HOMEWORK SET 3

Generating fuctions, random walk, Law of Rare Events, renewal processes

Further Topics in Probability, 1st teaching block, 2013

School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in class or in the blue locker with my name in the Main Maths Building before 11:59pm on Wednesday, 6th November. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

3.1 Car traffic of a street is modeled by

a) dividing the time into unit seconds;
b) assuming that with probability $0 < p < 1$ a car passes in a given second;
c) events (of cars passing or not) in different seconds are independent of each other.

A pedestrian wants to cross this street, and she can do so if no cars come in three consecutive seconds. (We assume that the pedestrian sees enough of that street to decide if she can cross safely.) Find the generating function and expectation of the time the pedestrian spends on waiting for the street to clear enough so that she can cross.

3.2 •••• A monkey repeatedly types in any of the 26 letters of the English alphabet independently with equal chance. Let T be the number of letters typed in when the word "DAD" first appears. Find the generating function and the expected value of T.

3.3 Little Johnny regularly speeds with his car, and police regularly stop him. Every such instance he pays £ 100 or £ 500 with probability 1/2-1/2, independently of everything else. Moreover, Johnny handles such situations in a very bad manner, and the blustering fellow he is, every such occasion also results in his driving licence being revoked with probability p. Find the generating function, expectation and variance of the total amount of fines paid by Johnny.

3.4 ••• (*Resnick.*) **A Skip Free Negative Random Walk.** Suppose $\{X_n, n \ge 1\}$ is independent, identically distributed. Define $S_0 = X_0 = 1$ and for $n \ge 1$

$$S_n = X_0 + X_1 + \cdots + X_n.$$

For $n \ge 1$ the distribution of X_n is specified by

$$\mathbf{P}\{X_n = j - 1\} = p_j, \qquad j = 0, 1, \ldots$$

where

$$\sum_{j=0}^{\infty} p_j = 1, \qquad f(s) = \sum_{j=0}^{\infty} p_j s^j, \quad 0 \le s \le 1.$$

(The random walk starts at 1; when it moves in the negative direction, it does so only by jumps of -1. The walk cannot jump over states when moving in the negative direction.) Let

$$N = \inf n \,:\, S_n = 0.$$

If $P(s) = \mathbf{E}s^N$, show $P(s) = sf(P(s))$. (Note what happens at the first step: Either the random walk goes from 1 to 0 with probability p_0 or from 1 to j with probability p_j.) If $f(s) = p/(1 - qs)$ corresponding to a geometric distribution, find the smallest solution.

3.5 Consider the simple random walk that starts from the root of a rooted tree of degree $d \geq 3$. That is, in each step the walker picks independently any of the d neighbors of a vertex to step on with equal chance. Compute the generating function of

a) the hitting time of a given neighbor of the root;

b) the generating function of the first return time to the origin;

c) the generating function of the last return time to the origin.

3.6 ••• (*Resnick.*) For a simple random walk $\{S_n\}$ let $u_0 = 1$ and for $n \geq 1$, let

$$u_n = \mathbf{P}\{S_n = 0\}.$$

Compute by combinatorics the value of u_n. Find the generating function

$$U(s) = \sum_{n=0}^{\infty} u_n s^n$$

in closed form. To get this in closed form you need the identity

$$\binom{2n}{n} = 4^n (-1)^n \binom{-\frac{1}{2}}{n}.$$

The binomial coefficient for general $\alpha \in \mathbb{R}$ and positive integer k is defined as

$$\binom{\alpha}{k} = \frac{\alpha \cdot (\alpha - 1) \cdot \dots \cdot (\alpha - k + 1)}{k!}.$$

With this definition the Binomial Theorem holds:

$$\forall x,\, y \in \mathbb{R}, \qquad (x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha - k}.$$

3.7 Let τ be the hitting time of level 1 in a symmetric simple random walk. Find its mass function $\mathbf{P}\{\tau = k\}$ and the limit $\lim_{k\to\infty} k^{3/2} \cdot \mathbf{P}\{\tau = k\}$. You may want to use the Binomial Theorem for the root of a sum, and the Stirling formula to take the above limit.

3.8 ••• Let X_r be a sequence of pessimistic Negative Binomial(r, $p(r)$) random variables with $p(r)$ such that $r \cdot (1 - p(r)) \to \lambda$ with $0 < \lambda < \infty$ as $r \to \infty$. Show that X_r converges to Poisson(λ) in distribution. (Notice that, as the pessimistic Negative Binomial is the sum of independent pessimistic Geometrics, this statement is similar in flavour to the Law of Rare Events.)

3.9 •••• (*Resnick.*) Find the renewal function $U(t) = \mathbf{E}N_t$ corresponding to

a) $F'(x) = \lambda \mathrm{e}^{-\lambda x}, \quad \alpha > 0, x > 0$. (This is the Poisson process.)

b) $F'(x) = \lambda^2 x \mathrm{e}^{-\lambda x}, \quad \alpha > 0, x > 0$. (This is a Gamma density.)

In both examples, verify $\lim_{t\to\infty} U(t)/t = 1/\mu$ directly. *HINT for b): What is the Taylor series of* $\mathrm{e}^y \pm \mathrm{e}^{-y}$*?*