

Problems with  $\bullet$ 's are to be handed in. These are due in the blue locker marked "Further Topics in Probability" on the ground floor of the Main Maths Building before 12:00pm on Thursday, 4<sup>th</sup> April. Please show your work leading to the result, not only the result. Each problem worth the number of  $\bullet$ 's you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

5.1  $\bullet\bullet$

a) Prove that Markov's inequality is *sharp* in the following sense: fixing  $0 < m \leq \lambda$ , there exists a non-negative random variable  $X$  with expectation  $\mathbf{E}X = m$  and 'saturated' Markov's inequality:  $\mathbf{P}\{X \geq \lambda\} = m/\lambda$ .

b) Prove that Markov's inequality is *not sharp*, in the following sense: for any fixed non-negative random variable  $X$  with finite mean,  $\lim_{\lambda \rightarrow \infty} \lambda \mathbf{P}\{X \geq \lambda\} / \mathbf{E}X = 0$ .

5.2 Show that  $\mathbf{E}X^2 < \infty$  if and only if  $\sum_{n=1}^{\infty} n \cdot \mathbf{P}\{|X| > n\} < \infty$ .

5.3  $\bullet\bullet\bullet$  Let the random variables  $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n, \dots, X$  and  $Y$  be defined on a common probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , and suppose  $X_n \xrightarrow{\mathbf{P}} X$  and  $Y_n \xrightarrow{\mathbf{P}} Y$ . Prove

- a)  $X_n + Y_n \xrightarrow{\mathbf{P}} X + Y$ ,
- b)  $X_n - Y_n \xrightarrow{\mathbf{P}} X - Y$ .

5.4 Let the random variables  $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n, \dots, X$  and  $Y$  be defined on a common probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , and suppose  $X_n \xrightarrow{\mathbf{P}} X$  and  $Y_n \xrightarrow{\mathbf{P}} Y$ . Prove

- a)  $X_n Y_n \xrightarrow{\mathbf{P}} XY$ ,
- b) if  $Y_n \neq 0$  and  $Y \neq 0$  a.s., then  $X_n/Y_n \xrightarrow{\mathbf{P}} X/Y$ .

5.5 Let  $X_1, X_2, \dots$  be independent. Prove that  $\sup_n X_n < \infty$  a.s. if and only if  $\sum_{n=1}^{\infty} \mathbf{P}\{X_n > A\} < \infty$  for some positive finite  $A$ .

5.6 Prove that for any sequence  $X_1, X_2, \dots$  of random variables there exists a deterministic sequence  $c_1, c_2, \dots$  of real numbers for which  $\frac{X_n}{c_n} \xrightarrow{\text{a.s.}} 0$ .

5.7  $\bullet\bullet\bullet\bullet$  Formulate necessary and sufficient conditions for  $\alpha_i < \beta_i$  such that independent (but not identically distributed) Uniform( $\alpha_i, \beta_i$ ) variables  $X_i$  converge to 0

- a) in distribution;
- b) almost surely.

5.8 Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential( $\lambda_i$ ) variables  $X_i$  to converge to 0

- a) in distribution;
- b) almost surely.

5.9  $\bullet\bullet\bullet$  We perform infinitely many independent experiments. The  $n^{\text{th}}$  one is successful with probability  $n^{-\alpha}$  and fails with probability  $1 - n^{-\alpha}$ ,  $0 < \alpha$ . Let  $k \geq 1$ . We are happy if we see  $k$  consecutive successes infinitely often. What is the probability of this?

5.10 (The longest run of heads, I.)

Let  $X_1, X_2, \dots$  be iid. random variables with  $\mathbf{P}\{X_k = 1\} = p$ ,  $\mathbf{P}\{X_k = 0\} = q$ , where  $p + q = 1$ . Fix a parameter  $\lambda > 1$ , and denote by  $A_k^{(\lambda)}$  the following events for  $k = 0, 1, 2, \dots$ :

$$A_k^{(\lambda)} := \left\{ \exists r \in \left[ \lfloor \lambda^k \rfloor, \lfloor \lambda^{k+1} \rfloor - k \right] \cap \mathbb{N} : X_r = X_{r+1} = \dots = X_{r+k-1} = 1 \right\}.$$

In plain words:  $A_k^{(\lambda)}$  means that somewhere between  $\lfloor \lambda^k \rfloor$  and  $\lfloor \lambda^{k+1} \rfloor - 1$  there is a sequence of  $k$  consecutive 1's. Prove that

- a) If  $\lambda < p^{-1}$ , then a.s. only finitely many of the events  $A_k^{(\lambda)}$  occur.
- b) If  $\lambda > p^{-1}$ , then a.s. infinitely many of the events  $A_k^{(\lambda)}$  occur.
- c) What happens for  $\lambda = p^{-1}$ ?

5.11 (The longest run of heads, II.)

