Fluctuations of asymmetric interacting systems in one dimension

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UW-Madison
Work supervised by Bálint Tóth

Mathematical Biosciences Institute
2005

1. Deposition processes and particle systems

2. Examples

3. The second class particle

4. Growth/current fluctuations

5. The role of the second class particle

6. The speed of the second class particle

(7. A few words on hydrodynamics)

1. Deposition processes and particle systems

Totally asymmetric simple exclusion:

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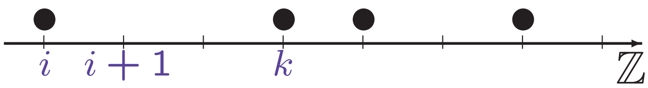
$\underline{\eta} = (\eta_i)_{i \in \mathbb{Z}}$



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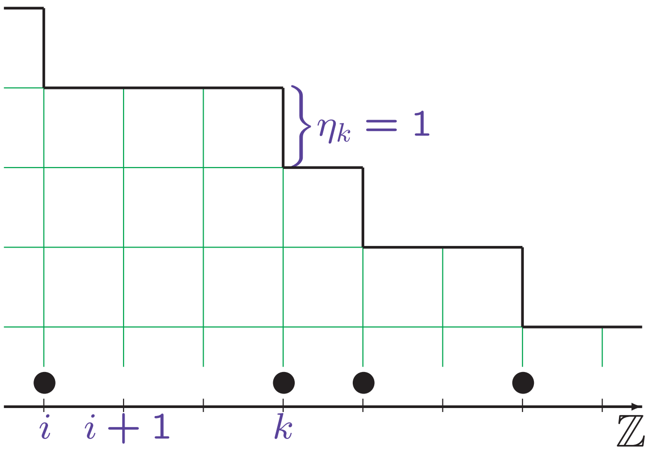

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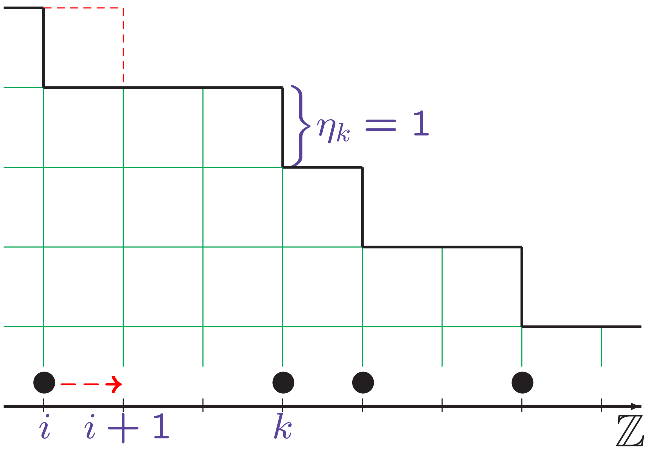

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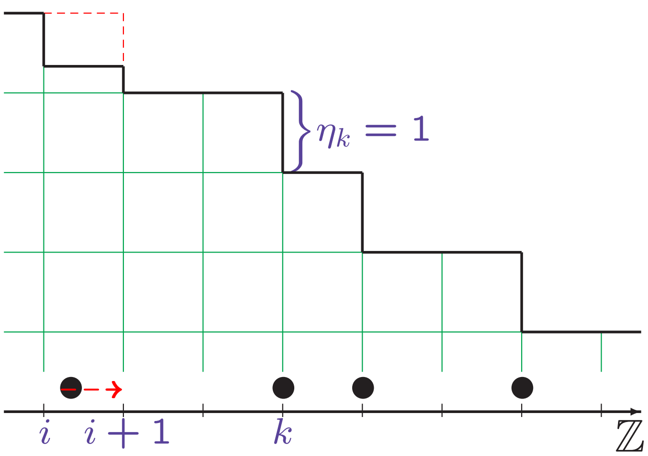

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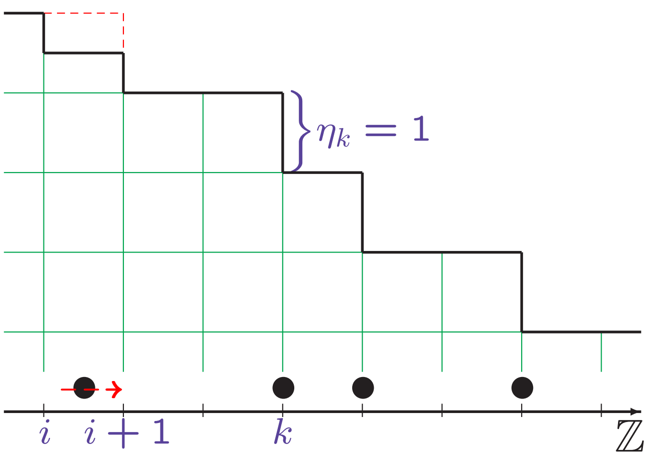

