Homework set 3

Convergence of random variables
(and a bit of martingales)
Martingale Theory with Applications, 1^{st} teaching block, 2021
School of Mathematics, University of Bristol

Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 4^{th} November. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it.

3.1 Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra on the probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$. If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X \,|\, \mathcal{G}) = X$ which suggests that the map $X \mapsto \mathbb{E}(X \,|\, \mathcal{G})$ is a projection. Show that indeed: this map is an orthogonal projection in the Hilbert space $\mathcal{L}^2(\Omega,\, \mathcal{F},\, \mathbb{P})$ (with inner product $\langle X,\, Y\rangle_{\mathbb{P}} = \mathbb{E}(XY)$) onto the subspace $\mathcal{L}^2(\Omega,\, \mathcal{G},\, \mathbb{P})$.

3.2 Let $X_1,\, X_2,\, \dots$ be independent. Prove that $\sup_n X_n < \infty$ a.s. if and only if $\sum_{n=1}^{\infty} \mathbf{P}\{X_n > A\} < \infty$ for some positive finite A.

3.3 Prove that for any sequence $X_1,\, X_2,\, \dots$ of random variables there exists a deterministic sequence $c_1,\, c_2,\, \dots$ of real numbers for which $\frac{X_n}{c_n} \xrightarrow{\text{a.s.}} 0$.

3.4 Let the random variables $X_1,\, X_2,\, \dots,\, X_n,\, Y_1,\, Y_2,\, \dots,\, Y_n,\, \dots,\, X$ and Y be defined on a common probability space $(\Omega,\, \mathcal{F},\, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_n + Y_n \xrightarrow{\mathbf{P}} X + Y$,

b) $X_n - Y_n \xrightarrow{\mathbf{P}} X - Y$.

3.5 Let the random variables $X_1,\, X_2,\, \dots,\, X_n,\, Y_1,\, Y_2,\, \dots,\, Y_n,\, \dots,\, X$ and Y be defined on a common probability space $(\Omega,\, \mathcal{F},\, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_n Y_n \xrightarrow{\mathbf{P}} XY$,

b) if $Y_n \neq 0$ and $Y \neq 0$ a.s., then $X_n / Y_n \xrightarrow{\mathbf{P}} X/Y$.

3.6 Formulate necessary and sufficient conditions for $\alpha_i < \beta_i$ such that independent (but not identically distributed) Uniform$(\alpha_i,\, \beta_i)$ variables X_i converge to 0

a) in distribution;

b) almost surely.

3.7 $\bullet\bullet\bullet\bullet\bullet\bullet$ Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential(λ_i) variables X_i to converge to 0

a) in distribution;

b) almost surely.

3.8 Let $\xi_1,\, \xi_2,\, \dots$ be iid. Poisson(1) random variables. (Recall their moment generating function: $\mathbb{E}(\mathrm{e}^{t\xi_i}) = \mathrm{e}^{\mathrm{e}^t - 1}$.) Let $a,\, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad \text{and} \qquad X_n = \mathrm{e}^{aS_n - bn}.$$

Show that

$$X_n \to 0 \text{ a.s.} \Leftrightarrow b > a,$$

but for any $r \geq 1$

$$X_n \to 0 \text{ in } \mathcal{L}^r \Leftrightarrow b > \frac{e^{ra} - 1}{r}.$$

3.9 ••••• Let $\xi_1, \xi_2, \dots$ be iid. standard normal random variables. (Recall their moment generating function: $\mathbb{E}(e^{\lambda \xi_i}) = e^{\lambda^2/2}$.) Let $a, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad \text{and} \qquad X_n = e^{aS_n - bn}.$$

Show that

$$X_n \to 0 \text{ a.s.} \Leftrightarrow b > 0,$$

but for any $r \geq 1$

$$X_n \to 0 \text{ in } \mathcal{L}^r \Leftrightarrow r < \frac{2b}{a^2}.$$

3.10 Let S and T be stopping times w.r.t. the filtration $\mathcal{F}_n$. Which of these are stopping times? Explain.

$S \wedge T := \min(S, T)$, $\quad S \vee T := \max(S, T)$, $\quad T + S$, $\quad T - S$ (assume $T \geq S$ here.)

