Homework set 2

Probabilistic tools

Martingale Theory with Applications, 1^{st} teaching block, 2022

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Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 20^{th} October. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it.

2.1 ••• Consider the example from the exercise class: $\Omega = \{1, 2, \ldots, 12\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathbb{P}$ is uniform on Ω, and the random variables X and Y are defined by $X(\omega) = \lceil \frac{\omega}{2} \rceil$, $Y(\omega) = \lceil \frac{\omega}{4} \rceil$, $\mathcal{G} = \sigma(Y)$. Show by explicit calculations in this example that $\mathbb{E}(XY \,|\, \mathcal{G}) = Y\,\mathbb{E}(X \,|\, \mathcal{G})$. This is referred to as '*taking out what's known*' or '*given Y, Y is not random*'.

2.2 ••• In the example of Problem 2.1, let $\mathcal{H} := \sigma(X)$. Calculate each of

- $\mathbb{E}\big(\mathbb{E}(X \,|\, \mathcal{G}) \,|\, \mathcal{H}\big)$,
- $\mathbb{E}\big(\mathbb{E}(X \,|\, \mathcal{H}) \,|\, \mathcal{G}\big)$,
- $\mathbb{E}\big(\mathbb{E}(Y \,|\, \mathcal{G}) \,|\, \mathcal{H}\big)$,
- $\mathbb{E}\big(\mathbb{E}(Y \,|\, \mathcal{H}) \,|\, \mathcal{G}\big)$.

Compare with $\mathbb{E}(X \,|\, \mathcal{G})$, $\mathbb{E}(X \,|\, \mathcal{H})$, $\mathbb{E}(Y \,|\, \mathcal{G})$, $\mathbb{E}(Y \,|\, \mathcal{H})$. *It is important here that one of the two σ-algebras contains the other!*

2.3 However, give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, sub-$\sigma$ algebras $\mathcal{F}_1 \subset \mathcal{F}, \mathcal{F}_2 \subset \mathcal{F}$, and a random variable X such that

$$\mathbb{E}\big(\mathbb{E}(X \,|\, \mathcal{F}_1) \,|\, \mathcal{F}_2\big) \neq \mathbb{E}\big(\mathbb{E}(X \,|\, \mathcal{F}_2) \,|\, \mathcal{F}_1\big).$$

Why is it not a contradiction with the previous problem?

2.4 (*Monty Hall problem with σ-algebras.*) The famous Monty Hall problem goes like this:

We have three doors. Behind one of them is a car, behind the others, goats.

1. You pick a door, let us assume it's door number 1.
2. Monty opens *another door with a goat behind it.*
3. Now you pick one of the two closed doors (repeat your choice, or switch to the other one).
4. Whatever is behind this door is yours.

Make the natural assumptions about the probabilities of the location of the car and the choice of door Monty opens (if he *has* a choice). Would you repeat your choice or switch?

(a) Write the full probability space of the experiment that involves the first two steps above.

(b) In this sample space, write the event $A = \{\text{door 3 has a goat}\}$, and its generated σ-algebra $\mathcal{F} = \sigma(A)$.

(c) Let $X = 1, 2, 3$ be the location of the car. Calculate $\mathbb{E}(X \,|\, \mathcal{F})$.

(d) Now write the event $B = \{\text{Monty opens door 3}\}$, and its generated σ-algebra $\mathcal{G} = \sigma(B)$.

(e) Calculate $\mathbb{E}(X \mid \mathcal{G})$.

(f) Conclude the optimal strategy for the player in this problem.

2.5 ••• Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} = \sigma(Y)$. Show that X is independent of $\mathcal{G}$ if and only if for any bounded and measurable functions f and g, we have $\mathbb{E}\big(f(X) \cdot g(Y)\big) = \mathbb{E}\, f(X) \cdot \mathbb{E}\, g(Y)$ (the *Probability 1* definition of independence).

2.6 Let A and B be two events in a probability space, B of positive probability. Derive the *Probability 1* definition of the conditional probability $\mathbb{P}\{A \mid B\}$ from our definition of conditional expectations.

2.7 Based on your definition above, show that for any fixed event B of positive probability in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the set function $\mathbb{Q}(\cdot) := \mathbb{P}\{\cdot \mid B\}$ is a probability measure.

2.8 Continuing the previous problem, show that for any events B and C with a positive probability intersection,

$$\mathbb{Q}(\cdot \mid C) = \mathbb{P}(\cdot \mid B \cap C).$$

2.9 Fix $0 < p = 1 - q < 1$ reals, let $\Omega = \{0, 1, 2, \dots\}$ be the non-negative integers, $\mathcal{F} = \mathcal{P}(\Omega)$ the power set, and $\mathbb{P}$ the probability measure that assigns $\mathbb{P}\{n\} = q^n p$ to $n \geq 0$. Define the function $X : \Omega \to \{0, 1, 2, \dots\}$ to be the identity function. Notice that so far this is a way to describe a Pessimistic Geometric(p) random variable (counting the failures before the first success). Define also

$$Y := (X \mod 2) = \mathbf{1}\{X \text{ is odd}\} = \begin{cases} 0, & \text{if } X \text{ is even,} \\ 1, & \text{if } X \text{ is odd.} \end{cases}$$

(a) What is the σ-algebra generated by X?

