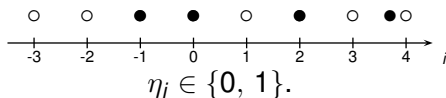
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



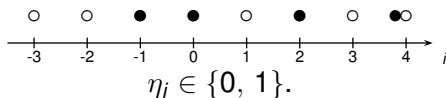
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



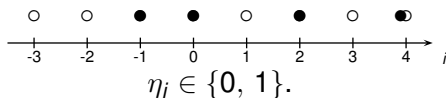
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



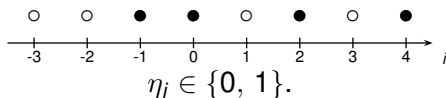
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



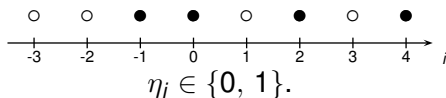
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



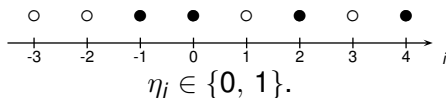
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



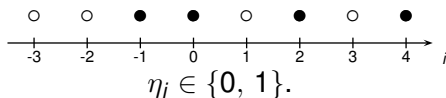
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



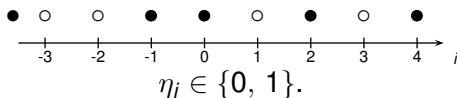
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



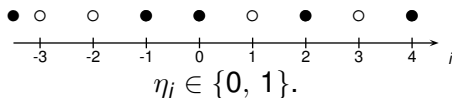
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



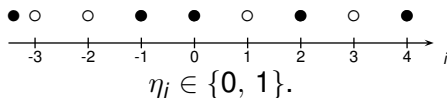
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



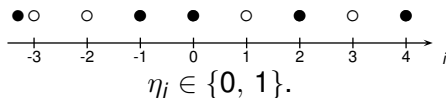
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



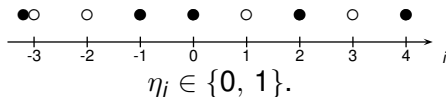
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



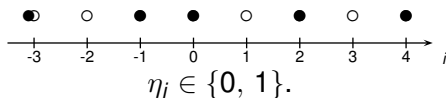
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



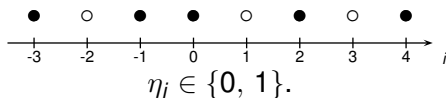
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



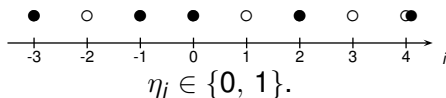
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



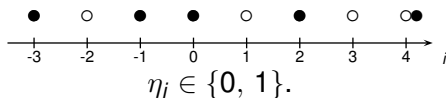
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



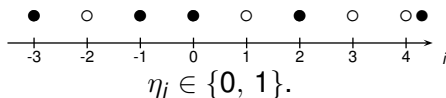
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



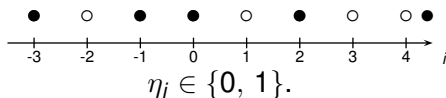
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



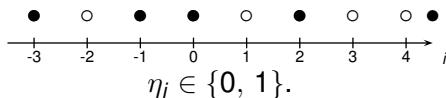
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



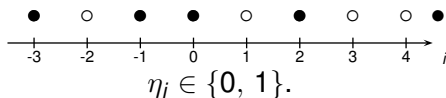
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



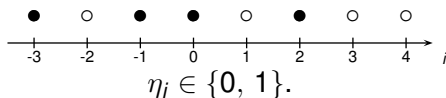
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



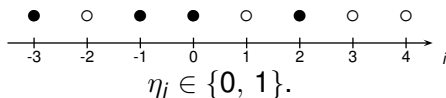
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



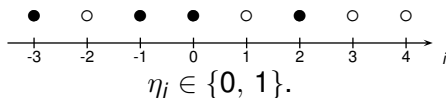
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



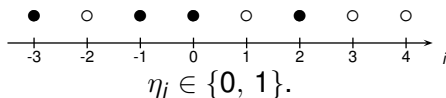
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



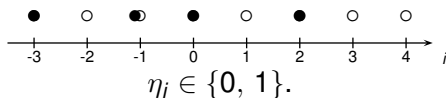
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



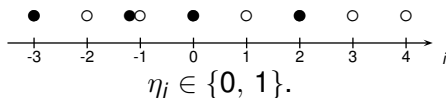
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



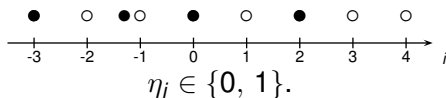
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



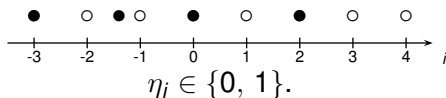
Particles try to jump

to the right **with rate** p ,

to the left **with rate** $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



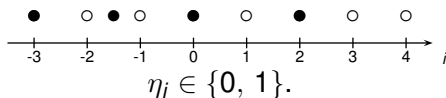
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



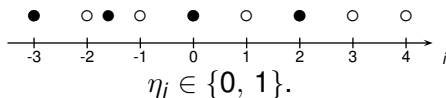
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



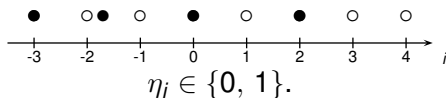
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



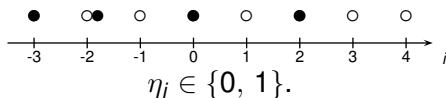
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



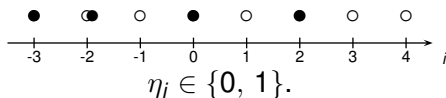
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



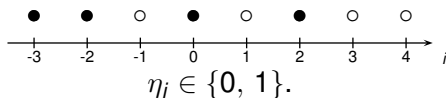
Particles try to jump

to the right with rate p ,

to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



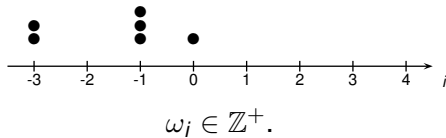
Particles try to jump

to the right with rate p ,

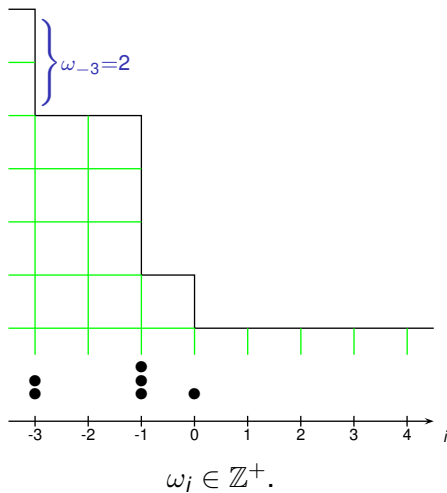
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