Let

$$R_n := \sup\{k \geq 0 : X_n = X_{n+1} = \dots = X_{n+k-1} = 1\}.$$

That is:  $R_n$  is the length of the run of consecutive 1's that starts at  $n$ . (If  $X_n = 0$ , then set  $R_n = 0$ .) Prove that

$$\mathbf{P} \left\{ \limsup_{n \rightarrow \infty} \frac{R_n}{\log n} = |\log p|^{-1} \right\} = 1.$$

HINT: For a fixed parameter  $\alpha > 0$ , let

$$B_n^{(\alpha)} := \{R_n > \alpha \log n / |\log p|\}.$$

If  $\alpha > 1$ , then by the first Borel-Cantelli Lemma and direct computation, only finitely many of the  $B_n^{(\alpha)}$ 's occur a.s. If  $\alpha \leq 1$ , then from the previous exercise it follows that a.s. infinitely many of the  $B_n^{(\alpha)}$ 's occur.

5.12 On the (simplified version of the) game Roulette, a player bets £1, and loses his bet with probability 19/37, but is given his bet and an extra pound back with probability 18/37. Use the Weak Law of Large Numbers to find the probability that the casino loses money with this game on the (very) long run. Explain your answer.

5.13  $\bullet\bullet$  Rolling a die 100 times, denote the outcome of roll  $i$  by  $X_i$ . Estimate the probability

$$\mathbf{P} \left\{ \prod_{i=1}^{100} X_i \leq a^{100} \right\}$$

for real  $1 < a < 6$ .

5.14 (The simplest form of the McMillan Theorem.)

Let  $\mathbf{p} = (p_1, p_2, \dots, p_r)$ , where  $p_i, i = 1, 2, \dots, r$  are positive numbers with  $p_1 + p_2 + \dots + p_r = 1$ . That is: given is a probability distribution on the set  $\{1, 2, \dots, r\}$ . The entropy of the distribution  $\mathbf{p}$  is defined by  $H(\mathbf{p}) := -\sum_{j=1}^r p_j \log p_j$ . Let  $X_1, X_2, \dots$  be iid. random variables from this distribution  $\mathbf{p}$ . Define the random variables  $R_n := \prod_{k=1}^n p_{X_k}$ : this is the a priori probability of the observed sequence  $X_1, X_2, \dots, X_n$  of outcomes. Prove that

$$\mathbf{P} \left\{ \lim_{n \rightarrow \infty} n^{-1} \log R_n = -H(\mathbf{p}) \right\} = 1.$$

5.15 Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Prove

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 f \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right) dx_1 dx_2 \cdots dx_n = f \left( \frac{1}{2} \right).$$

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 f \left( ((x_1 x_2 \cdots x_n)^{1/n}) \right) dx_1 dx_2 \cdots dx_n = f \left( \frac{1}{e} \right).$$

5.16 ••• Prove

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n = \frac{2}{3}.$$

5.17 Let  $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$  be the surface of the  $n$ -dimensional Euclidean unit sphere. There is a unique probability measure  $\nu^{(n-1)}$  on  $S^{n-1}$  that is invariant to orthogonal transformations of  $\mathbb{R}^n$ ; for any Borel-measurable  $A \subset S^{n-1}$  and  $H$  orthogonal transformation of  $\mathbb{R}^n$ ,  $\nu^{(n-1)}(HA) = \nu^{(n-1)}(A)$ . (This measure is called the *Haar measure* on  $S^{n-1}$ , it is actually the uniform measure on  $S^{n-1}$ .)

a) Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be the vector of iid. Standard Normal components in  $\mathbb{R}^n$ . Prove that applying an arbitrary orthogonal transformation  $H$  on  $\mathbf{X}$ , the resulting vector  $\mathbf{Y} := H\mathbf{X}$  again has iid. Standard Normal components  $Y_1, Y_2, \dots, Y_n$ . From this and the uniqueness of the Haar measure prove that  $\mathbf{X}/|\mathbf{X}| \in S^{n-1}$  has the uniform distribution  $\nu^{(n-1)}$  on the surface of the sphere. (That is, for any Borel measurable  $A \subset S^{n-1}$ , we have  $\mathbf{P}(\mathbf{X}/|\mathbf{X}| \in A) = \nu^{(n-1)}(A)$ .)

b) Let  $X_1, X_2, \dots$  be iid. Standard Normal random variables, and

$$R_n := (X_1^2 + X_2^2 + \cdots + X_n^2)^{1/2}.$$

Prove  $R_n/\sqrt{n} \xrightarrow{\mathbf{P}} 1$  as  $n \rightarrow \infty$ .

c) Pick now a uniform random point  $P$  on the surface  $S^{n-1}$  of the unit sphere, and denote its coordinates in  $\mathbb{R}^n$  by  $(Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)})$ . Use the above a) and b) to prove the following limit theorems for  $P$ :

$$\lim_{n \rightarrow \infty} \mathbf{P} \left( \sqrt{n} Y_1^{(n)} < y \right) = \Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx,$$

$$\lim_{n \rightarrow \infty} \mathbf{P} \left( \sqrt{n} Y_1^{(n)} < y_1; \sqrt{n} Y_2^{(n)} < y_2 \right) = \Phi(y_1) \Phi(y_2).$$

$$\text{HINT: } \mathbf{P} \left( \sqrt{n} Y_1^{(n)} < y \right) = \mathbf{P} \left( \sqrt{n} X_1 / R_n < y \right).$$