Homework set 3

Generating functions

Further Topics in Probability, 2nd teaching block, 2019

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Problems with •'s are to be handed in. These are due in class or in the blue locker marked "Further Topics in Probability" on the ground floor of the Main Maths Building before 12:00pm on Thursday, 7th March. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

3.1 ••• Let $X_1, X_2, \dots$ be iid. Optimistic Geometric(p_1) random variables, and Z an independent Optimistic Geometric(p_2) random variable. Prove, using generating functions, that

$$\sum_{i=1}^{Z} X_i \sim \text{Geometric}(p_1 p_2).$$

Give a probabilistic interpretation of this fact.

3.2 The distribution of a random variable X is called *infinitely divisible*, if for every $n \geq 1$ integer there exist $Y_1^{(n)}, \dots, Y_n^{(n)}$ iid. random variables such that

$$\sum_{i=1}^{n} Y_i^{(n)} \sim X.$$

a) Is the Poisson distribution infinitely divisible?

b) Is the Binomial distribution infinitely divisible?

c) Prove that for every $0 < p < 1$ there exists a probability mass function $p_1, p_2, \dots$ and $\lambda > 0$ with which $\sum_{i=1}^{Z} X_i \sim$ Pessimistic Geometric(p), where $X_1, X_2, \dots$ are iid., for $k \geq 1$ $\mathbf{P}\{X_i = k\} = p_k$, and Z is an independent Poisson(λ) random variable.

d) Show, with the help of c), that the Geometric distribution is infinitely divisible.

3.3 a) •• (*Resnick.*) In a branching process

$$P(s) = as^2 + bs + c$$

where $a > 0$, $b > 0$, $c > 0$, $P(1) = 1$. Compute π. Give a condition for sure extinction.

b) •• Little Johnny is salesman in a store where servicing a customer takes exactly 1 minute. Under this time, with probability 0.6 two new customers arrive in the queue, with probability 0.2 one new customer arrives, with probability 0.2 no new customers appear. Little Johnny can have his coffee if the queue empties out. What is the probability that this ever happens after the first customer steps in the store? Explain.

3.4 In a branching process let $X = \sum_{n=0}^{\infty} Z_n$ be the number of individuals that ever existed. You can use (we saw a little bit of this in class for binary branching, feel free to prove it in general!) that its generating function is

$$Q(s) = \left(\frac{s}{P(s)}\right)^{-1} \quad \leftarrow\text{function inverse}.$$

a) Find $Q(s)$ when the distribution of the number of children of an individual is Bernoulli(p).

b) Find $Q(s)$ when the distribution of the number of children of an individual is Pessimistic Geometric(p).

c) Find $\mathbf{E}X$ in both cases.

3.5 Car traffic of a street is modeled by

a) dividing the time into unit seconds;

b) assuming that with probability $0 < p < 1$ a car passes in a given second;

c) events (of cars passing or not) in different seconds are independent of each other.

A pedestrian wants to cross this street, and she can do so if no cars come in three consecutive seconds. (We assume that the pedestrian sees enough of that street to decide if she can cross safely.) Find the generating function and expectation of the time the pedestrian spends on waiting for the street to clear enough so that she can cross.

3.6 •••• A monkey repeatedly types in any of the 26 letters of the English alphabet independently with equal chance. Let T be the number of letters typed in when the word "DAD" first appears. Find the generating function and the expected value of T.

3.7 Little Johnny regularly speeds with his car, and police regularly stop him. Every such instance he pays £ 100 or £ 500 with probability 1/2-1/2, independently of everything else. Moreover, Johnny handles such situations in a very bad manner, and the blustering fellow he is, every such occasion also results in his driving licence being revoked with probability p. Find the generating function, expectation and variance of the total amount of fines paid by Johnny.

3.8 ••• (*Resnick.*) **A Skip Free Negative Random Walk.** Suppose $\{X_n,\, n \geq 1\}$ is independent, identically distributed. Define $S_0 = X_0 = 1$ and for $n \geq 1$

$$S_n = X_0 + X_1 + \cdots + X_n.$$

For $n \geq 1$ the distribution of X_n is specified by

$$\mathbf{P}\{X_n = j - 1\} = p_j, \qquad j = 0,\, 1,\, \ldots$$

where

$$\sum_{j=0}^{\infty} p_j = 1, \qquad f(s) = \sum_{j=0}^{\infty} p_j s^j, \quad 0 \leq s \leq 1.$$

(The random walk starts at 1; when it moves in the negative direction, it does so only by jumps of -1. The walk cannot jump over states when moving in the negative direction.) Let

$$N = \inf\{n \,:\, S_n = 0\}.$$

If $P(s) = \mathbf{E}s^N$, show $P(s) = sf(P(s))$. (Note what happens at the first step: Either the random walk goes from 1 to 0 with probability p_0 or from 1 to j with probability p_j.) If $f(s) = p/(1 - qs)$ corresponding to a geometric distribution, find the smallest solution.

3.9 Consider the simple random walk that starts from the root of a rooted tree of degree $d \geq 3$. That is, in each step the walker picks independently any of the d neighbors of a vertex to step on with equal chance. Compute the generating function of

a) the hitting time of a given neighbor of the root;

b) the generating function of the first return time to the origin;

c) the generating function of the last return time to the origin.

3.10 ••• (*Resnick.*) For a simple random walk $\{S_n\}$ let $u_0 = 1$ and for $n \geq 1$, let

$$u_n = \mathbf{P}\{S_n = 0\}.$$

Compute by combinatorics the value of u_n. Find the generating function

$$U(s) = \sum_{n=0}^{\infty} u_n s^n$$

in closed form. To get this in closed form you need the identity

$$\binom{2n}{n} = 4^n (-1)^n \binom{-\frac{1}{2}}{n}.$$

The binomial coefficient for general $\alpha \in \mathbb{R}$ and positive integer k is defined as

$$\binom{\alpha}{k} = \frac{\alpha \cdot (\alpha - 1) \cdot \cdots \cdot (\alpha - k + 1)}{k!}.$$

With this definition and assuming $|x| < |y|$, the Binomial Theorem holds:

$$\forall x,\, y \in \mathbb{R}, \qquad (x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha - k}.$$