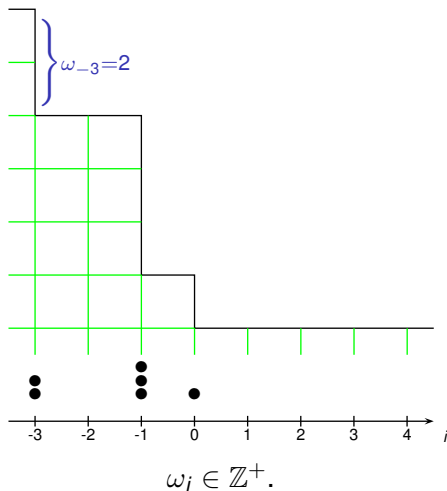
The asymmetric zero range process



The asymmetric zero range process

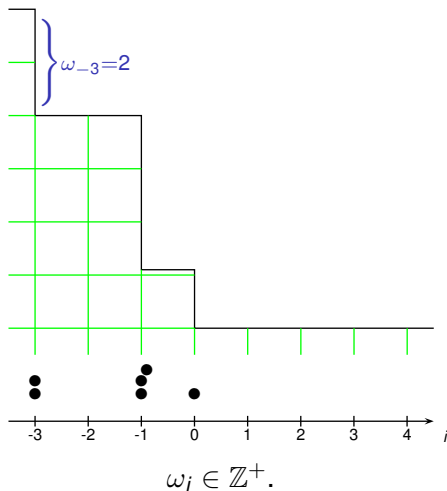


The asymmetric zero range process



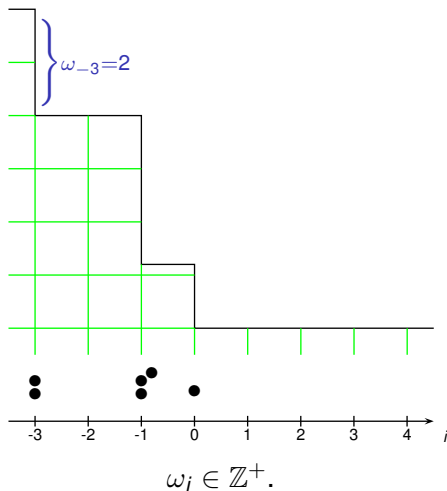
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



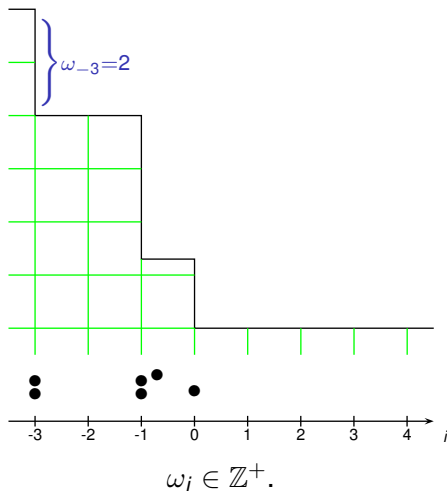
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



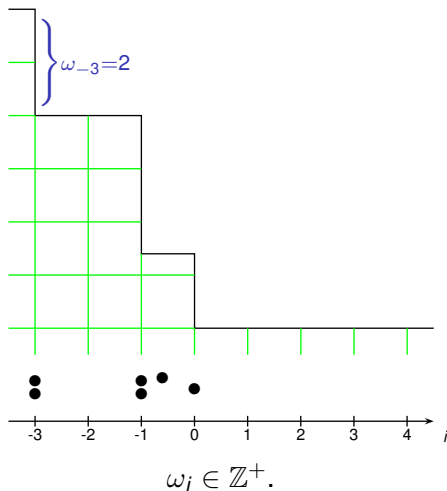
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



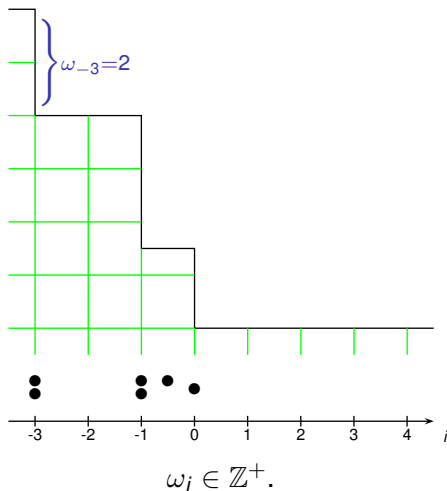
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



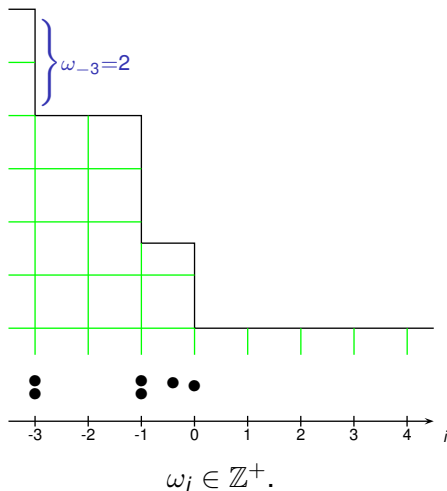
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



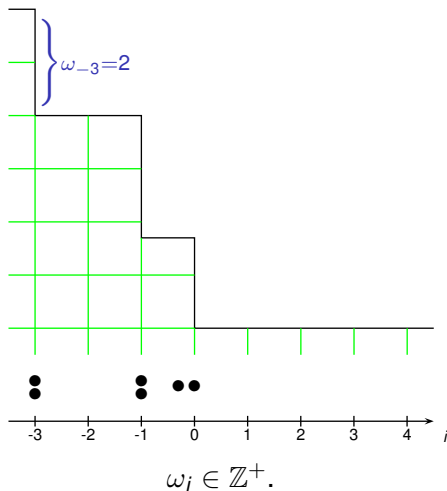
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



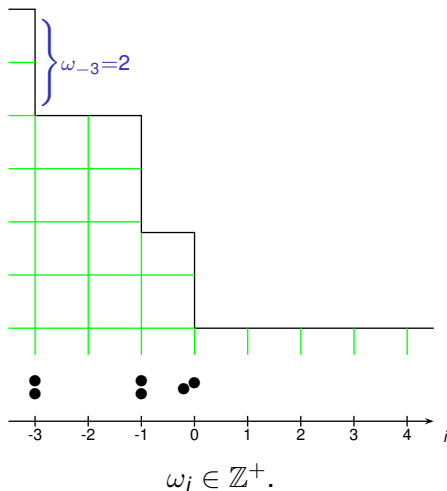
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



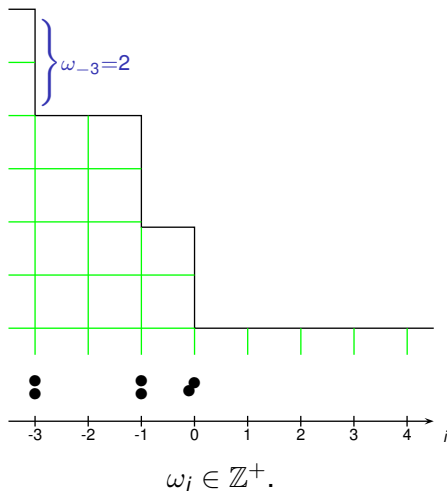
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



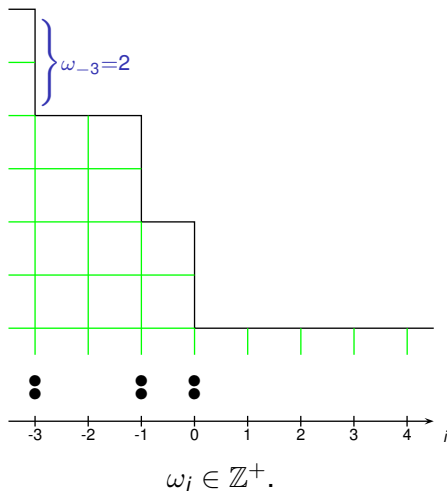
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



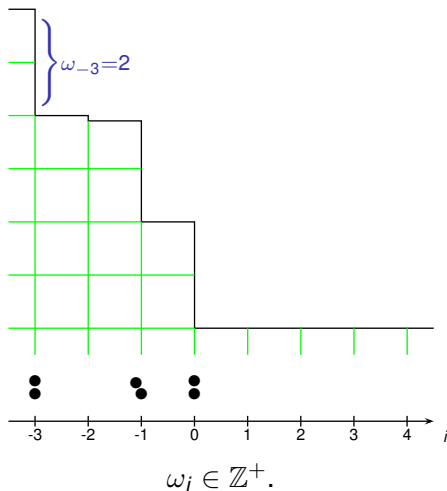
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