(b) What is the σ-algebra generated by Y?

(c) Use Kolmogorov's theorem on conditional expectations to calculate $\mathbb{E}(X \mid Y)$.

2.10 ••••• Let X and Y be two i.i.d. Exp(1) random variables, and $Z = X + Y$.

(a) Write down an actual probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to model this situation.

(b) Fix any $t \in \mathbb{R}$ with $0 < |t| < 1$ and show that the random variable $\frac{\mathrm{e}^{tZ}-1}{tZ}$ satisfies Kolmogorov's theorem on conditional expectations for $\mathbb{E}(\mathrm{e}^{tX} \mid Z)$, hence by uniqueness $\frac{\mathrm{e}^{tZ}-1}{tZ}$ is a version of this conditional expectation.
Hint1: recall from earlier studies or accept the fact that the expectation of a function of X and Y is calculated as a double integral of that function with the joint density, which latter in our case is just the product of the marginal Exponential densities due to independence.
Hint2: $\sigma(Z)$ is generated by events of the form $G = \{Z \leq z\}$, hence it is enough to work with these.
Hint3: A substitution of $v = x + y$, then swapping integrals might prove useful.

(c) Conclude that $(X \mid Z) \sim \text{Uniform}(0, Z)$. In other words, X given the sum $Z = X+Y$ has the uniform distribution on the allowed range $(0, X + Y)$.

2.11 Let $X_1, X_2, \dots, X_n$ be iid. random variables with finite mean, and S_n their sum. Calculate $\mathbb{E}(X_1 \mid S_n)$.

2.12 Let X and Y be random variables with finite mean on a probability space. Prove that if $\mathbb{E}(X \mid Y) = Y$ and $\mathbb{E}(Y \mid X) = X$, then $X = Y$ a.s.

2.13 Let X and Y be random variables with finite second moment on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-σ algebra of $\mathcal{F}$. Suppose that $\mathbb{E}(X \mid \mathcal{G}) = Y$ and $\mathbb{E}\, X^2 = \mathbb{E}\, Y^2$. Prove that $X = Y$ a.s.

2.14 Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X \mid \mathcal{G}) = X$ which suggests that the map $X \mapsto \mathbb{E}(X \mid \mathcal{G})$ is a projection. Show that indeed: this map is an orthogonal projection in the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ (with inner product $\langle X, Y \rangle_{\mathbb{P}} = \mathbb{E}(XY)$) onto the subspace $\mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$.

2.15 ••• We perform infinitely many independent experiments. The n^{th} one is successful with probability $n^{-\alpha}$ and fails with probability $1 - n^{-\alpha}$, $0 < \alpha$. Let $k \geq 1$. We are happy if we see k consecutive successes infinitely often. What is the probability of this?

2.16 (The longest run of heads, I.)
Let $X_1, X_2, \ldots$ be iid. random variables with $\mathbf{P}\{X_k = 1\} = p$, $\mathbf{P}\{X_k = 0\} = q$, where $p + q = 1$. Fix a parameter $\lambda > 1$, and denote by $A_k^{(\lambda)}$ the following events for $k = 0, 1, 2, \ldots$:

$$A_k^{(\lambda)} := \Big\{ \exists r \in \big[\lfloor \lambda^k \rfloor, \lfloor \lambda^{k+1} \rfloor - k \big] \cap \mathbb{N} \,:\, X_r = X_{r+1} = \cdots = X_{r+k-1} = 1 \Big\}.$$

In plain words: $A_k^{(\lambda)}$ means that somewhere between $\lfloor \lambda^k \rfloor$ and $\lfloor \lambda^{k+1} \rfloor - 1$ there is a sequence of k consecutive 1's. Prove that

a) If $\lambda < p^{-1}$, then a.s. only finitely many of the events $A_k^{(\lambda)}$ occur.
b) If $\lambda > p^{-1}$, then a.s. infinitely many of the events $A_k^{(\lambda)}$ occur.
c) What happens for $\lambda = p^{-1}$?

2.17 (The longest run of heads, II.)
Let

$$R_n := \sup\{k \geq 0 \,:\, X_n = X_{n+1} = \cdots = X_{n+k-1} = 1\}.$$

That is: R_n is the length of the run of consecutive 1's that starts at n. (If $X_n = 0$, then set $R_n = 0$.) Prove that

$$\mathbf{P}\left\{ \limsup_{n \to \infty} \frac{R_n}{\log n} = |\log p|^{-1} \right\} = 1.$$

HINT: For a fixed parameter $\alpha > 0$, let

$$B_n^{(\alpha)} := \{R_n > \alpha \log n / |\log p|\}.$$

If $\alpha > 1$, then by the first Borel-Cantelli Lemma and direct computation, only finitely many of the $B_n^{(\alpha)}$'s occur a.s. If $\alpha \leq 1$, then from the previous exercise it follows that a.s. infinitely many of the $B_n^{(\alpha)}$'s occur.