HOMEWORK SET 4

Branching process, random walk

Further Topics in Probability, 2^{nd} teaching block, 2022

School of Mathematics, University of Bristol

Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 17^{th} March. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

4.1 ••••• Let X_i be i.i.d. Bernoulli(p) random variables, and $S_n = \sum_{i=1}^n X_i$.

(a) Prove for $0 \le k \le n+1$ that $\mathbb{P}\{S_{n+1} = k\} = p\mathbb{P}\{S_n = k-1\} + q\mathbb{P}\{S_n = k\}$.

(b) Use this recursion to find the generating function of S_n.

4.2 ••• Let X_r be a sequence of pessimistic Negative Binomial(r, $p(r)$) random variables (number of failures before the r^{th} success) with $p(r)$ such that $r\cdot(1-p(r)) \to \lambda$ with $0 < \lambda < \infty$ as $r \to \infty$. Show that X_r converges to Poisson(λ) in distribution. (Notice that, as the pessimistic Negative Binomial is the sum of independent pessimistic Geometrics, this statement is similar in flavour to the Law of Rare Events.)

4.3 ••••• Enthusiastic students run a club that was founded by just one single student. Membership in this club automatically terminates after one year. Academics can also be members but they are rather busy hence less enthusiastic. When a student's membership expires, they will recruit two students with probability $\frac{1}{4}$ or one student and one academic with probability $\frac{2}{3}$ or two academics with probability $\frac{1}{12}$. When an academic's membership expires, they will not recruit anyone (remember, they are too busy).

(a) After n years, what is the probability that no academics will have yet been recruited?

(b) What is the probability that the club will eventually have no members?

4.4 Let Z_n be the population of generation n in a Galton-Watson branching process with offspring distribution $\mathbb{P}\{\xi = i\} = p(i)$ and $Z_0 = 1$.

(a) Argue that $W_n := Z_{2n}$ is a branching process and find its offspring distribution in terms of $p(i)$.

(b) Show analytically that Z_n and W_n have the same extinction probability.

4.5 An amoeba with alternating life cycle has offspring distribution with generating function P in the first generation. Independently, each individual in the next generation has offspring distribution with generating function Q. It then alternates, the next generation is back to P, then Q again, etc. Determine the mean number of individuals in the $2n^{\text{th}}$ generation and the extinction probability of the process.

4.6 Let S_n be a simple symmetric random walk on the segment $1, 2, \ldots, L$. This walker steps from 1 to 2 and from L to $L-1$ with probability one, and makes left and right moves with equal chance from any other site. Given the location, the next move is independent of the walker's past. Let τ be the hitting time of position L i.e., $\tau = \inf\{n \ge 0 : S_n = L\}$, and $P_i(s)$ the generating function of τ if the walker starts from position i. Determine $P_i(s)$ and $E_i := \mathbb{E}(\tau \mid S_0 = i)$. *Hint: search for the solution as an exponential of i.*

4.7 Let τ be the hitting time of level 1 in a symmetric simple random walk. Find its mass function $\mathbb{P}\{\tau = k\}$ and the limit $\lim_{k\to\infty} k^{3/2} \cdot \mathbb{P}\{\tau = k\}$. You may want to use the Binomial Theorem for the root of a sum, and the Stirling formula to take the above limit.

4.8 •••• Let Y_1, Y_2, ... be i.i.d. non-negative random variables with finite mean $\mu := \mathbb{E}(Y_i) < \infty$. Define, for $n \geq 1$, $S_n := \sum_{i=1}^{n} Y_i$ and $N(t) := \max\{n : S_n \leq t\}$. We know by the Weak Law of Large Numbers that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big\{\Big|\frac{S_n}{n} - \mu\Big| > \varepsilon\Big\} = 0.$$

Prove that the dual statement is also true: for any $\delta > 0$,

$$\lim_{t\to\infty} \mathbb{P}\Big\{\Big|\frac{N(t)}{t} - \mu^{-1}\Big| > \delta\Big\} = 0.$$