HOMEWORK SET 6

Characteristic functions, CLT

Further Topics in Probability, 2nd teaching block, 2019

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Problems with •'s are to be handed in. These are due in the blue locker marked "Further Topics in Probability" on the ground floor of the Main Maths Building before 12:00pm on Thursday, 9th May. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Introducing: ◖ for half a mark. Random variables are defined on a common probability space unless otherwise stated.

6.1 Determine the characteristic functions of

a) ◖ the Bernoulli(p),
b) ◖ the Binomial(n, p),
c) ◖ the Poisson(λ),
d) ◖ the Optimistic and Pessimistic Geometric(p),
e) the Normal(μ, σ^2) from Standard Normal,
f) ◖ the Exponential(λ),
g) ◖ the Uniform(a, b),
h) the Cauchy(b, a) if you haven't seen it on Thursday

distributions.

6.2 •••• From convolutions it's easy to see that for integer $n > 0$

$$\varphi_{\text{Gamma}(n,\,\lambda)}(t) = \Big(\frac{\lambda}{\lambda - it}\Big)^n.$$

In fact for any real $\alpha > 0$ and any $t \in \mathbb{R}$,

$$\varphi_{\text{Gamma}(\alpha,\,\lambda)}(t) = \Big(\frac{\lambda}{\lambda - it}\Big)^\alpha.$$

This problem asks you to prove this latter for $|t| < \lambda$. *Hint: 1) write the Taylor series for* e^{itx}, *2) check if the sum and the integral can be swapped, 3) do the integral by comparing with the Gamma density; 4) get a binomial coefficient by recursion on the Gamma function, 5) transform your binomial into* $\binom{-\alpha}{n}$, *6) apply the generalised Binomial Theorem.*

6.3 Let φ be the characteristic function of a probability distribution. Are $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ characteristic functions?

6.4 Let $f(x) = 1 - |x - 1|$ for $0 \le x \le 2$, and 0 otherwise. Determine the characteristic function of the distribution with density f.

6.5 Explain, using characteristic functions, the identities

$$\frac{\sin t}{t} = \frac{\sin(t/2)}{t/2} \cdot \cos(t/2) \qquad \text{and} \qquad \frac{\sin t}{t} = \prod_{k=1}^{\infty} \cos\Big(\frac{t}{2^k}\Big).$$

6.6 Show that if φ is the characteristic function of an integer-valued distribution, then the mass function is of the form

$$p(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi(t)\,dt, \qquad \forall k \in \mathbb{Z}.$$

6.7 Let μ_n be the Exponential(λ_n) probability distribution, $n = 1, 2, \dots$. Prove that the sequence $\{\mu_n\}_n$ is tight if and only if the positive number sequence λ_n is bounded away from zero.

6.8 Let μ_n be the Normal(m_n, σ_n^2) probability distribution, $n = 1, 2, \dots$. Prove that the sequence $\{\mu_n\}_n$ is tight if and only if the number sequences m_n and σ_n^2 are both bounded.

6.9 ••• Let μ_n be the Uniform(α_n, β_n) probability distribution for $\alpha_n < \beta_n$, $n = 1, 2, \dots$. What condition on these two sequences makes $\{\mu_n\}_n$ relatively compact?

Below you will need the Continuity Lemma. It states

Theorem 1 *Let* μ_n, $n = 1, 2, \dots$ *be a sequence of distributions, and* φ_n *the associated characteristic functions.*

1. *If* $\mu_n \xrightarrow{w} \mu$, *then for all* $t \in \mathbb{R}$ $\varphi_n(t) \to \varphi(t)$, *where* φ *is the characteristic function of* μ.
2. *If for all* $t \in \mathbb{R}$ $\varphi(t) := \lim_{n\to\infty} \varphi_n(t)$ *exists, and is continuous at* $t = 0$, *then* φ *is the characteristic function of a distribution* μ, *and* $\mu_n \xrightarrow{w} \mu$.

6.10 Let X_p be Pessimistic Geometric(p) distributed. Prove, via characteristic functions and the Continuity Lemma, that $p \cdot X_p$ converges to Exponential(1) in distribution as $p \searrow 0$.

6.11 Let X be Poisson(λ) distributed. Prove, via characteristic functions and the Continuity Lemma, that

$$\frac{X - \lambda}{\sqrt{\lambda}} \xrightarrow{d} \text{Normal}(0,\,1)$$

as $\lambda \to \infty$.

6.12 •••• Let X be Gamma(α, λ) distributed (notice that α might be a non-integer). Prove, via characteristic functions and the Continuity Lemma, that

$$\frac{\lambda X - \alpha}{\sqrt{\alpha}} \xrightarrow{d} \text{Normal}(0,\,1)$$

as $\alpha \to \infty$ and λ is fixed.

6.13 Let $Y_1, Y_2, \dots$ be iid. Uniform(0, 1) random variables, and $X_k = k \cdot Y_k$, $S_n = X_1 + X_2 + \dots + X_n$. Prove that

$$\frac{S_n}{\frac{n^2}{4}} \xrightarrow[n\to\infty]{w} 1, \qquad \text{and} \qquad \frac{S_n - \frac{n^2}{4}}{\frac{1}{6} n^{\frac{3}{2}}} \xrightarrow[n\to\infty]{w} \text{Normal}(0,\,1).$$

Below you will need the Central Limit Theorem:

Theorem 2 *Let* X_i *be iid. random variables with finite mean* m *and variance* σ^2. *Then for every* $a \in \mathbb{R}$,

$$\mathbf{P}\Big\{\frac{X_1 + X_2 + \dots + X_n - n \cdot m}{\sigma\sqrt{n}} \le a\Big\} \overset{n\to\infty}{\longrightarrow} \phi(a) \qquad \textit{(Standard Normal distribution).}$$

6.14 We keep rolling a fair die until the sum of the numbers shown on it exceeds 300. Estimate the probability that at least 80 rolls are needed for this.

6.15 ••• We round 48 real numbers to integers, then sum them up. Suppose that rounding makes a Uniform(-0.5, 0.5) error independently, for each of the 48 numbers. Estimate the probability that our result differs from the true sum by more than 3.