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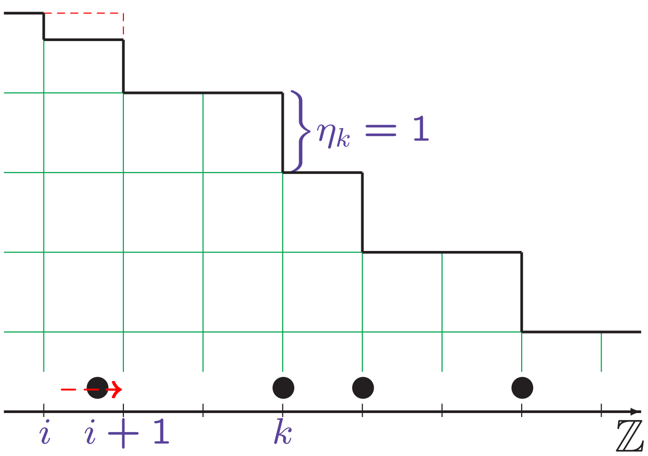

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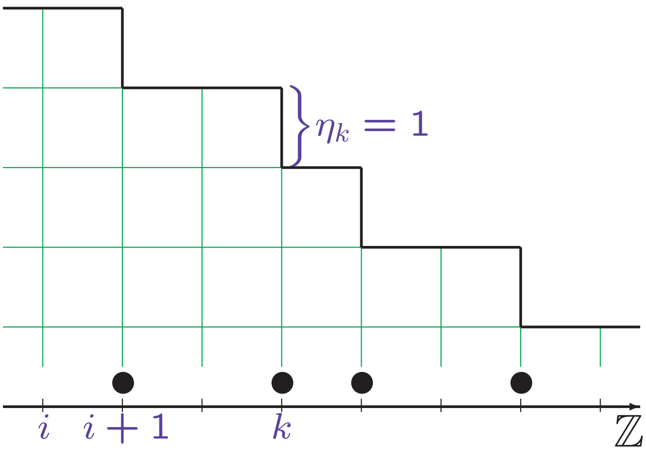

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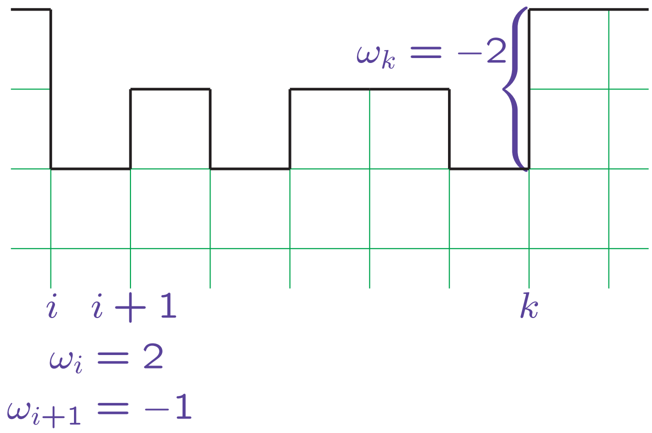
ω_i = negative discrete gradient



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Attractivity: $r(\cdot, \cdot)$ is

non-decreasing in the first / non-increasing in the second variable.

Higher neighbors $\rightsquigarrow$ higher growth rates.

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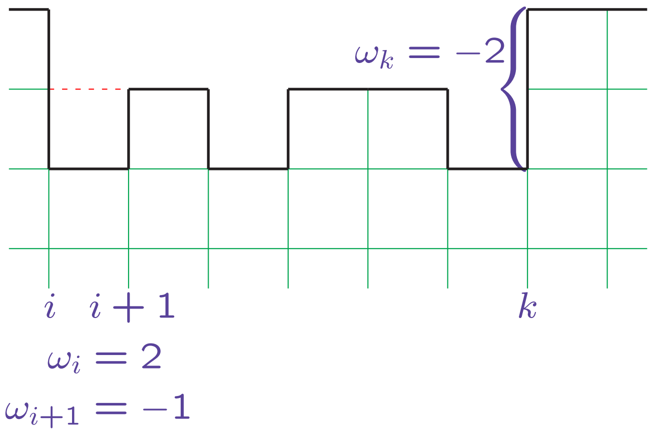