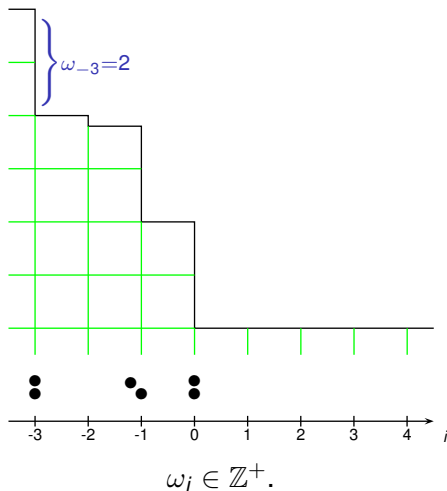
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



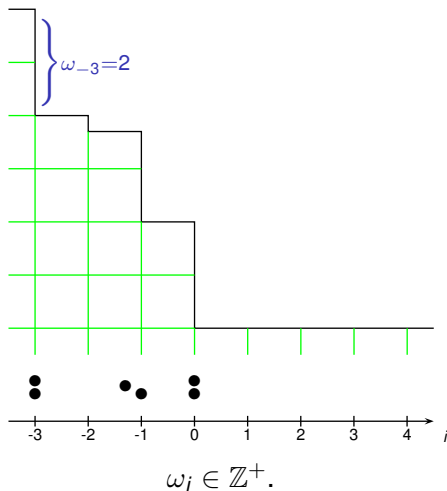
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



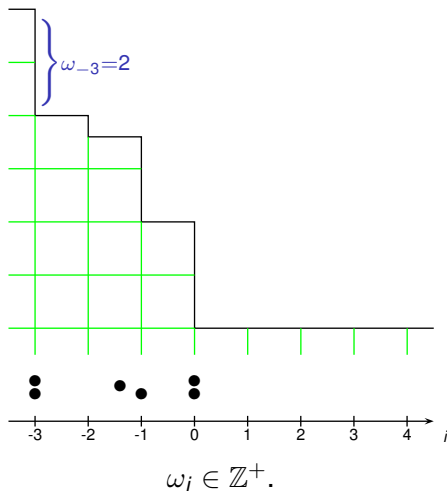
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



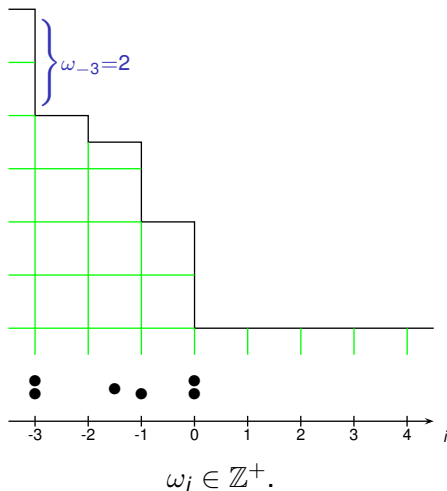
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



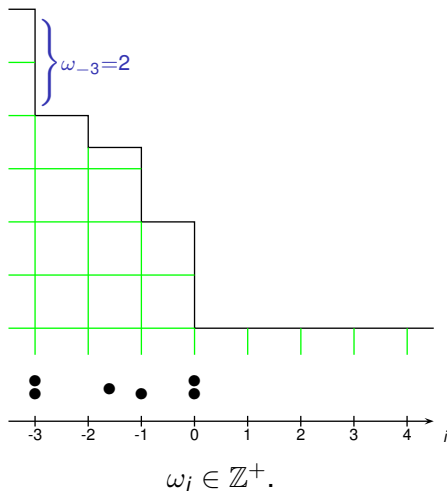
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



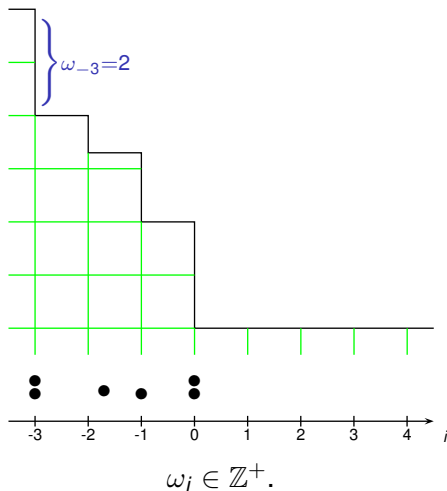
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



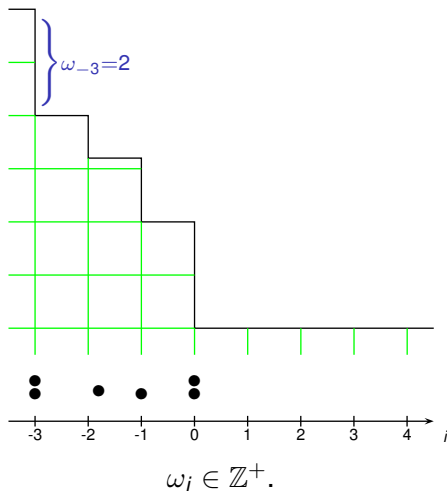
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



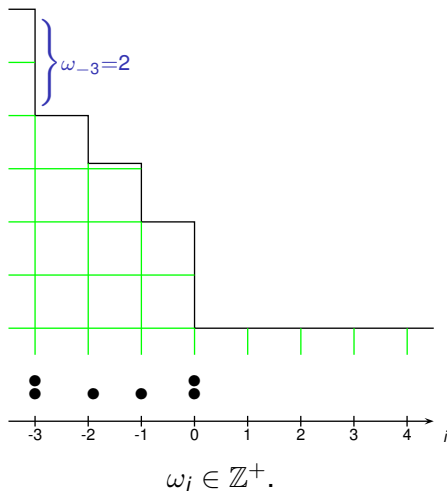
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



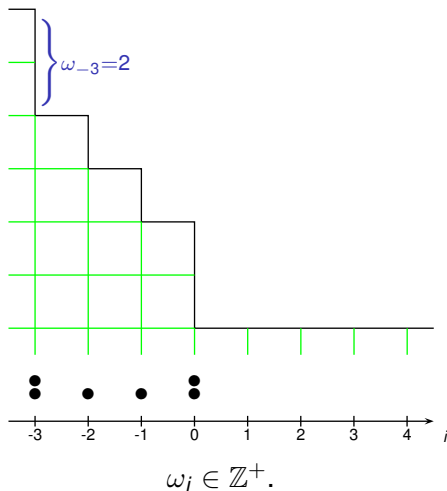
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



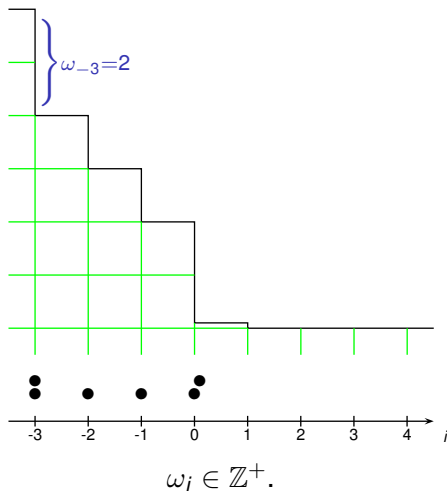
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



