HOMEWORK SET 3

Martingales, stopping times, convergence

Martingale Theory with Applications, 1st teaching block, 2019

School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in the locker with "Martingale Theory" on it in G.90 Fry Building before 12:00pm on Monday, 11th November. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Hence, for example, Problem 3.4 worth four marks.

3.1 *Bellman's Optimality Principle.* We model a sequence of gamblings as follows. Let $\xi_1, \xi_2, \dots$ be iid. random variables with $\mathbb{P}\{\xi_n = +1\} = p$, $\mathbb{P}\{\xi_n = -1\} = q$, where $p = 1 - q > 1/2$. Define the *entropy* of this distribution by

$$\alpha = p\ln\Big(\frac{p}{1/2}\Big) + q\ln\Big(\frac{q}{1/2}\Big) = p\ln p + q\ln q + \ln 2.$$

A gambler starts playing with initial fortune $Y_0 > 0$. Her return at time n on a *unit bet* is the random variable ξ_n, and she plays C_n in round n. In other words, with probability p she doubles her bet and with probability q she looses it. Therefore her fortune after round n is

$$Y_n = Y_{n-1} + C_n\xi_n.$$

The bet C_n may depend on the values $\xi_1, \xi_2, \dots, \xi_{n-1}$, and has bounds $0 \le C_n \le Y_{n-1}$. The expected rate of winnings up to time n is

$$r_n := \mathbb{E}\ln\Big(\frac{Y_n}{Y_0}\Big),$$

which the gambler wishes to maximise.

(a) Prove that no matter what strategy C the gambler chooses,

$$X_n := \ln Y_n - n\alpha$$

is a supermartingale, hence her expected average winning rate, $\frac{r_n}{n} \le \alpha$.

(b) However, there exists a gambling strategy that makes the above X a martingale, hence realises the average expected winning rate α. Find this strategy.

3.2 *Galton-Watson Branching Process.* Let $\xi_{n,k}$, $n = 1, 2, \dots$, $k = 1, 2, \dots$ be iid. non-negative integer random variables with finite mean μ and variance σ^2. Define the Galton-Watson branching process

$$Z_0 := 1, \qquad Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n+1,k},$$

and let $\mathcal{F}_n := \sigma(Z_j : 0 \le j \le n)$, $n \ge 0$ be the natural filtration.

(a) Prove that $M_n := Z_n/\mu^n$, $n \ge 0$ is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(b) Prove that $\mathbb{E}(Z_{n+1}^2 \,|\, \mathcal{F}_n) = \mu^2 Z_n^2 + \sigma^2 Z_n$.

(c) Using the result from (b) prove that

$$N_n := \begin{cases} M_n^2 - \dfrac{\sigma^2}{\mu^{n+1}}\dfrac{\mu^n - 1}{\mu - 1} M_n, & \text{if } \mu \ne 1, \\ M_n^2 - n\sigma^2 M_n, & \text{if } \mu = 1 \end{cases}$$

is also an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(d) Using the result from (c) prove that if $\mu > 1$ then M_n is bounded in $\mathcal{L}^2$, while if $\mu \le 1$ then $\lim_{n\to\infty} \mathbb{E}M_n^2 = \infty$.

3.3 *Gambler's ruin.* Let $X_1, X_2, \dots$ be iid. random variables with $\mathbb{P}\{X_i = 1\} = p = 1 - q = 1 - \mathbb{P}\{X_i = -1\}$. Fix also $0 < a < b$ integers, and

$$S_n := a + \sum_{k=1}^n X_k, \qquad T := \inf\{n : S_n = 0 \text{ or } S_n = b\}.$$

(We think about S_n as a gambler's money at time n; the gambler starts at a, and is either ruined ($S_T = 0$) or wins it all ($S_T = b$).)

(a) Show that $\mathbb{E}T < \infty$. *Hint: we had a lemma for this...*

(b) Show that both

$$M_n := S_n - n(p - q) \qquad \text{and} \qquad N_n := \begin{cases} S_n^2 - n, & \text{if } p = q = \dfrac{1}{2}, \\ \Big(\dfrac{q}{p}\Big)^{S_n}, & \text{if } p \ne q \end{cases}$$

are martingales w.r.t. the natural filtration.