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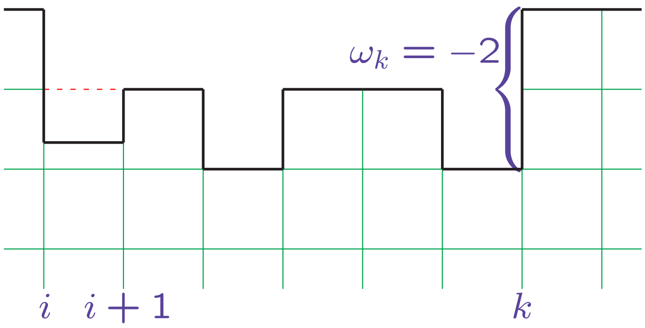

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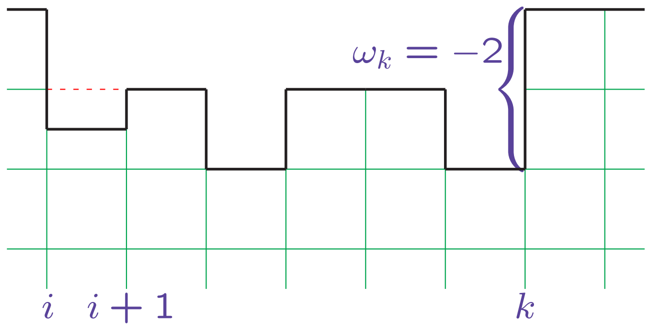

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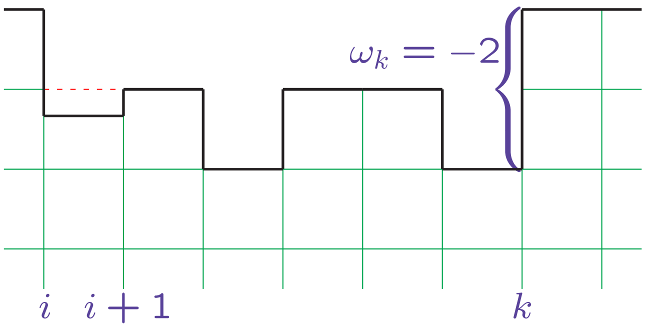

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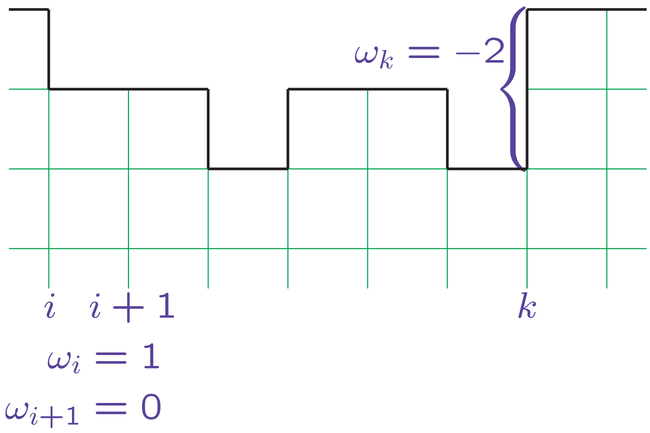

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Equilibrium

We need a well-behaved equilibrium distribution. The only case we can really handle is the *product measure*, i.e. when ω_i's are iid.

Technical assumptions for the equilibrium being product:

$$\begin{aligned} & r(x,\, y) + r(y,\, z) + r(z,\, x) \\ = & \, r(x,\, z) + r(z,\, y) + r(y,\, x) \end{aligned}$$

and

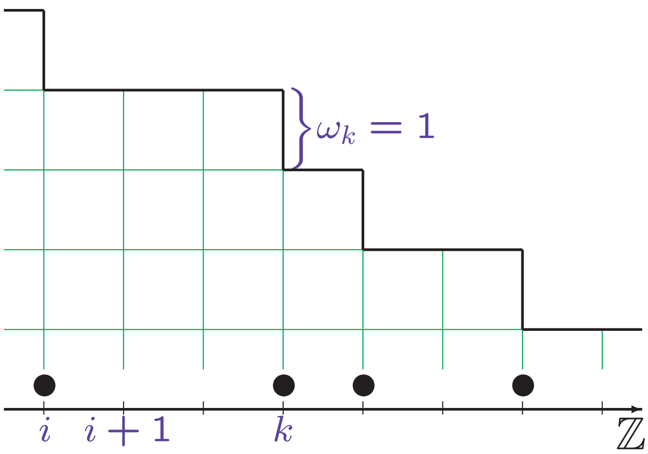
$$\begin{aligned} & r(x, y-1) \cdot r(y, z-1) \cdot r(z, x-1) \\ = & \, r(x, z-1) \cdot r(z, y-1) \cdot r(y, x-1) \end{aligned}$$