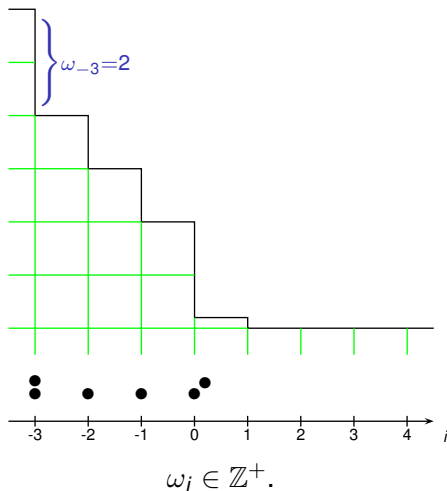
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



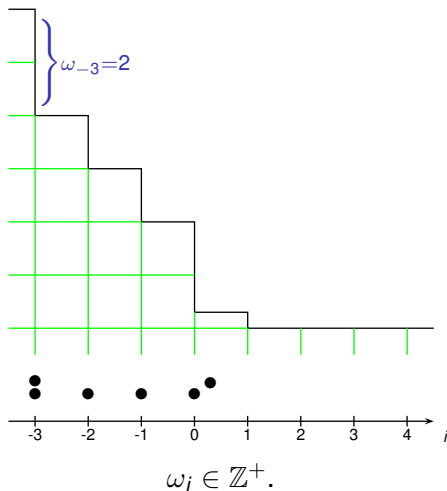
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



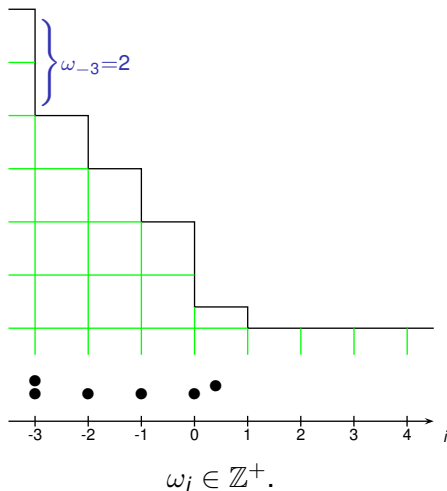
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



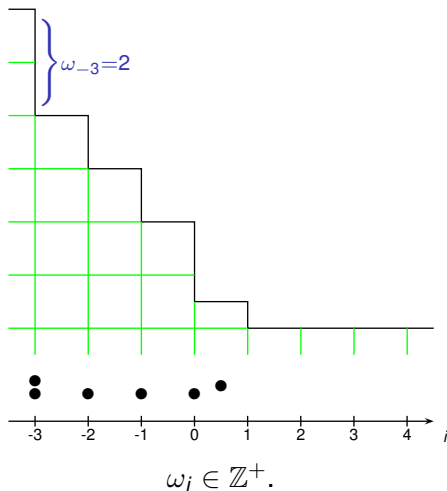
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



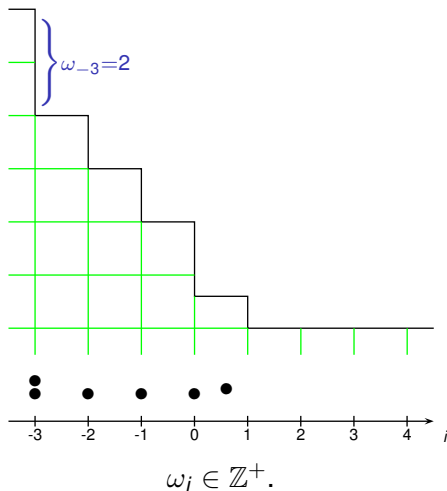
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



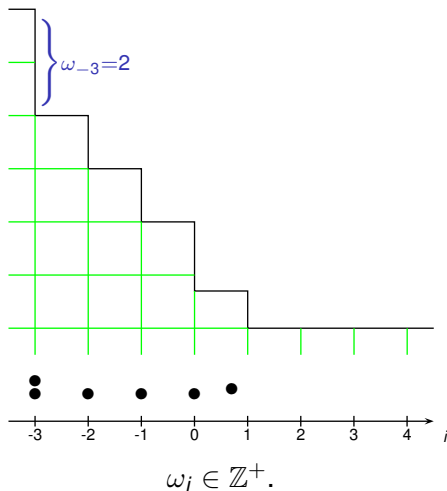
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



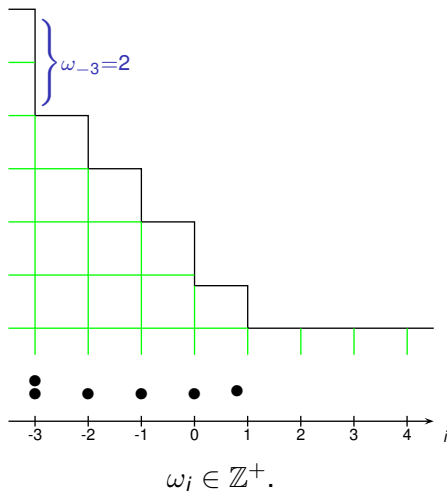
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



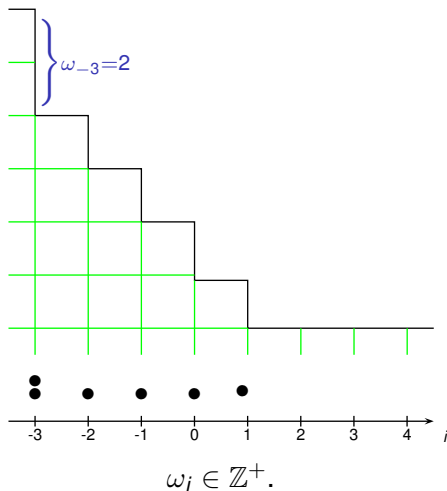
Particles jump to the right with rate $p \cdot r(\omega_i)$
 to the left with rate $q \cdot r(\omega_i)$.

The asymmetric zero range process



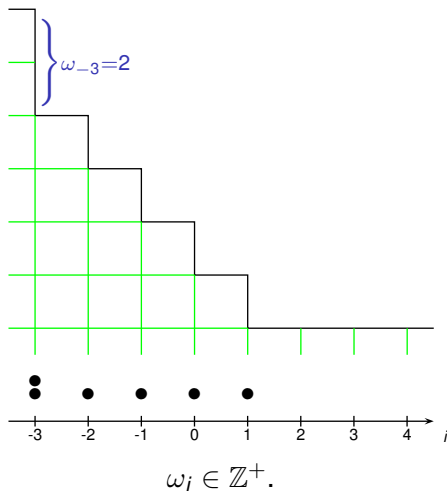
Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process



Particles jump to the right with rate $p \cdot r(\omega_j)$
 to the left with rate $q \cdot r(\omega_j)$.

The asymmetric zero range process

We need r **non-decreasing** and assume, as before,
 $q = 1 - p < p$.

Examples:

- ▶ 'Classical' ZRP: $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$.
- ▶ Independent walkers: $r(\omega_i) = \omega_i$.

