HOMEWORK SET 4

Renewal processes, DeMoivre-Laplace CLT, measue theory

Further Topics in Probability, 1st teaching block, 2013

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Problems with •'s are to be handed in. These are due in class or in the blue locker with my name in the Main Maths Building before 11:59pm on Wednesday, 20th November. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

4.1 (*Durrett.*) Let $A_t = t - T_{N(t)-1}$ (age) and $B_t = T_{N(t)} - t$ (residual time) in a renewal process.

a) ••• Show that for every $x,\, y > 0$,

$$\mathbf{P}\{A_t > x,\ B_t > y\} \stackrel{t\to\infty}{\longrightarrow} \frac{1}{\mu}\int_{x+y}^{\infty}(1 - F(t))\,\mathrm{d}t.$$

b) • Recall from class that A_t and B_t have the same limit distribution

$$\lim_{t\to\infty}\mathbf{P}\{A_t > x\} = \lim_{t\to\infty}\mathbf{P}\{B_t > x\} = \frac{1}{\mu}\int_{x}^{\infty}(1 - F(t))\,\mathrm{d}t.$$

In which case are A_t and B_t asymptotically independent? *HINT: For a non-increasing nonnegative function g, $g(x + y) = g(x)\cdot g(y)$ $\forall x,\, y$ implies it's exponential.*

4.2 (*Durrett.*) Let $X_1,\, X_2,\, \dots$ be iid. with distribution F_X, $Y_1,\, Y_2,\, \dots$ be iid. and independent of the X_i's with distribution F_Y. Let $T_0 = 0$ and for $k \geq 1$ let $S_k = T_{k-1} + X_k$ and $T_k = S_k + Y_k$. In words, we have a machine that works for an amount of time X_k, breaks down, and then requires Y_k units of time to be repaired. Let $H(t)$ be the probability the machine is working at time t.

a) • Show that if $F_X * F_Y$ is nonarithmetic, then as $t \to \infty$

$$H(t) \to \frac{\mu_X}{\mu_X + \mu_Y},$$

where $\mu_X = \mathbf{E}X_1$ and $\mu_Y = \mathbf{E}Y_1$.

b) • Show that in a renewal process

$$\mathbf{P}\{\text{the number of renewals in } [0,\, t] \text{ is odd}\} \to \frac{1}{2}.$$

HINT: This is a special case of a).

c) •• Let now X_i's and Y_i's each have the Exponential(λ) distribution. Compute $H(t)$ directly for finite $t > 0$. *HINT: From a previous problem we know that in a renewal process with Gamma($2,\, \lambda$) interarrival distribution we have $U(t) = 3/4 + \lambda t/2 + \mathrm{e}^{-2\lambda t}/4$. Be very careful with $dU(0)$!*

4.3 (*Resnick.*) In a renewal process, find the limit of

$$H(t) = \mathbf{P}\Big\{\bigcup_{n=0}^{\infty}\{S_n \in (t,\, t + x]\}\Big\}$$

as $t \to \infty$ with a fixed $x > 0$. What is $H(t)$ in words?

4.4 Let $Y_1, Y_2, \ldots$ be iid. Uniform(0, 1) random variables. Define $S_n := \sum_{i=1}^{n} Y_i$ $(n \geq 1)$, and $M := \min\{n : S_n > 1\}$.

a) Fix $0 \leq z \leq 1$. Prove inductively that $\mathbf{P}(S_n < z) = z^n/n!$. (This is not true for $z > 1$!)

b) Rewrite $\mathbf{E}M$ in terms of the tail probabilities $\mathbf{P}(M > n)$. Then rewrite these in terms of S_n-probabilities and find $\mathbf{E}M$.

c) Since M is a stopping time, use Wald's identity: $\mathbf{E}S_M = \mathbf{E}M \cdot \mathbf{E}Y_1$ to find $\mathbf{E}(S_M - 1)$, the expected time to wait for the first renewal after 1.

4.5 ••• (*Ross.*) Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas, if it lands heads less than 525 times, then we shall conclude that it is the fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

4.6 •••• A fraction p of citizens in a city smoke. We are to determine the value of p by making a survey involving n citizens whom we select randomly. If k of these n people smoke, then $p' = k/n$ will be our result. How large should we choose n if we want our result p' to be closer to the real value p than 0.005 with probability at least 0.95? In other words: determine the smallest number n_0 such that $P(|p' - p| \leq 0.005) \geq 0.95$ for any $p \in (0, 1)$ and $n \geq n_0$.

4.7 Given are two very similar insurance companies, each with 10 000 customers. In the beginning of the year, each customer pays £ 500, and during the year each customer independently claims £ 1500 with probability 1/3. The two companies start the year with capital £ 50 000 each. A company becomes bankrupt if it cannot pay the insurance claims during the year. Should the two companies unite to help avoiding bankruptcy? Explain.

4.8 Use normal approximation to find the numerical value of

$$\binom{3600}{2376} \cdot 0.64^{2376} \cdot 0.36^{1224}.$$

4.9 •• (*Shiryaev.*) Let Ω be a countable set and $\mathcal{F}$ the collection of all its subsets. Put $\mu(A) = 0$ if A is finite and $\mu(A) = \infty$ if A is infinite. Show that the set function μ is finitely additive but not σ-additive.