for any $x,\, y,\, z \in \mathbb{Z}$.

$\rightsquigarrow$ Then ω_i's being independent and $\mu^{(\theta)}$-distributed is an equilibrium with some $\mu^{(\theta)}$ depending on the form of the rates $r(\cdot,\, \cdot)$. The parameter θ of μ sets $\mathbf{E}(\omega_i)$, i.e. the average (negative) slope of the wall.

2. Examples

Totally asymmetric simple exclusion (TASE):



$\omega_i \in \{0, 1\}$

$$r(\omega_i, \omega_{i+1}) = \begin{cases} 1 \text{ if } (\omega_i, \omega_{i+1}) = (1, 0) \\ 0 \text{ else} \end{cases}$$

Equilibrium:

Bernoulli measure with density ϱ (instead of θ).

Constructed e.g. in Liggett's 1985 book.

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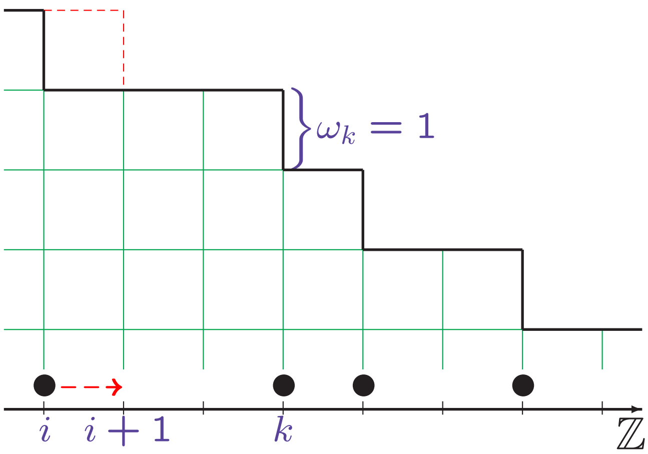

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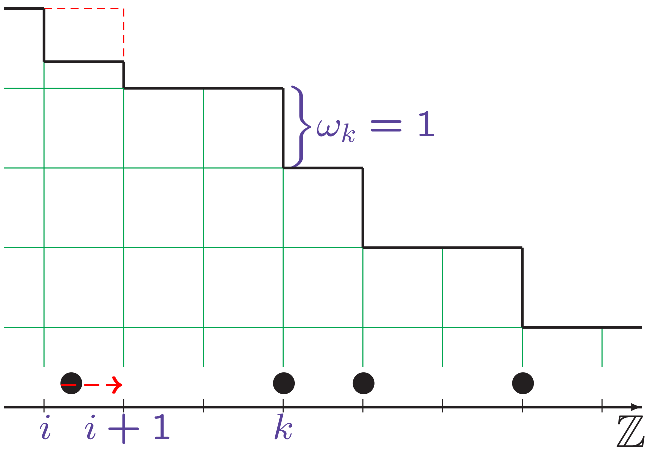

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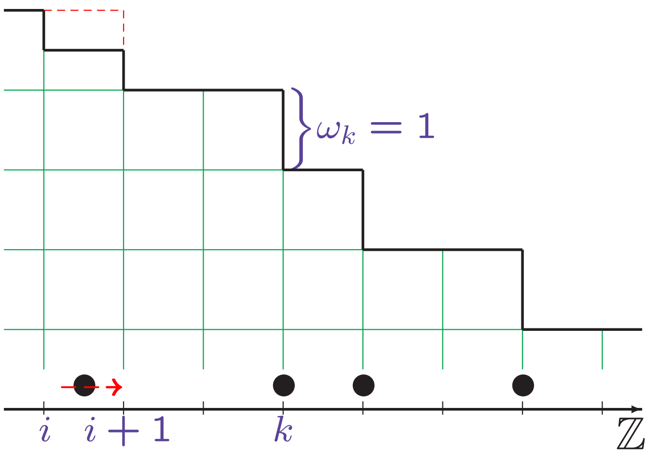

