

Problems with •'s are to be handed in. These are due in the blue locker marked "Further Topics in Probability" on the ground floor of the Main Maths Building before 12:00pm on Thursday, 21st March. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

4.1 Let τ be the hitting time of level 1 in a symmetric simple random walk. Find its mass function $\mathbf{P}\{\tau = k\}$ and the limit $\lim_{k \rightarrow \infty} k^{3/2} \cdot \mathbf{P}\{\tau = k\}$. You may want to use the Binomial Theorem for the root of a sum, and the Stirling formula to take the above limit.

4.2 ••• Let X_r be a sequence of pessimistic Negative Binomial($r, p(r)$) random variables (number of failures before the r^{th} success) with $p(r)$ such that $r \cdot (1 - p(r)) \rightarrow \lambda$ with $0 < \lambda < \infty$ as $r \rightarrow \infty$. Show that X_r converges to Poisson(λ) in distribution. (Notice that, as the pessimistic Negative Binomial is the sum of independent pessimistic Geometrics, this statement is similar in flavour to the Law of Rare Events.)

4.3 a) Prove that Markov's inequality is *sharp* in the following sense: fixing $0 < m < \lambda \leq \lambda$, there exists a non-negative random variable X with expectation $\mathbf{E}X = m$ and 'saturated' Markov's inequality: $\mathbf{P}\{X \geq \lambda\} = m/\lambda$.

b) Prove that Markov's inequality is *not sharp*, in the following sense: for any fixed non-negative random variable X with finite mean, $\lim_{\lambda \rightarrow \infty} \lambda \mathbf{P}\{X \geq \lambda\} / \mathbf{E}X = 0$.

4.4 Show that $\mathbf{E}X^2 < \infty$ if and only if $\sum_{n=1}^{\infty} n \cdot \mathbf{P}\{|X| > n\} < \infty$.

4.5 (Ross.) Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas, if it lands heads less than 525 times, then we shall conclude that it is the fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

4.6 •••• A fraction p of citizens in a city smoke. We are to determine the value of p by making a survey involving n citizens whom we select randomly. If k of these n people smoke, then $p' = k/n$ will be our result. How large should we choose n if we want our result p' to be closer to the real value p than 0.005 with probability at least 0.95? In other words: determine the smallest number n_0 such that $P(|p' - p| \leq 0.005) \geq 0.95$ for any $p \in (0, 1)$ and $n \geq n_0$.

4.7 ••• Given are two very similar insurance companies, each with 10 000 customers. In the beginning of the year, each customer pays £500, and during the year each customer independently claims, at most once, £1 500 with probability 1/3. The two companies start the year with capital £50 000 each. A company becomes bankrupt if it cannot pay the insurance claims during the year. Should the two companies unite to help avoiding bankruptcy? Explain.

4.8 Use normal approximation to find the numerical value of

$$\binom{3600}{2376} \cdot 0.64^{2376} \cdot 0.36^{1224}.$$

4.9 ••• Normal approximation of Poisson. Let $X \sim \text{Poi}(\lambda)$. Use asymptotics on the mass function directly to prove that, as $\lambda \rightarrow \infty$,

$$\sqrt{\lambda} \cdot \mathbf{P}\{X = \lfloor \lambda + x\sqrt{\lambda} \rfloor\} = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) + \mathcal{O}(\lambda^{-1/2}),$$

where the error term is uniformly small if x stays in a bounded range. *HINT: First argue that the floor is not important, then apply Stirling's formula. And be patient.*

As a corollary, prove

$$\mathbf{P}\left\{a < \frac{X - \lambda}{\sqrt{\lambda}} \leq b\right\} \rightarrow \int_a^b \varphi(y) dy = \phi(b) - \phi(a)$$

as $\lambda \rightarrow \infty$. (Notice $\mathbf{E}X = \lambda$ and $\text{SD}X = \sqrt{\lambda}$.)

4.10 ••• (Shiryayev.) Let Ω be a countable set and \mathcal{F} the collection of all its subsets. Put $\mu(A) = 0$ if A is finite and $\mu(A) = \infty$ if A is infinite. Show that the set function μ is finitely additive but not σ -additive.