(c) Calculate the ruin probability $\mathbb{P}\{S_T = 0\}$ and the expected duration $\mathbb{E}T$ of the game.

3.4 •••• *Extending Doob's Optional Stopping Theorem.* Let $\tau \ge 0$ be a stopping time, $\mathbb{E}\tau < \infty$.

(a) Show $\{\tau \ge k\} \in \mathcal{F}_{k-1}$.

(b) Based on the identity

$$|X_{\tau\wedge n} - X_0| = \Big|\sum_{k=1}^n (X_k - X_{k-1}) \cdot \mathbf{1}\{\tau \ge k\}\Big| \le \sum_{k=1}^\infty |X_k - X_{k-1}| \cdot \mathbf{1}\{\tau \ge k\}$$

and the proof of Doob's Optional Stopping Theorem, part (iii), show the following: If X a supermartingale for which there exists a $C \in \mathbb{R}$ with

$$\mathbb{E}(|X_k - X_{k-1}| \,|\, \mathcal{F}_{k-1}) \le C \qquad \forall k > 0, \text{ a.s.},$$

then $\mathbb{E}X_\tau \le \mathbb{E}X_0$. Of course we have equality in case X is a martingale.

(c) Prove that for any process $(M_n)_{n=0}^\infty$ with $M_0 = 0$,

(1)

$$M_{\tau\wedge n}^2 = \sum_{k=1}^n \big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \ge k\} + 2\sum_{1\le i<j\le n} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \ge j\},$$

(2)

$$M_\tau^2 = \sum_{k=1}^\infty \big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \ge k\} + 2\sum_{1\le i<j<\infty} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \ge j\},$$

holds. (Notice that a.s. finitely many terms are non-zero in each of these sums.).

(d) Show that the first sum on the right hand-side of (1) converges monotonically to that on the right hand-side of (2).

(e) Let M be a martingale with $M_0 = 0$ and a $C \in \mathbb{R}$ such that

$$\big|M_k - M_{k-1}\big| \le C \qquad \forall k > 0.$$

From now on, let us assume that the stopping time τ is in $\mathcal{L}^2$. Show that the second sums on the right hand-sides of both (1) and (2) are mean zero. *Hint: Fubini's Theorem.*

(f) With the condition as in the previous part, conclude $\lim\limits_{n\to\infty} \mathbb{E}M^2_{\tau\wedge n} = \mathbb{E}M^2_\tau$.

3.5 •••• *Wald's identities. (Notice: In contrary to this problem, Ábrahám Wald invented these without the use of martingales.)* Let $Y_1, Y_2, \ldots$ be iid. random variables in $\mathcal{L}^1$, $\mu := \mathbb{E}Y_i$, and $\tau \ge 1$ a stopping time (w.r.t. the natural filtration), $\mathbb{E}\tau < \infty$. Let $S_n = \sum\limits_{i=1}^n Y_i$. Show that

(a) $\mathbb{E}S_\tau = \mu \cdot \mathbb{E}\tau$. *Hint: use a martingale and the previous problem.*

(b) If Y_i is bounded and, additionally, $\mathbb{E}\tau^2 < \infty$ also holds, then with $\sigma^2 := \mathbf{Var}Y_i$ we have $\mathbb{E}(S_\tau - \mu\tau)^2 = \sigma^2 \cdot \mathbb{E}\tau$. (*This one is often used in case $\mu = 0$.*) *Hint: find a martingale for S^2_τ, and use the previous problem on your martingale from part (a). Do it for $\mu = 0$ first.*

3.6 ••• *First mark after 1 in a Uniform renewal process.* Let $Y_1, Y_2, \ldots$ be iid. Uniform(0, 1) random variables, let $S_n := \sum\limits_{i=1}^n Y_i$, and $\tau := \min\{n : S_n > 1\}$.

