Homework set 5

Borel-Cantelli Lemmas, Convergence, Law of Large Numbers

Further Topics in Probability, 2nd teaching block, 2021

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Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 29th April. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

5.1 ••

a) Prove that Markov's inequality is *sharp* in the following sense: fixing $0 < m \le \lambda$, there exists a non-negative random variable X with expectation $\mathbf{E}X = m$ and 'saturated' Markov's inequality: $\mathbf{P}\{X \ge \lambda\} = m/\lambda$.

b) Prove that Markov's inequality is *not sharp*, in the following sense: for any fixed non-negative random variable X with finite mean, $\lim\limits_{\lambda\to\infty} \lambda\mathbf{P}\{X \ge \lambda\}/\mathbf{E}X = 0$.

5.2 Show that $\mathbf{E}X^2 < \infty$ if and only if $\sum_{n=1}^{\infty} n \cdot \mathbf{P}\{|X| > n\} < \infty$.

5.3 ••• Let the random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, \ldots, X$ and Y be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_n + Y_n \xrightarrow{\mathbf{P}} X + Y$,

b) $X_n - Y_n \xrightarrow{\mathbf{P}} X - Y$.

5.4 Let the random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, \ldots, X$ and Y be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_nY_n \xrightarrow{\mathbf{P}} XY$,

b) if $Y_n \ne 0$ and $Y \ne 0$ a.s., then $X_n/Y_n \xrightarrow{\mathbf{P}} X/Y$.

5.5 Let $X_1, X_2, \ldots$ be independent. Prove that $\sup_n X_n < \infty$ a.s. if and only if $\sum_{n=1}^{\infty} \mathbf{P}\{X_n > A\} < \infty$ for some positive finite A.

5.6 Prove that for any sequence $X_1, X_2, \ldots$ of random variables there exists a deterministic sequence $c_1, c_2, \ldots$ of real numbers for which $\frac{X_n}{c_n} \xrightarrow{\text{a.s.}} 0$.

5.7 •••• Formulate necessary and sufficient conditions for $\alpha_i < \beta_i$ such that independent (but not identically distributed) Uniform(α_i, β_i) variables X_i converge to 0

a) in distribution;

b) almost surely.

5.8 Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential(λ_i) variables X_i to converge to 0

a) in distribution;

b) almost surely.

5.9 ••• We perform infinitely many independent experiments. The n^{th} one is successful with probability $n^{-\alpha}$ and fails with probability $1 - n^{-\alpha}$, $0 < \alpha$. Let $k \geq 1$. We are happy if we see k consecutive successes infinitely often. What is the probability of this?

5.10 (The longest run of heads, I.)
Let $X_1, X_2, \ldots$ be iid. random variables with $\mathbf{P}\{X_k = 1\} = p$, $\mathbf{P}\{X_k = 0\} = q$, where $p + q = 1$. Fix a parameter $\lambda > 1$, and denote by $A_k^{(\lambda)}$ the following events for $k = 0, 1, 2, \ldots$:

$$A_k^{(\lambda)} := \Big\{\exists r \in \big[\lfloor \lambda^k \rfloor, \lfloor \lambda^{k+1} \rfloor - k\big] \cap \mathbb{N} : X_r = X_{r+1} = \cdots = X_{r+k-1} = 1\Big\}.$$

In plain words: $A_k^{(\lambda)}$ means that somewhere between $\lfloor \lambda^k \rfloor$ and $\lfloor \lambda^{k+1} \rfloor - 1$ there is a sequence of k consecutive 1's. Prove that

a) If $\lambda < p^{-1}$, then a.s. only finitely many of the events $A_k^{(\lambda)}$ occur.
b) If $\lambda > p^{-1}$, then a.s. infinitely many of the events $A_k^{(\lambda)}$ occur.
c) What happens for $\lambda = p^{-1}$?

5.11 (The longest run of heads, II.)
Let

$$R_n := \sup\{k \geq 0 : X_n = X_{n+1} = \cdots = X_{n+k-1} = 1\}.$$

That is: R_n is the length of the run of consecutive 1's that starts at n. (If $X_n = 0$, then set $R_n = 0$.) Prove that

$$\mathbf{P}\left\{\limsup_{n\to\infty} \frac{R_n}{\log n} = |\log p|^{-1}\right\} = 1.$$

HINT: For a fixed parameter $\alpha > 0$, let

$$B_n^{(\alpha)} := \{R_n > \alpha \log n / |\log p|\}.$$

If $\alpha > 1$, then by the first Borel-Cantelli Lemma and direct computation, only finitely many of the $B_n^{(\alpha)}$'s occur a.s. If $\alpha \leq 1$, then from the previous exercise it follows that a.s. infinitely many of the $B_n^{(\alpha)}$'s occur.