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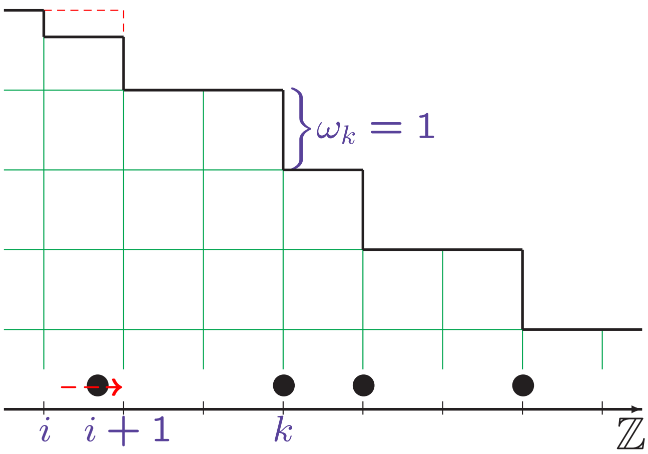

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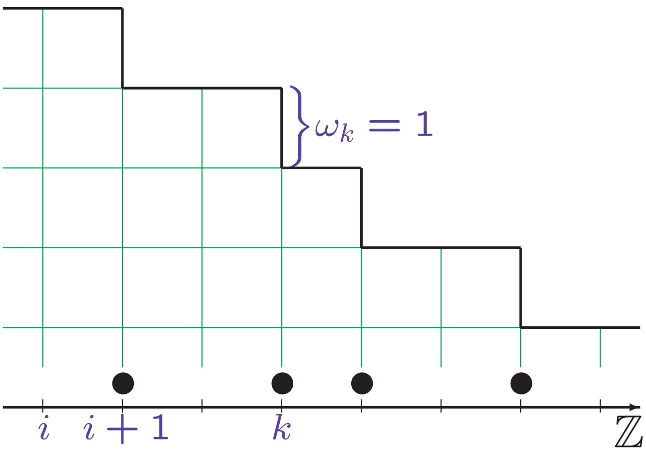

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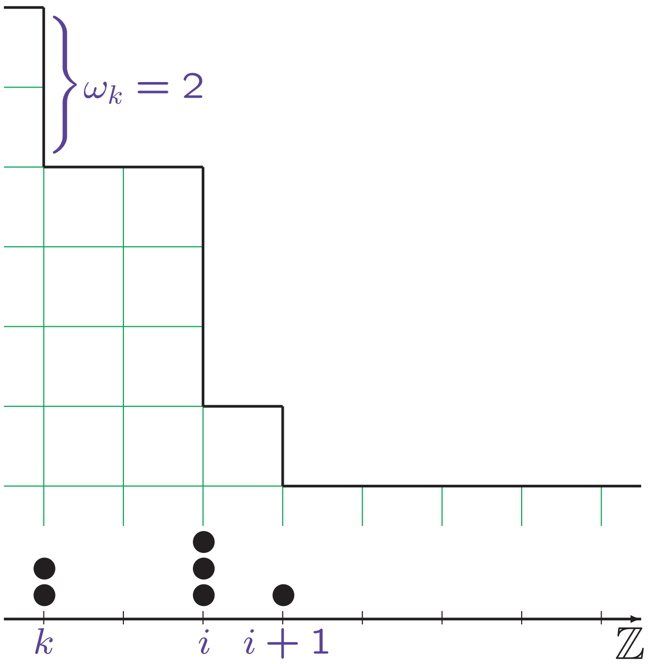

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Equilibrium:

Product of modified Poisson-distributions with a parameter θ.

Special case:

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Constructed by Andjel 1981 if $f(z+1) - f(z) \leq K$.

