Homework set 1

Convolutions

Further Topics in Probability, 2nd teaching block, 2019

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Problems with $^\bullet$'s are to be handed in. These are due in the blue locker marked "Further Topics in Probability" on the ground floor of the Main Maths Building before 12:00pm on Thursday, 7th February. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Hence, for example, Problem 1.1 worth three marks. Random variables are defined on a common probability space unless otherwise stated.

1.1 $^{\bullet\bullet\bullet}$ Determine the mass function of the sum of the numbers shown by two fair dice after rolling them.

1.2 $^{\bullet\bullet\bullet}$ Let X be a discrete uniform random variable on the set $\{0, 1, \dots, n-1\}$ (that is, it takes on any of these values with probability $1/n$). Argue that if n is not a prime number then the distribution of X can be written as the convolution of two integer-valued distributions.

1.3 Show analytically that $*^r\mathrm{Geom}(p) = \mathrm{NegBinom}(r, p)$. *HINT: Can use induction on the sum of binomial coefficients that emerges.*

1.4 $^{\bullet\bullet\bullet}$ Let $X \sim \mathrm{Binomial}(n, p)$ and $Y \sim \mathrm{Uniform}(0, 1)$ be independent random variables. Find the distribution function of the random variable $Z := X + Y$.

1.5 $^{\bullet\bullet\bullet\bullet}$ A man takes the train and then transfers to the bus when commuting to work each day. It takes two minutes for him to walk from the train to the bus at the transfer station. *In principle* the train arrives at 7:30am, and the bus leaves at 7:37am. The fact is that the train arrives at a normally distributed random time, having mean 7:30am and standard deviation 4 minutes. Independently, the bus leaves at a normally distributed random time with mean 7:37am and standard deviation 3 minutes. What is the probability that our man misses his bus at most once on the five working days of the week? *HINT: Use what you know about the sum of independent normal variables.*

1.6 *(This is not convolution, just a bit of outlook.)* Let X and Y be iid. Uniform random variables on the interval $(0, 1)$. Find the density of their product XY.

1.7 $^{\bullet\bullet\bullet\bullet}$ Let X and Y be iid. random variables, each with density $f(x) = 2x \cdot \mathbf{1}\{x \in [0, 1]\}$ ($\mathbf{1}$ stands for the indicator function here). Determine the density of the random variables

a) $U := X + Y$;

b) $V := X - Y$.

1.8 Let $X_1, X_2, \dots, X_n, \dots$ be iid. random variables, each of Uniform$(0, 1)$ distribution. Denote by $f_n(x)$ the density of the random variable $S_n := \sum_{k=1}^n X_k$. Prove

$$f_n(x) = \frac{1}{(n-1)!}\sum_{k=0}^{\lfloor x \rfloor}(-1)^k\binom{n}{k}(x-k)^{n-1}.$$

Plot, using a computer,

$$\widetilde{f}_n(x) := \sqrt{\frac{n}{12}} f_n\Big(\frac{n}{2} + \sqrt{\frac{n}{12}}x\Big)$$

for $n = 1, 2, \dots, 10$. What do you see?

1.9 Let $X_1, X_2, \dots, X_n, \dots$ be iid. random variables, each of distribution $\mathbf{P}\{X_i = 0\} = \mathbf{P}\{X_i = 1\} = \frac{1}{2}$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. Argue that Y is uniformly distributed on the real interval $[0, 1]$. *HINT: It is enough to show that probabilities of Y falling in intervals agree with those of the uniform distribution. The two ends of an interval can be approximated by numbers in the base $\frac{1}{2}$ number system.*

1.10 Let $X_1, X_2, \dots, X_n, \dots$ be iid. random variables, each of distribution $\mathbf{P}\{X_i = 0\} = p \neq \mathbf{P}\{X_i = 1\} = 1 - p$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. Argue that Y is *singular* w.r.t. the uniform distribution on the interval $[0, 1]$, that is,

$$\forall \varepsilon > 0 \quad \exists A \subset \mathbb{R} \quad : \quad \mathbf{P}\{Y \in A\} \geq 1 - \varepsilon, \qquad \mathbf{P}\{U \in A\} \leq \varepsilon,$$

where U is uniformly distributed over $(0, 1)$. (In fact, A has to be Borel measurable, but don't worry about this.) *HINT: Again, think in the base $\frac{1}{2}$ number system. The Weak Law of Large Numbers says that the probability that the average of the first n X_i's is off from p by more than δ converges to zero as $n \to \infty$.*