HOMEWORK SET 5

Martingale convergence (and Doob's decomposition)

Martingale Theory with Applications, 1st teaching block, 2022

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Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 1st December. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it.

5.1 ••••• *Pólya urn.* At time $n = 0$, an urn contains $B_0 = 1$ blue and $R_0 = 1$ red balls. At each time $n > 0$ a ball is chosen uniformly at random from the urn and returned to the urn, together with a new ball of the same colour. We denote by B_n and R_n the number of blue, respectively, red balls in the urn after the n^{th} turn of this procedure. Notice that $B_n + R_n = n + 2$. Let

$$M_n := \frac{B_n}{B_n + R_n}$$

be the proportion of blue balls in the urn just after turn n.

(a) Show that M_n is a martingale w.r.t. the natural filtration of the process.

(b) Show that B_n is discrete uniform: $\mathbb{P}\{B_n = k\} = \frac{1}{n+1}$ for $1 \le k \le n + 1$.

(c) Show that $M_\infty := \lim_{n\to\infty} M_n$ exists a.s. What is its distribution?

(d) Let T be the time the first blue ball is drawn. Show that $T < \infty$ a.s. *Hint: show that the events $\{T > n\}$ are decreasing and find the limit of their probabilities.*

(e) Show that $\mathbb{E}\,\frac{1}{T+2} = \frac{1}{4}$.

5.2 •••• Let $X_1, X_2 \ldots$ be independent random variables with

$$\mathbb{P}\{X_n = i\} = \begin{cases} \mathrm{e}^{-n}, & \text{if } i = 0, \\ 1 - 2\mathrm{e}^{-n}, & \text{if } i = 1, \\ \mathrm{e}^{-n}, & \text{if } i = 2, \end{cases}$$

and $M_n = \prod_{k=1}^n X_k$.

(a) Show that M_n is a martingale w.r.t. the natural filtration.

(b) Show that M_n has an almost sure limit M_∞ that is almost surely finite.

(c) Construct a random variable Z that bounds M_n for each n, and is of finite mean.

(d) Show that $\mathbb{E}\,M_\infty = 1$.

5.3 Let $X_1, X_2, \ldots$ be strictly positive i.i.d. random variables such that $\mathbb{E}\,X_1 = 1$ and $\mathbb{P}\{X_1 = 1\} < 1$.

(a) Show that $M_n = \prod_{i=1}^n X_i$ is a martingale w.r.t. the natural filtration.

(b) Deduce that there exists a real valued random variable L such that $M_n \to L$ a.s. as $n \to \infty$.

(c) Show that $\mathbb{P}\{L = 0\} = 1$. *Hint: argue by contradiction and note that if $M_n, M_{n+1} \in (a - \varepsilon, a + \varepsilon)$ then $X_{n+1} \in \left(\frac{a-\varepsilon}{a+\varepsilon}, \frac{a+\varepsilon}{a-\varepsilon}\right)$.*

(d) Use the Strong Law of Large Numbers to show that there exists $c \in \mathbb{R}$ such that $\frac{1}{n} \ln M_n \to c$ a.s. as $n \to \infty$. Use Jensen's inequality to show that $c < 0$.

5.4 Let $X_n \in [0, 1]$ be adapted to $\mathcal{F}_n$. Let $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and suppose

$$\mathbb{P}\{X_{n+1} = \alpha + \beta X_n \,|\, \mathcal{F}_n\} = X_n, \quad \mathbb{P}\{X_{n+1} = \beta X_n \,|\, \mathcal{F}_n\} = 1 - X_n.$$

Show:

(a) $\mathbb{P}\{\lim_n X_n = 0 \text{ or } 1\} = 1$. *Hint: Use martingale convergence, and try to find an independent sequence U_n that generates X_n.*

(b) If $X_0 = \theta$ then $\mathbb{P}\{\lim_n X_n = 1\} = \theta$.

5.5 Recall the ABRACADABRA problem, and the variable X_n being the total wealth of all gamblers in play after the n^{th} letter has been typed. Perform Doob's decomposition on X_n.

5.6 Let $\Gamma_0 = 0$, and $\Gamma_1, \Gamma_2, \dots$ be the marks of a rate λ homogeneous Poisson process on $\mathbb{R}^+$.

(a) Perform Doob's decomposition on Γ_n.

(b) Show that the martingale M you found above is $\mathcal{L}^2$. Find its brackets process $\langle M \rangle$ and the Doob decomposition of M^2.

5.7 Fix $\beta < \lambda$, and for the Poisson process of Problem 5.6 define $X_n := \mathrm{e}^{\beta \Gamma_n}$. Perform Doob's decomposition on this process. (*It is not going to be very nice.*)

5.8 •••• *Decomposing in a product sense.* Let X_n be an adapted process, and assume that it is bounded: $\sup_n X_n < K < \infty$ and bounded away from zero: $\inf_n X_n > \delta > 0$ for some non-random K and δ. Show that

(a) there is

- a martingale $(M_n)_{n \geq 0}$ with $M_0 = 1$,
- a predictable process $(B_n)_{n \geq 0} > 0$ with $B_0 = 1$

such that $X_n = X_0 \cdot M_n \cdot B_n$. This decomposition is almost everywhere unique in the sense that for any other pair $(\widehat{M}_n, \ \widehat{B}_n)_{n \geq 0}$ with the above properties we have $\mathbb{P}\{M_n = \widehat{M}_n \text{ and } B_n = \widehat{B}_n \text{ for all } n \geq 0\} = 1$.

(b) $(X_n)_{n \geq 0}$ is a submartingale if and only if $\mathbb{P}\{B_n \leq B_{n+1} \text{ for all } n \geq 0\} = 1$ in the above decomposition.

5.9 •••• Fix $\beta < \lambda$, and for the Poisson process of Problem 5.6 define $X_n := \mathrm{e}^{\beta \Gamma_n}$.

(a) Find a predictable process $B_n > 0$ for which $X_n = M_n \cdot B_n$ where M_n is a martingale. (*Hint: while X_n does not have the bounds assumed in 5.8, the calculation still works. Check that the martingale is in $\mathcal{L}^1$ separately.*)

(b) Fix $t > 0$ and $T := \inf\{n : \Gamma_n > t\}$. Assume $\beta > 0$ and use the memoryless property of the Exponential distribution to bound the stopped martingale M^T from above by a random variable of finite mean. Hence show that optional stopping applies.

(c) Show from the above that the number of marks before t is Poisson$(\lambda \cdot t)$ distributed (just by assuming the i.i.d. Exponential(λ) interarrival times in the Poisson(λ) process). (*Hint: use its generating function.*)

5.10 Let Y_i, $i = 1, 2, \dots$ be i.i.d. variables in $\mathcal{L}^2$ with mean μ and variance σ^2. Define $X_0 = 0$, $X_k := \frac{1}{k} \sum_{i=1}^{k} Y_i$ $(k \geq 1)$.

(a) Perform Doob's decomposition on X.

(b) Show that the martingale M you found above is $\mathcal{L}^2$. Find its brackets process $\langle M \rangle$ and the Doob decomposition of M^2.

(c) Show that the martingale M converges a.s.