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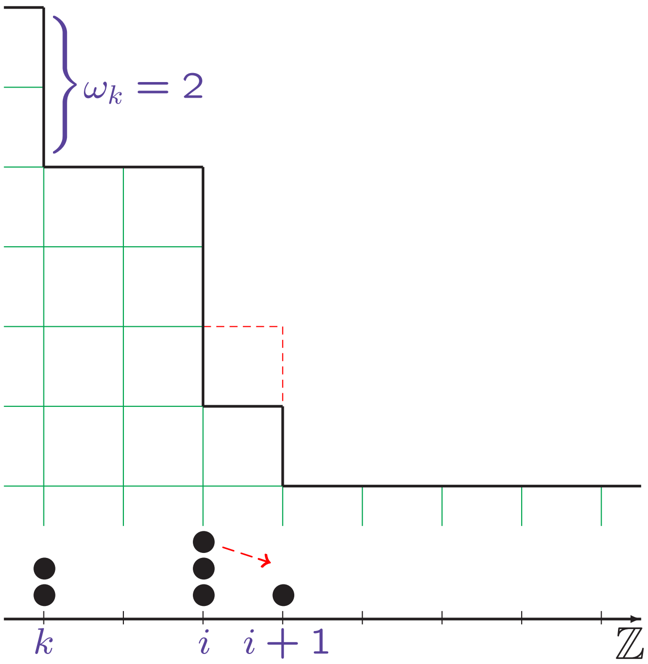

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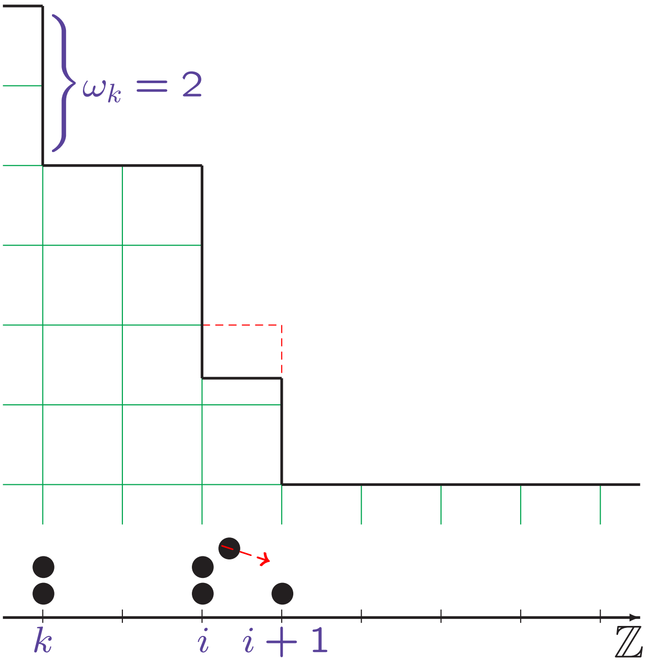

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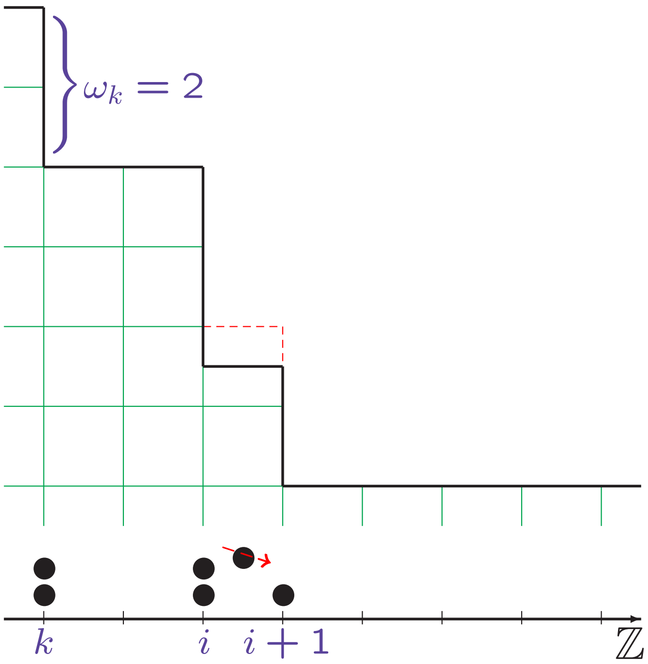

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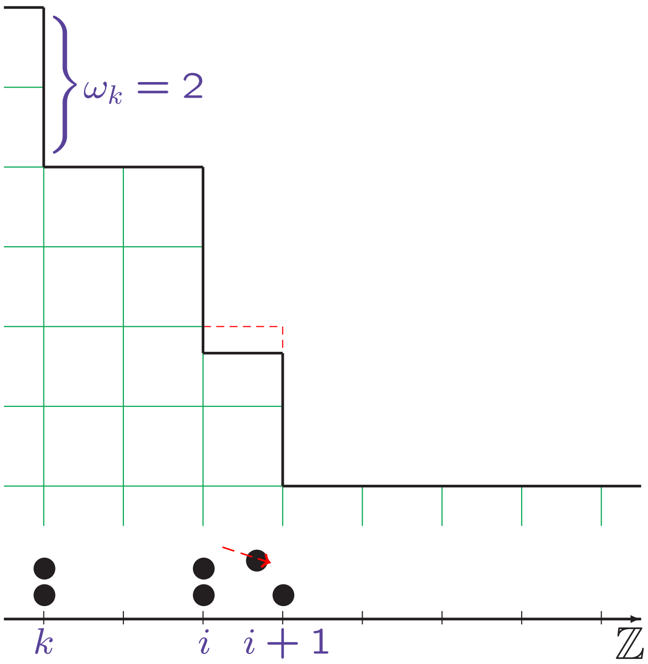