Product blocking measures

Can we have a reversible stationary distribution in product form:

$$\underline{\mu}(\underline{\omega}) = \bigotimes_i \mu_i(\omega_i);$$

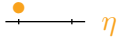
$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \rightsquigarrow i+1}) = \underline{\mu}(\underline{\omega}^{i \rightsquigarrow i+1}) \cdot \text{rate}(\underline{\omega}^{i \rightsquigarrow i+1} \rightarrow \underline{\omega}) \quad ?$$

Here

$$\underline{\omega}^{i \rightsquigarrow i+1} = \underline{\omega} - \underline{\delta}_i + \underline{\delta}_{i+1}.$$

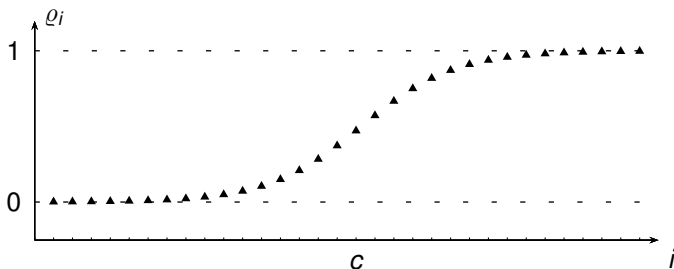
Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}(\underline{\eta} \rightarrow \underline{\eta}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\eta}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\eta}^{i \curvearrowright i+1} \rightarrow \underline{\eta})$$

ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$;  η

$$\varrho_i(1 - \varrho_{i+1}) \cdot p = (1 - \varrho_i)\varrho_{i+1} \cdot q$$

Solution:
$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$



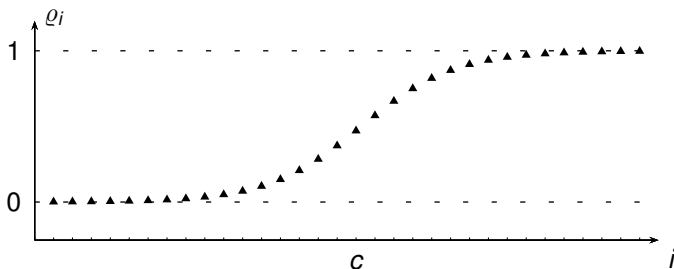
Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}(\underline{\eta} \rightarrow \underline{\eta}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\eta}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\eta}^{i \curvearrowright i+1} \rightarrow \underline{\eta})$$

ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$; $\text{---} \overset{\bullet}{\text{---}} \underline{\eta}^{i \curvearrowright i+1}$

$$\varrho_i(1 - \varrho_{i+1}) \cdot p = (1 - \varrho_i)\varrho_{i+1} \cdot q$$

Solution:
$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$



Asymmetric zero range process

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\omega}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\omega}^{i \curvearrowright i+1} \rightarrow \underline{\omega}) \quad ?$$

AZRP:

$$\mu_i(\omega_i) \mu_{i+1}(\omega_{i+1}) \cdot p \mathbf{1}\{\omega_i > 0\} = \mu_i(\omega_i - 1) \mu_{i+1}(\omega_{i+1} + 1) \cdot q$$

Solution: $\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i - \text{const}}\right).$

State space: ASEP

Notice:

$$\mathbf{P}\{\eta_i = 0\} = 1 - \varrho_i = \frac{1}{1 + \left(\frac{\rho}{q}\right)^{i-c}} \quad \text{as } i \rightarrow \infty$$

$$\mathbf{P}\{\eta_i = 1\} = \varrho_i = \frac{1}{\left(\frac{q}{\rho}\right)^{i-c} + 1} \quad \text{as } i \rightarrow -\infty$$

are both summable. Hence by Borel-Cantelli there is $\underline{\mu}$ -a.s. a rightmost hole and a leftmost particle,

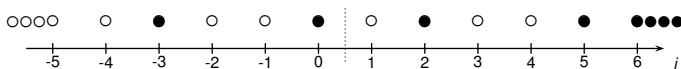
$$N := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

is $\underline{\mu}$ -a.s. finite.

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

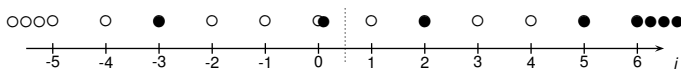
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

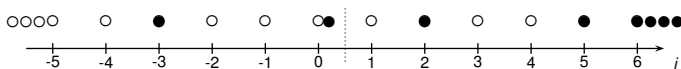
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

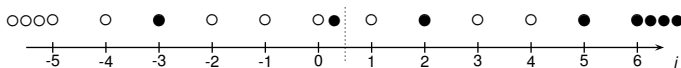
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

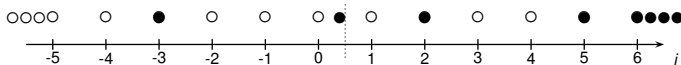
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

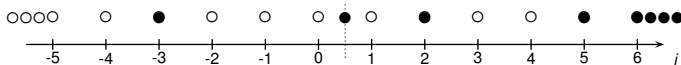
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

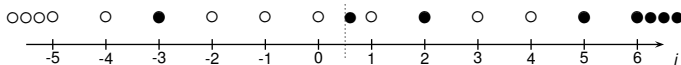
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

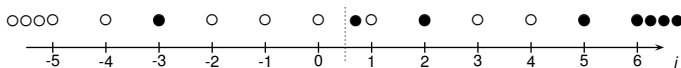
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

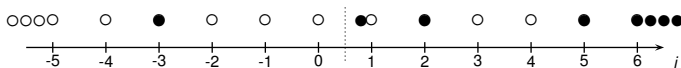
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

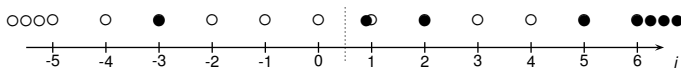
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

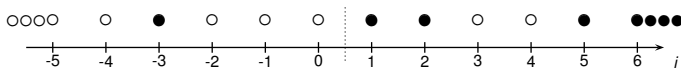
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 2 - 1$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

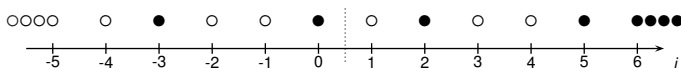
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

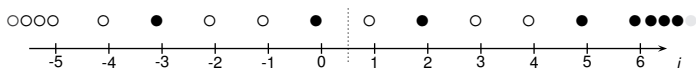
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

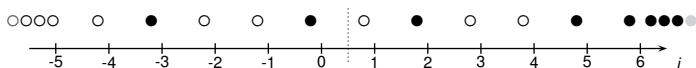
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

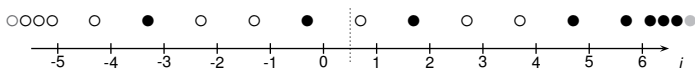
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

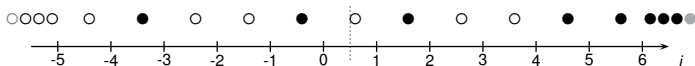
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

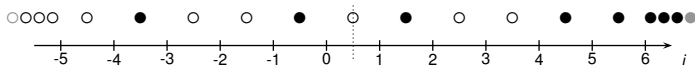
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

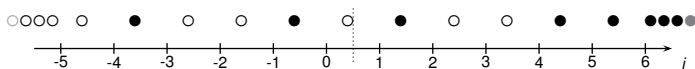
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

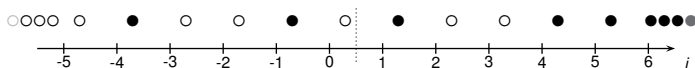
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

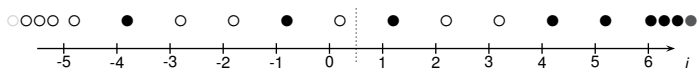
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

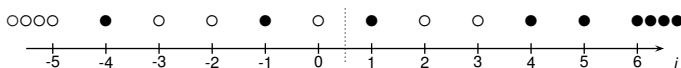
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

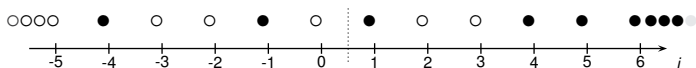
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

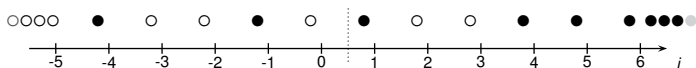
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

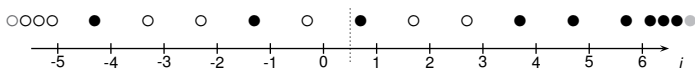
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

