Homework set 1

Measure theory background

Martingale Theory with Applications, 1^{st} teaching block, 2021

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Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 7^{th} October. Please show your work leading to the result, not only the result. Each problem is worth the number of •'s you see right next to it. Hence, for example, Problem 1.1 is worth three marks.

1.1 ••• Let $(\Omega, \mathcal{F})$ be a measurable space. Prove that if $A, B \in \mathcal{F}$, then

$$A \cap B, \qquad A - B \text{ (set-difference,)} \qquad A \Delta B \text{ (symmetric set-difference)}$$

are also in $\mathcal{F}$.

1.2 •••• Is the union of two σ-algebras (on the same set) also a σ-algebra? If yes, prove it, if no, give a counterexample.

1.3 •••• Is the intersection of two σ-algebras (on the same set) also a σ-algebra? If yes, prove it, if no, give a counterexample.

1.4 ••• Define the Borel σ-algebra on $\mathbb{R}$ as we did in class:

$$\mathfrak{B}(\mathbb{R}) := \sigma\Big\{\bigcup_{i=1}^{n}(a_i,\, b_i] \,:\, n < \infty,\ a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n \text{ in } \mathbb{R} \cup \{\infty\} \cup \{-\infty\}\Big\}.$$

Show that each of

$$(a,\, b), \qquad [a,\, b), \qquad [a,\, b], \qquad \{a\}, \qquad (a,\, \infty)$$

are in $\mathfrak{B}(\mathbb{R})$ for any $a < b$ in $\mathbb{R}$.

1.5 (*Shiryaev.*) Let Ω be a countable set and $\mathcal{F}$ the collection of all its subsets. Put $\mu(A) = 0$ if A is finite and $\mu(A) = \infty$ if A is infinite. Show that the set function μ is finitely additive but not σ-additive.

1.6 ••• (*Shiryaev.*) Let μ be the Lebesgue-Stieltjes measure generated by a continuous distribution function. Show that if the set A is at most countable, then $\mu(A) = 0$.

1.7 (*Construction of the Vitali set – an example that cannot be Lebesgue measurable.*) Let $\Omega := [0,\, 1)$ and define on Ω the following equivalence relation:

$$x \sim y \text{ iff } x - y \in \mathbb{Q} \text{ (the rational numbers).}$$

Let $V \subset [0,\, 1)$ consist of *exactly one representative element from each equivalence class of* $\sim$. (Notice: this construction relies on the Axiom of Choice.) For $q \in \mathbb{Q} \cap [0,\, 1)$, denote

$$V_q := \{x + q \pmod 1 \,:\, x \in V\}.$$

Prove that

(a) The sets V_q are congruent: for any $q,\, q' \in \mathbb{Q} \cap [0,\, 1)$, $V_{q'} = (q' - q) + V_q \pmod 1$.

(b) If $q \neq q'$ in $\mathbb{Q} \cap [0,\, 1)$, then $V_q \cap V_{q'} = \emptyset$.

(c) $\bigcup_{q \in \mathbb{Q} \cap [0,\,1)} V_q = [0,\, 1)$.

Conclude that the Vitali set V cannot be Lebesgue measurable.