3.11 Let $X_1, X_2, \dots$ be iid. Exponential(1) random variables, $S_n = X_1 + \cdots + X_n$, and $\{\mathcal{F}_n\}$ the natural filtration. Show that

$$\frac{n!}{(1 + S_n)^{n+1}} e^{S_n}$$

is a martingale w.r.t. $\{\mathcal{F}_n\}$.

3.12 An urn contains n white and n black balls. We draw them one by one without replacement. We receive £1 for any white ball, while nothing happens upon drawing a black one. Denote by X_i our money after the i^{th} draw ($X_0 = 0$). Let

$$Y_i = \frac{2X_i - i}{2n - i} \quad (1 \leq i \leq 2n - 1), \qquad \text{and}$$
$$Z_i = \frac{2n - i}{2n - i - 1} Y_i^2 - \frac{1}{2n - i - 1} \quad (1 \leq i \leq 2n - 2).$$

(a) Show that both Y_i and Z_i are martingales.

(b) Calculate the mean and variance of X_i.

3.13 •••••• An urn contains n white and n black balls. We draw them one by one without replacement. We pay £1 for any black ball drawn but receive £1 for any white one. Denote by X_i our money after the i^{th} draw ($X_0 = 0$). Let

$$Y_i = \frac{X_i}{2n - i} \quad (1 \leq i \leq 2n - 1), \qquad \text{and} \qquad Z_i = \frac{X_i^2 - (2n - i)}{(2n - i)(2n - i - 1)} \quad (1 \leq i \leq 2n - 2).$$

(a) Show that both Y_i and Z_i are martingales.

(b) Calculate the variance of X_i.

3.14 Let X_j, $j \geq 1$, be absolutely integrable random variables, and $\mathcal{F}_n := \sigma(X_j,\ 1 \leq j \leq n)$, $n \geq 0$, their natural filtration. Define the new random variables

$$Z_0 := 0, \qquad Z_n := \sum_{j=0}^{n-1} \big(X_{j+1} - \mathbb{E}(X_{j+1} \,|\, \mathcal{F}_j) \big).$$

Prove that the process $n \mapsto Z_n$ is an $(\mathcal{F}_n)_{n \geq 0}$-martingale.

3.15 A biased coin shows HEAD with probability $\theta \in (0,\, 1)$, and TAIL with probability $1-\theta$. The value θ of the bias in *not known*. For $t \in [0,\, 1]$ and $n \in \mathbb{N}$ we define $p_{n,t} : \{0,\, 1\}^n \to [0,\, 1]$ by

$$p_{n,t}(x_1,\, x_2,\, \ldots,\, x_n) = t^{\sum_{j=1}^n x_j} \cdot (1-t)^{n - \sum_{j=1}^n x_j}.$$

We make two hypotheses about the possible value of θ: either $\theta = a$, or $\theta = b$, where $a,\, b \in [0,\, 1]$ and $a \neq b$. We toss the coin repeatedly and form the sequence of random variables

$$Z_n := \frac{p_{n,a}(\xi_1,\, \xi_2,\, \ldots,\, \xi_n)}{p_{n,b}(\xi_1,\, \xi_2,\, \ldots,\, \xi_n)},$$

where we write $\xi_j = 1$ if the j^{th} flip is HEAD and $\xi_j = 0$ if it is TAIL. Show that the process $n \mapsto Z_n$ is a martingale (w.r.t. the natural filtration generated by the coin tosses) if and only if the true bias of the coin is $\theta = b$.

3.16 Let η_n be a homogeneous Markov chain on the countable state space $S := \{0,\, 1,\, 2,\, \ldots\}$ and $\mathcal{F}_n := \sigma(\eta_j,\ 0 \leq j \leq n)$, $n \geq 0$ its natural filtration. For $i \in S$ denote by $Q(i)$ the probability that the Markov chain starting from site i *ever reaches* the point $0 \in S$:

$$Q(i) := \mathbb{P}\{\exists m < \infty : \eta_m = 0 \,|\, \eta_0 = i\}.$$

Prove that $Z_n := Q(\eta_n)$ is an $(\mathcal{F}_n)_{n \geq 0}$-martingale.