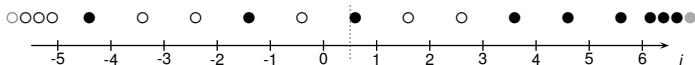
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

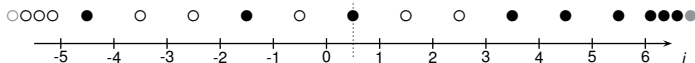
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

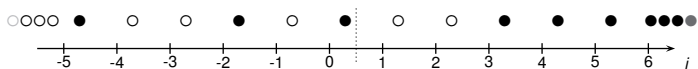
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

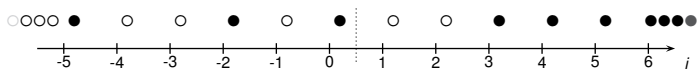
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

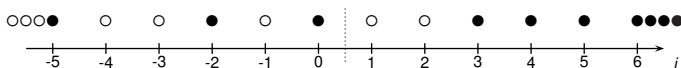
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 2 - 3 = -1$$

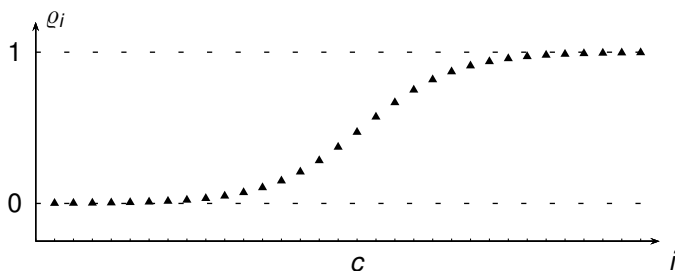


$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Recall

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}},$$

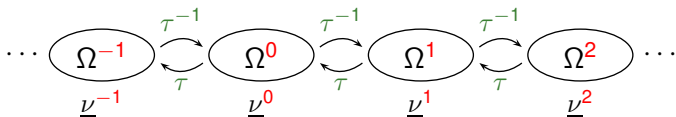


but

$$\underline{\nu}^n(\cdot) := \underline{\mu}(\cdot \mid N(\cdot) = n)$$

doesn't depend on c anymore. **Stationary measure on Ω^n .**

State space: ASEP



$$\underline{\mu}(\cdot) = \sum_{n=-\infty}^{\infty} \underline{\mu}(\cdot | N(\cdot) = n) \underline{\mu}(N(\cdot) = n) = \sum_{n=-\infty}^{\infty} \underline{\nu}^n(\cdot) \underline{\mu}(N(\cdot) = n).$$

Ergodic decomposition of $\underline{\mu}$.

Let's find the coefficients $\underline{\mu}(N(\cdot) = n)$!

State space: ASEP

Recall:

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$\underline{\mu}(\underline{\eta}) = \prod_{i \leq 0} \frac{\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{1 + \left(\frac{p}{q}\right)^{i-c}} \cdot \prod_{i > 0} \frac{\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

Recall:

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$\underline{\mu}(\underline{\eta}) = \prod_{i \leq 0} \frac{\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{1 + \left(\frac{p}{q}\right)^{i-c}} \cdot \prod_{i > 0} \frac{\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

$$\underline{\mu}(\underline{\tau\eta}) = \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}
 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{-c}
 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{-c} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}
 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{-c} \\
 &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c} \\
 &= \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}.
 \end{aligned}$$

State space: ASEP

So,

$$\underline{\mu}(N = n - 1) = \sum_{\underline{\eta}: N(\underline{\eta}) = n - 1} \underline{\mu}(\underline{\eta})$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\tau}\underline{\eta})=n-1} \underline{\mu}(\underline{\tau}\underline{\eta})\end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}
 \underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c}
 \end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c}\end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c} \\ &= \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.\end{aligned}$$

State space: ASEP

So,

$$\begin{aligned}
 \underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c} \\
 &= \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.
 \end{aligned}$$

Solution:

$$\underline{\mu}(N = n) = \frac{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}$$

discrete Gaussian.

State space: ASEP

and, if $N(\underline{\eta}) = n$,

$$\begin{aligned} \underline{\nu}^n(\underline{\eta}) &= \underline{\mu}(\underline{\eta} \mid N(\underline{\eta}) = n) = \frac{\underline{\mu}(\underline{\eta})}{\mu(N(\underline{\eta}) = n)} \\ &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}. \end{aligned}$$

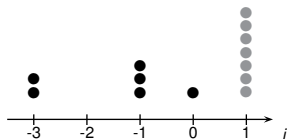
This is the unique stationary distribution on Ω^n .