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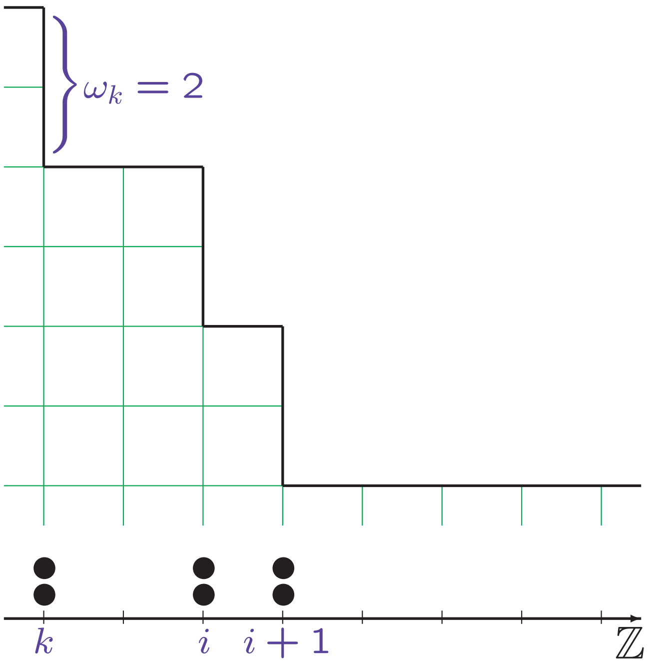

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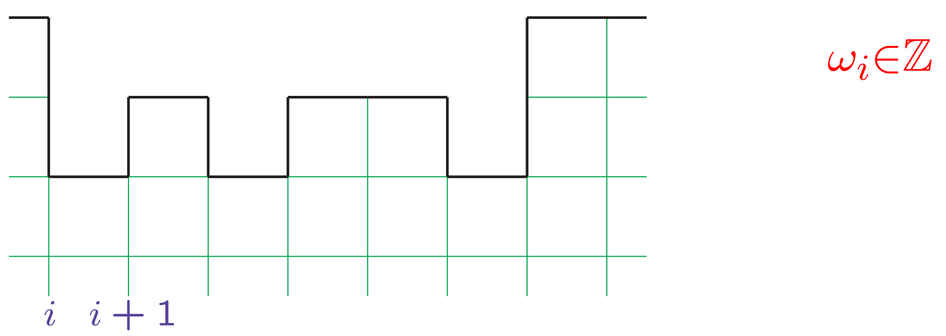

$$r(\omega_i, \omega_{i+1}) = f(\omega_i) + f(-\omega_{i+1})$$

with f non-decreasing, and $f(z) \cdot f(1-z) = 1$.

Equilibrium:

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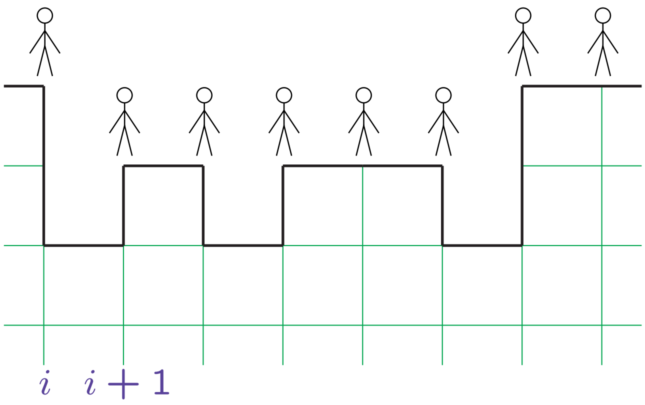