5.12 On the (simplified version of the) game Roulette, a player bets £1, and looses his bet with probability 19/37, but is given his bet and an extra pound back with probability 18/37. Use the Weak Law of Large Numbers to find the probability that the casino looses money with this game on the (very) long run. Explain your answer.

5.13 •• Rolling a die 100 times, denote the outcome of roll i by X_i. Estimate the probability

$$\mathbf{P}\Big\{\prod_{i=1}^{100} X_i \leq a^{100}\Big\}$$

for real $1 < a < 6$.

5.14 (The simplest form of the *McMillan Theorem.*)
Let $\mathbf{p} = (p_1, p_2, \ldots, p_r)$, where p_i, $i = 1, 2, \ldots, r$ are positive numbers with $p_1 + p_2 + \cdots + p_r = 1$. That is: given is a probability distribution on the set $\{1, 2, \ldots, r\}$. The *entropy* of the distribution $\mathbf{p}$ is defined by $H(\mathbf{p}) := -\sum_{j=1}^{r} p_j \log p_j$. Let $X_1, X_2, \ldots$ be iid. random variables from this distribution $\mathbf{p}$. Define the random variables $R_n := \prod_{k=1}^{n} p_{X_k}$: this is the a priori probability of the observed sequence $X_1, X_2, \ldots, X_n$ of outcomes. Prove that

$$\mathbf{P}\left\{\lim_{n\to\infty} n^{-1} \log R_n = -H(\mathbf{p})\right\} = 1.$$

5.15 Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n = f\left(\frac{1}{2}\right).$$

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n = f\left(\frac{1}{e}\right).$$

5.16 ••• Prove

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n = \frac{2}{3}.$$

5.17 Let $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ be the surface of the n-dimensional Euclidean unit sphere. There is a unique probability measure $\nu^{(n-1)}$ on S^{n-1} that is invariant to orthogonal transformations of $\mathbb{R}^n$: for any Borel-measurable $A \subset S^{n-1}$ and H orthogonal transformation of $\mathbb{R}^n$, $\nu^{(n-1)}(HA) = \nu^{(n-1)}(A)$. (This measure is called the *Haar measure* on S^{n-1}, it is actually the uniform measure on S^{n-1}.)

a) Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be the vector of iid. Standard Normal components in $\mathbb{R}^n$. Prove that applying an arbitrary orthogonal transformation H on $\mathbf{X}$, the resulting vector $\mathbf{Y} := H\mathbf{X}$ again has iid. Standard Normal components $Y_1, Y_2, \ldots, Y_n$. From this and the uniqueness of the Haar measure prove that $\mathbf{X}/|\mathbf{X}| \in S^{n-1}$ has the uniform distribution $\nu^{(n-1)}$ on the surface of the sphere. (That is, for any Borel measurable $A \subset S^{n-1}$, we have $\mathbf{P}\big(\mathbf{X}/|\mathbf{X}| \in A\big) = \nu^{(n-1)}(A)$.)

b) Let $X_1, X_2, \ldots$ be iid. Standard Normal random variables, and

$$R_n := \left(X_1^2 + X_2^2 + \cdots + X_n^2\right)^{1/2}.$$

Prove $R_n/\sqrt{n} \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$.

c) Pick now a uniform random point P on the surface S^{n-1} of the unit sphere, and denote its coordinates in $\mathbb{R}^n$ by $(Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)})$. Use the above a) and b) to prove the following limit theorems for P:

$$\lim_{n\to\infty} \mathbf{P}\left(\sqrt{n}Y_1^{(n)} < y\right) = \Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, \mathrm{d}x,$$

$$\lim_{n\to\infty} \mathbf{P}\left(\sqrt{n}Y_1^{(n)} < y_1;\ \sqrt{n}Y_2^{(n)} < y_2\right) = \Phi(y_1)\Phi(y_2).$$

HINT: $\mathbf{P}\big(\sqrt{n}Y_1^{(n)} < y\big) = \mathbf{P}\big(\sqrt{n}X_1/R_n < y\big)$.