(a) Show that for any fixed $0 \le z \le 1$, $\mathbb{P}\{S_n \le z\} = z^n/n!$ holds. *Be careful, this is not true for $z > 1$!*

(b) Find $\mathbb{E}\tau$. *Hint: τ is non-negative, therefore we can sum tail probabilities. These latter are in close connection with part (a).*

(c) Since τ is a stopping time, use Wald's identity to calculate $\mathbb{E}(S_\tau - 1)$, the residual time at 1 until the next mark in a Uniform renewal process.

3.7 ••• *Pólya urn.* At time $n = 0$, an urn contains $B_0 = 1$ blue and $R_0 = 1$ red balls. At each time $n > 0$ a ball is chosen uniformly at random from the urn and returned to the urn, together with a new ball of the same colour. We denote by B_n and R_n the number of blue, respectively, red balls in the urn after the n^{th} turn of this procedure. Notice that $B_n + R_n = n + 2$. Let

$$M_n := \frac{B_n}{B_n + R_n}$$

be the proportion of blue balls in the urn just after turn n.

(a) Show that M_n is a martingale w.r.t. the natural filtration of the process.

(b) Show that B_n is discrete uniform: $\mathbb{P}\{B_n = k\} = \frac{1}{n+1}$ for $1 \le k \le n+1$.

(c) Show that $M_\infty := \lim_{n\to\infty} M_n$ exists a.s. What is its distribution?

(d) Let T be the time the first blue ball is drawn. Show that $T < \infty$ a.s. *Hint: show that the events $\{T > n\}$ are decreasing and find the limit of their probabilities.*

(e) Show that $\mathbb{E}\frac{1}{T+2} = \frac{1}{4}$.

(f) Now take a look at the next problem. (*No need to hand in, just take a look.*)

3.8 *Beta function.* Prove that for any $a, b \ge 0$ integers,

$$I_{a,b} := \int_0^1 \theta^a (1-\theta)^b \,\mathrm{d}\theta = \frac{a! \cdot b!}{(a+b+1)!}.$$

Hint: show via integration by parts that for $b \ge 1$, $I_{a,b} = \frac{b}{a+1} \cdot I_{a+1,b-1}$, while the case $b = 0$ is easy. From here, a recursive argument does the trick.

3.9 ••• *Bayes urn.* Assume we have a *randomly biased* coin that shows HEAD with probability θ and TAIL with probability $1-\theta$. This parameter θ is random and has the Uniform(0, 1) distribution. We flip this coin repeatedly and record

$$B_0 := 1, \qquad B_n := 1 + \text{no. of HEADs in the first } n \text{ trials},$$
$$R_0 := 1, \qquad R_n := 1 + \text{no. of TAILs in the first } n \text{ trials}.$$

Notice that $B_n + R_n = n + 2$. Define the filtration generated by the first n flips, $\mathcal{F}_n = \sigma(B_1, B_2, \ldots, B_n)$, and mind that θ is *not* included in here.

(a) Determine the probability of a given sequence of flips,

$$\mathbb{P}\{B_1 = b_1, B_2 = b_2, \ldots, B_n = b_n\}.$$

Hint: Condition on θ and use Problem 3.8.

(b) Based on the previous part, find the distribution of B_{n+1}, given $\mathcal{F}_n$. Compare with the Pólya urn. Remember: θ is not included in $\mathcal{F}_n$.

(c) Show that, modulo zero measure sets, θ is $\mathcal{F}_\infty = \sigma\Big(\bigcup_{n=1}^\infty \mathcal{F}_n\Big)$-measurable.

(d) What is the conditional expectation of θ, given the first n flips? Explain. *Hint: the Pólya urn, and our theorem on uniformly integrable martingales…*

(e) *No marks for this, only for pride, as you might not have met conditional densities before.* Use the Bayes urn to find the conditional density of M_∞, given $\mathcal{F}_n$ in the Pólya urn.

