Homework set 2

Convolutions, generating fcts.

Further Topics in Probability, 2^{nd} teaching block, 2019

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Problems with $^\bullet$'s are to be handed in. These are due in the blue locker marked "Further Topics in Probability" on the ground floor of the Main Maths Building before 12:00pm on Thursday, 21^{st} February. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

2.1 Prove that if ξ is a Cauchy random variable with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, then $X := 1/\xi$, $Y := 2\xi/(1-\xi^2)$, and $Z := (3\xi - \xi^3)/(1-3\xi^2)$ are also Cauchy distributed. *HINT: use the trigonometric identities: if* $\xi = \tan(\alpha)$, *then* $1/\xi = \tan(\frac{\pi}{2} - \alpha)$, $2\xi/(1-\xi^2) = \tan(2\alpha)$, *and* $(3\xi - \xi^3)/(1-3\xi^2) = \tan(3\alpha)$.

2.2 Let X be a Cauchy random variable with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. We know that $\mathbf{E}|X| = \infty$, but $\mathbf{E}(|X|^{1-\varepsilon}) < \infty$ for any $\varepsilon > 0$. Determine the limit $\lim\limits_{\varepsilon \searrow 0} \varepsilon \mathbf{E}(|X|^{1-\varepsilon})$. *HINT: Split the integral at a well chosen point.*

2.3 Let X and Y be iid. standard Normal random variables. Determine the joint density of the pair $U = X$, $V = X/Y$. Then use it to show that X/Y has the standard Cauchy distribution.

2.4 We are given a biased coin that shows Heads with probability p and Tails with probability $1-p$. The coin initially shows Heads, and at time moments given by a Poisson(λ) process we flip the coin again and again. What is the probability that the coin shows Heads at time t?

2.5 $^{\bullet\bullet\bullet\bullet}$ (*With a good understanding of stuff, you can solve this problem without any computation.*) Let X_1, X_2, ... be iid. random variables each with density $x\mathrm{e}^{-x}$ for $x > 0$ and 0 otherwise. Let $S_0 = 0$ and $S_n := X_1 + \cdots + X_n$, and $N(t) := \max\{n : S_n < t\}$.

a) Determine the density of the random variable S_2.

b) Find the mass function of the random variable $N(t)$.

2.6 Show, using the convolution formula, that Gamma(α, λ) and Gamma(β, λ) convolve to Gamma($\alpha + \beta$, λ) for any positive reals α, β, λ. Along the way there will be a tricky integral, which doesn't have to be explicitly calculated as long as you can scale out its dependence on the important variable. You can then use the fact that Gamma densities are normalized (thus proving a representation of Beta functions this way).

2.7 $^{\bullet\bullet\bullet}$ Find the density of the difference of an Exponential(λ) and an independent Exponential(μ) random variable. Can you interpret the result? (What happens for $\lambda = \mu$?)

2.8 $^{\bullet\bullet\bullet\bullet}$ Are the following functions generating functions of non-negative integer probability distributions?

(a) $\exp\left(\frac{s-1}{\lambda}\right)$, $\lambda > 0$; (b) $\frac{(s+1)^7}{128}$;

(c) $\frac{2}{2-s}$; (d) $\frac{2}{1+s}$.

2.9 ••• Let X be a non-negative integer-valued random variable with generating function $P(s)$. Find the generating function for the distribution of $Y := X + 3$ and the one for $Z := 5X$.

2.10 ••• Let $X_1, X_2, X_3, \ldots$ be iid. random variables with common distribution function F. Let $\nu \geq 1$ an independent, integer-valued variable with generating function P, and define $Y := \min\{X_1, X_2, X_3, \ldots, X_\nu\}$. Show that Y has distribution function $1 - P\big(1 - F(\cdot)\big)$.