State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

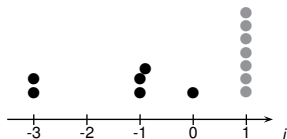


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

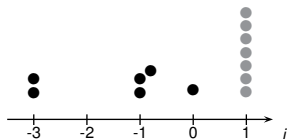


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

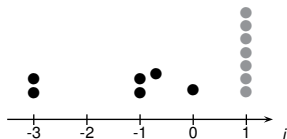


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

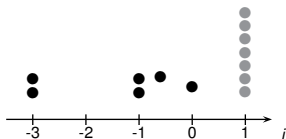


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

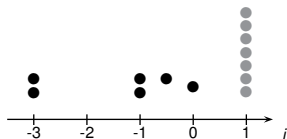


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

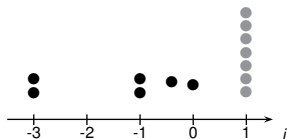


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

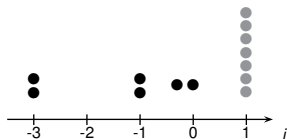


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

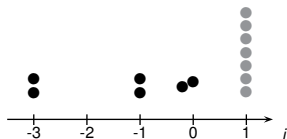


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

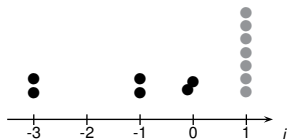


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

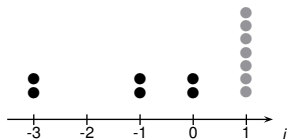


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

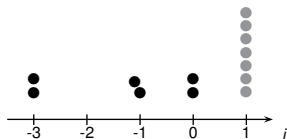


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

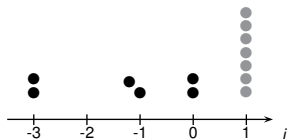


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

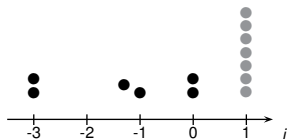


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

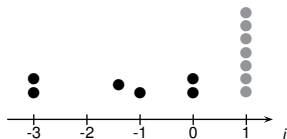


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

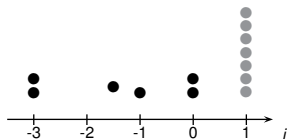


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

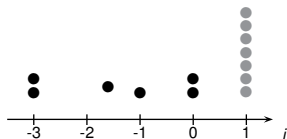


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

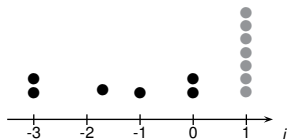


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

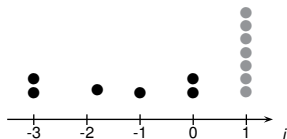


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

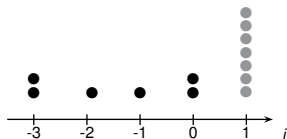


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

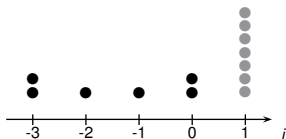


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

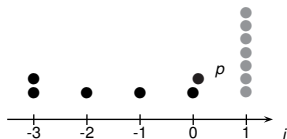


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

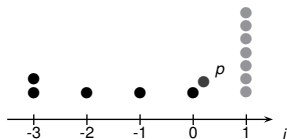


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

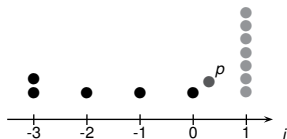


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

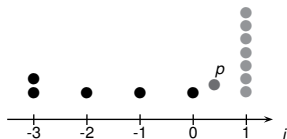


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

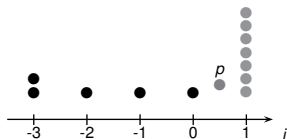


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

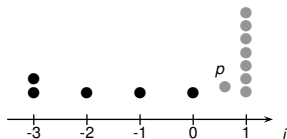


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

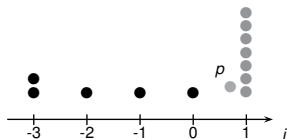


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

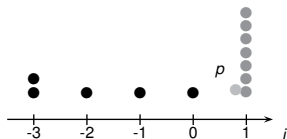


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

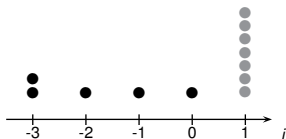


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

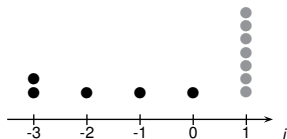


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

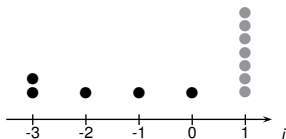


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

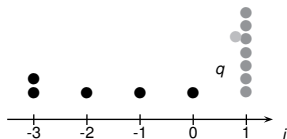


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

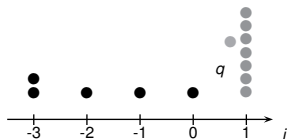


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

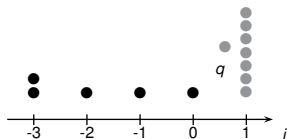


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

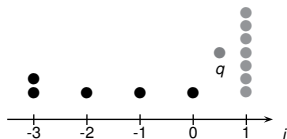


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

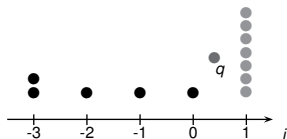


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

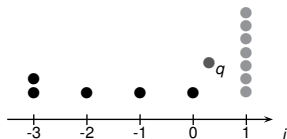


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

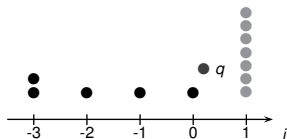


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

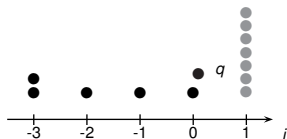


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

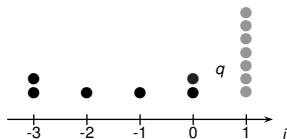


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

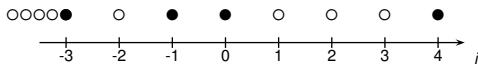
\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.



\rightsquigarrow The product measure stays stationary on the half-line.

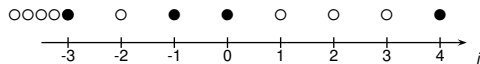
Lay down / stand up

ASEP



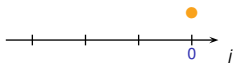
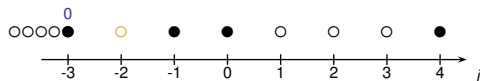
Lay down / stand up

ASEP



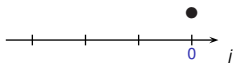
Lay down / stand up

ASEP



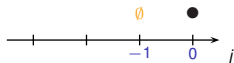
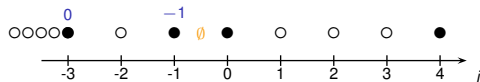
Lay down / stand up

ASEP



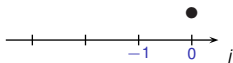
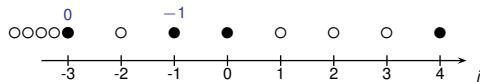
Lay down / stand up

ASEP



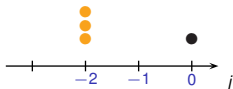
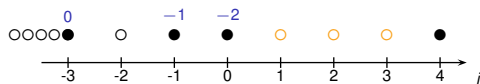
Lay down / stand up

ASEP



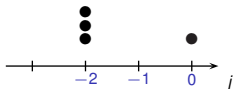
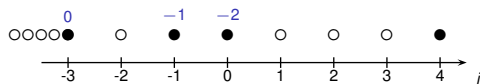
Lay down / stand up

ASEP



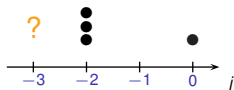
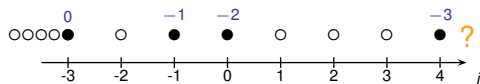
Lay down / stand up

ASEP



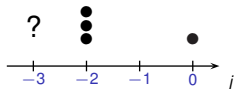
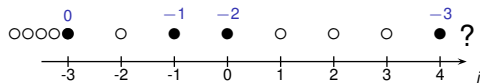
Lay down / stand up

ASEP



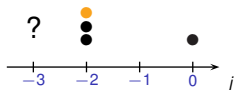
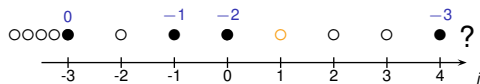
Lay down / stand up

ASEP



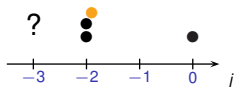
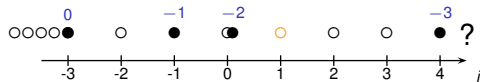
Lay down / stand up

ASEP



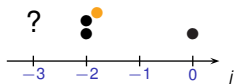
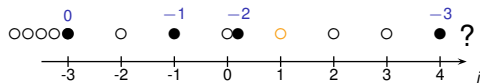
Lay down / stand up

ASEP



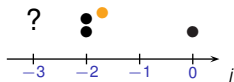
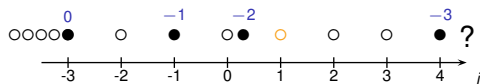
Lay down / stand up

ASEP



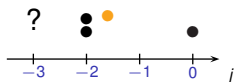
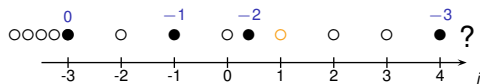
Lay down / stand up

ASEP



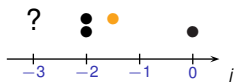
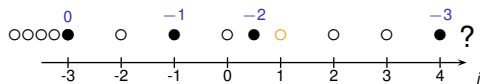
Lay down / stand up

ASEP



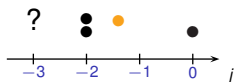
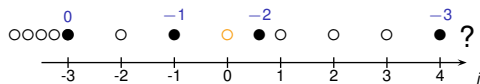
Lay down / stand up

ASEP



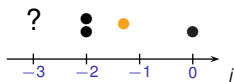
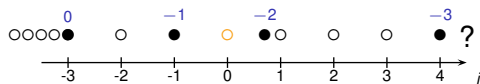
Lay down / stand up

ASEP



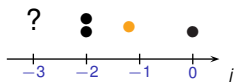
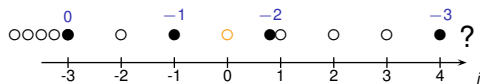
Lay down / stand up

ASEP



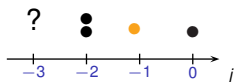
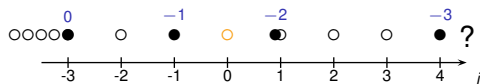
Lay down / stand up