3.10 Let $X_1, X_2, \ldots$ be strictly positive iid. random variables such that $\mathbb{E}X_1 = 1$ and $\mathbb{P}\{X_1 = 1\} < 1$.

(a) Show that $M_n = \prod_{i=1}^n X_i$ is a martingale w.r.t. the natural filtration.

(b) Deduce that there exists a real valued random variable L such that $M_n \to L$ a.s. as $n \to \infty$.

(c) Show that $\mathbb{P}\{L = 0\} = 1$. *Hint: argue by contradiction and note that if $M_n, M_{n+1} \in (a-\varepsilon, a+\varepsilon)$ then $X_{n+1} \in \big(\frac{a-\varepsilon}{a+\varepsilon}, \frac{a+\varepsilon}{a-\varepsilon}\big)$.*

(d) Use the Strong Law of Large Numbers to show that there exists $c \in \mathbb{R}$ such that $\frac{1}{n}\ln M_n \to c$ a.s. as $n \to \infty$. Use Jensen's inequality to show that $c < 0$.

3.11 Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$ a sub σ-algebra. Show that given $\mathcal{G}$, $\mathbb{E}(X \,|\, \mathcal{G})$ is the best predictor of X in the following sense: the minimum mean square error $\mathbb{E}(V - X)^2$ among $\mathcal{G}$-measurable random variables V is achieved for $V = \mathbb{E}(X \,|\, \mathcal{G})$. What is this minimal mean square error? *Hint: use a tower rule first, then minimise pointwise among $\mathcal{G}$-measurable functions.*

3.12 Given are N balls and K, initially empty, urns. We place the balls, one by one, into the urns without removing them. Each ball independently goes to a uniformly chosen urn from 1 to K. These choices are denoted by $X_1, X_2, \dots, X_N$, which are therefore iid. discrete uniform on the set $\{1, 2, \dots, K\}$. The generated filtration is $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ for $n = 0, 1, \dots, N$. Denote by Z the number of empty urns when all N balls have been placed, and Z_n the number of empty urns after the n^{th} step.

(a) Calculate the best prediction martingale at time n (see the previous problem) $M_n = \mathbb{E}(Z \,|\, \mathcal{F}_n)$, $(n = 0, 1, \dots, N)$ explicitly, and show its martingale property via direct computation. *Hint: use indicators for urns to stay empty.*

(b) What is M_0 and what is M_N?

(c) Find $\mathbb{E}Z_n$ $(0 \le n \le N)$ and $\mathbb{E}Z$.

3.13 Let M be a uniformly integrable martingale in the filtration $(\mathcal{F}_n)_{n\ge 0}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $S \le T$ a.s. be finite stopping times. We denote by $\mathcal{F}_T$ the collection of all events $A \in \mathcal{F}$ such that $A \cap \{T = n\} \in \mathcal{F}_n$ for all n, which can be thought of as the set of events whose occurrence or non-occurrence is known by time T.

(a) Prove that $\mathcal{F}_T$ is a σ-algebra.

(b) Prove that $M_T = \mathbb{E}(M_\infty \,|\, \mathcal{F}_T)$ and that $M_S = \mathbb{E}(M_T \,|\, \mathcal{F}_S)$. *Hint: observe that $\mathcal{F}_T$ is generated by sets $A \cap \{T = n\}$ where $A \in \mathcal{F}$ and $n \in \mathbb{Z}^+$.*

3.14 Let $X_n \in [0, 1]$ be adapted to $\mathcal{F}_n$. Let $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and suppose

$$\mathbb{P}\{X_{n+1} = \alpha + \beta X_n \,|\, \mathcal{F}_n\} = X_n, \quad \mathbb{P}\{X_{n+1} = \beta X_n \,|\, \mathcal{F}_n\} = 1 - X_n.$$

Show:

(a) $\mathbb{P}\{\lim_n X_n = 0 \text{ or } 1\} = 1$. *Hint: Use martingale convergence, and try to find an independent sequence U_n that generates X_n.*

(b) If $X_0 = \theta$ then $\mathbb{P}\{\lim_n X_n = 1\} = \theta$.