ASEP



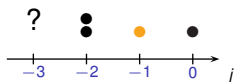
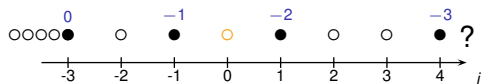
Lay down / stand up

ASEP



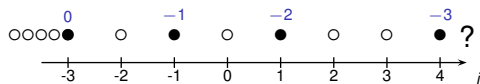
Lay down / stand up

ASEP

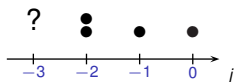


Lay down / stand up

ASEP

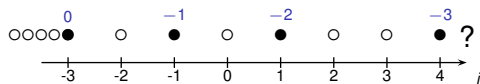


AZRP

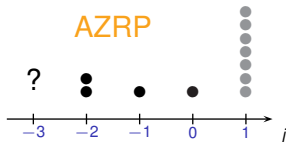


Lay down / stand up

ASEP

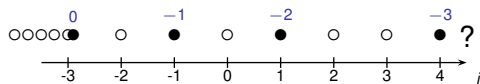


AZRP

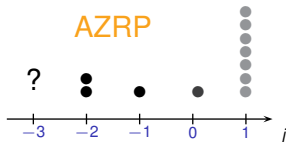


Lay down / stand up

ASEP

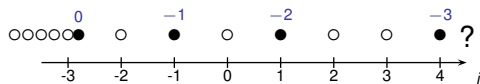


AZRP

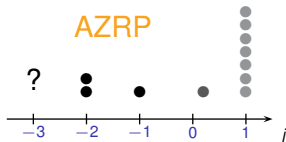


Lay down / stand up

ASEP

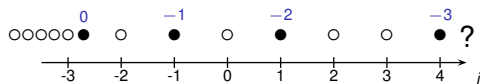


AZRP

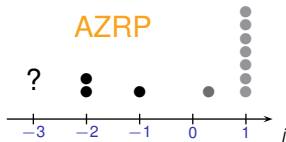


Lay down / stand up

ASEP

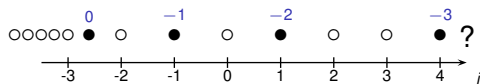


AZRP

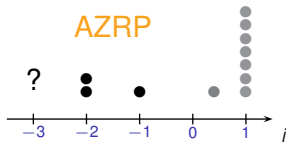


Lay down / stand up

ASEP

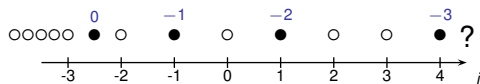


AZRP

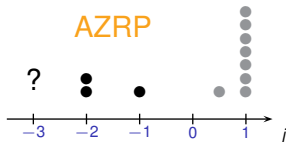


Lay down / stand up

ASEP

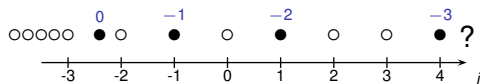


AZRP

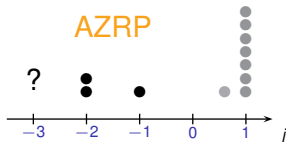


Lay down / stand up

ASEP

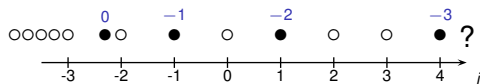


AZRP

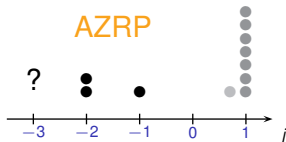


Lay down / stand up

ASEP

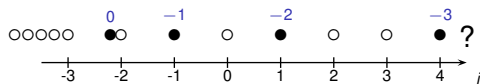


AZRP

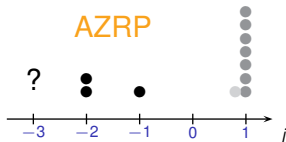


Lay down / stand up

ASEP

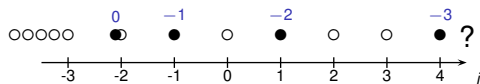


AZRP

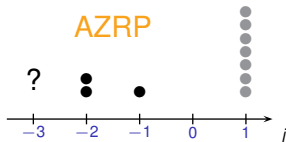


Lay down / stand up

ASEP

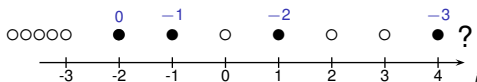


AZRP

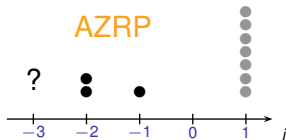


Lay down / stand up

ASEP



AZRP

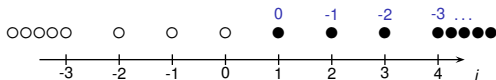
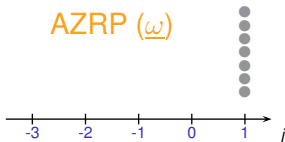


$$\text{ASEP} \stackrel{T^n}{=} \text{AZRP}$$

$$\underline{\nu}^n \stackrel{T^n}{=} \prod_{i \leq 0} \text{Geometric} \left(1 - \left(\frac{p}{q} \right)^{i-1} \right)$$

since stationary distributions of countable irreducible Markov chains are unique.

Jacobi triple product

ASEP ($\underline{\eta}$)AZRP ($\underline{\omega}$)

$$\eta_i = \mathbf{1}\{i \geq 1\}, \quad N(\underline{\eta}) = 0, \quad \omega_i \equiv 0.$$

$$\underline{\nu}^0(\underline{\eta}) = \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c) \cdot 0}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c) \cdot (1-1)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{0^2+0}{2} - c \cdot 0}}$$

$$\underline{\mu}(\underline{\omega}) = \prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right)$$

Jacobi triple product

$$\prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right) \cdot \prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right) \cdot \prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) = \sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}$$

LHS:

$$\begin{aligned} \prod_{i=1}^{\infty} \left(1 - \left(\frac{q}{p}\right)^i\right) \cdot \left(1 + \left(\frac{q}{p}\right)^{i-1+c}\right) \cdot \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) \\ = \prod_{i=1}^{\infty} \left(1 - x^{2i}\right) \left(1 + \frac{x^{2i-1}}{y^2}\right) \left(1 + x^{2i-1}y^2\right) \end{aligned}$$

with $x = \left(\frac{q}{p}\right)^{\frac{1}{2}}$, $y = \left(\frac{q}{p}\right)^{\frac{1}{4}-\frac{c}{2}}$.

RHS:

$$\sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2}{2}} \left(\frac{q}{p}\right)^{m(\frac{1}{2}-c)} = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$



Further models

Product blocking measures are very general.

Further models

Product blocking measures are very general.

- ▶ ASEP

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)
- ▶ Misanthrope / bricklayers processes

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)
- ▶ Misanthrope / bricklayers processes

Other models can be *stood up*:

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)
- ▶ Misanthrope / bricklayers processes

Other models can be *stood up*:

- ▶ ASEP

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)
- ▶ Misanthrope / bricklayers processes

Other models can be *stood up*:

- ▶ ASEP
- ▶ q -exclusion

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)
- ▶ **Misanthrope** / bricklayers processes

Other models can be *stood up*:

- ▶ ASEP
- ▶ q -exclusion
- ▶ **Katz-Lebowitz-Spohn model**

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)
- ▶ Misanthrope / bricklayers processes

Other models can be *stood up*:

- ▶ ASEP
- ▶ q -exclusion
- ▶ Katz-Lebowitz-Spohn model

Further models

Product blocking measures are very general.

- ▶ ASEP
- ▶ K -exclusion (!)
- ▶ All zero range processes (“classical”, independent walkers, q -zero range)
- ▶ Misanthrope / bricklayers processes

Other models can be *stood up*:

- ▶ ASEP
- ▶ q -exclusion
- ▶ Katz-Lebowitz-Spohn model

The point was that ASEP is in both lists.

Thank you.