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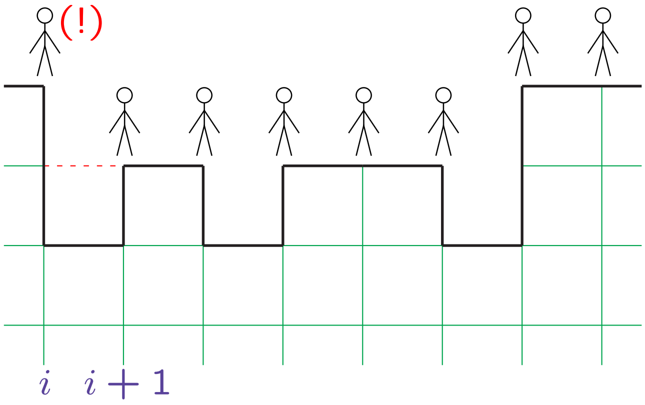

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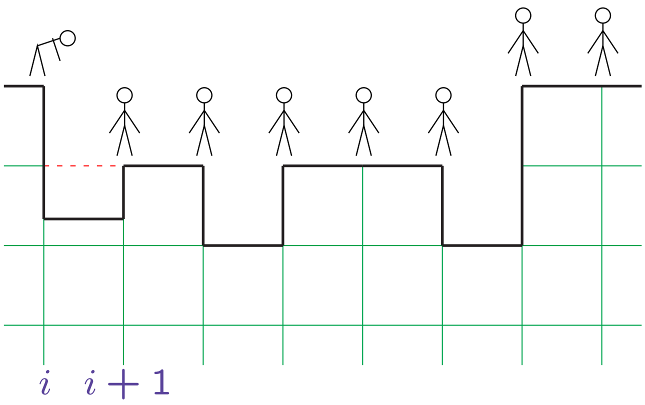

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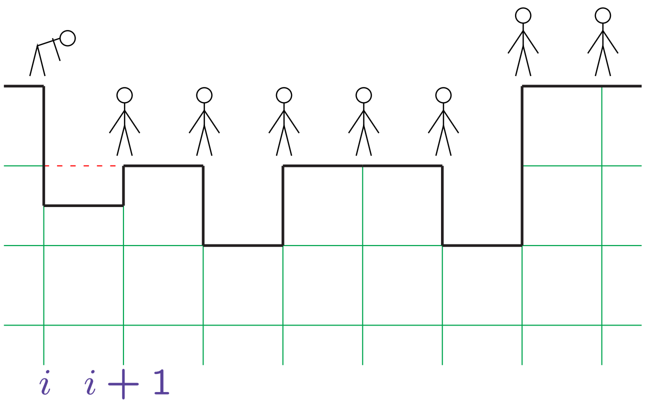

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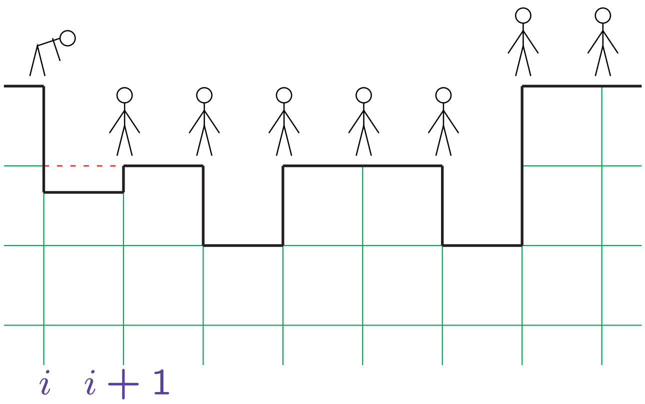

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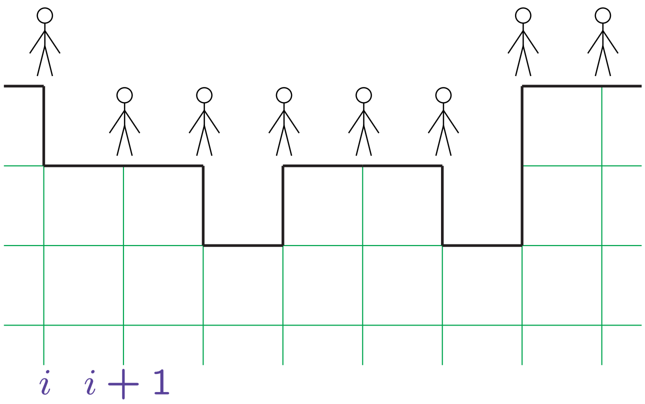

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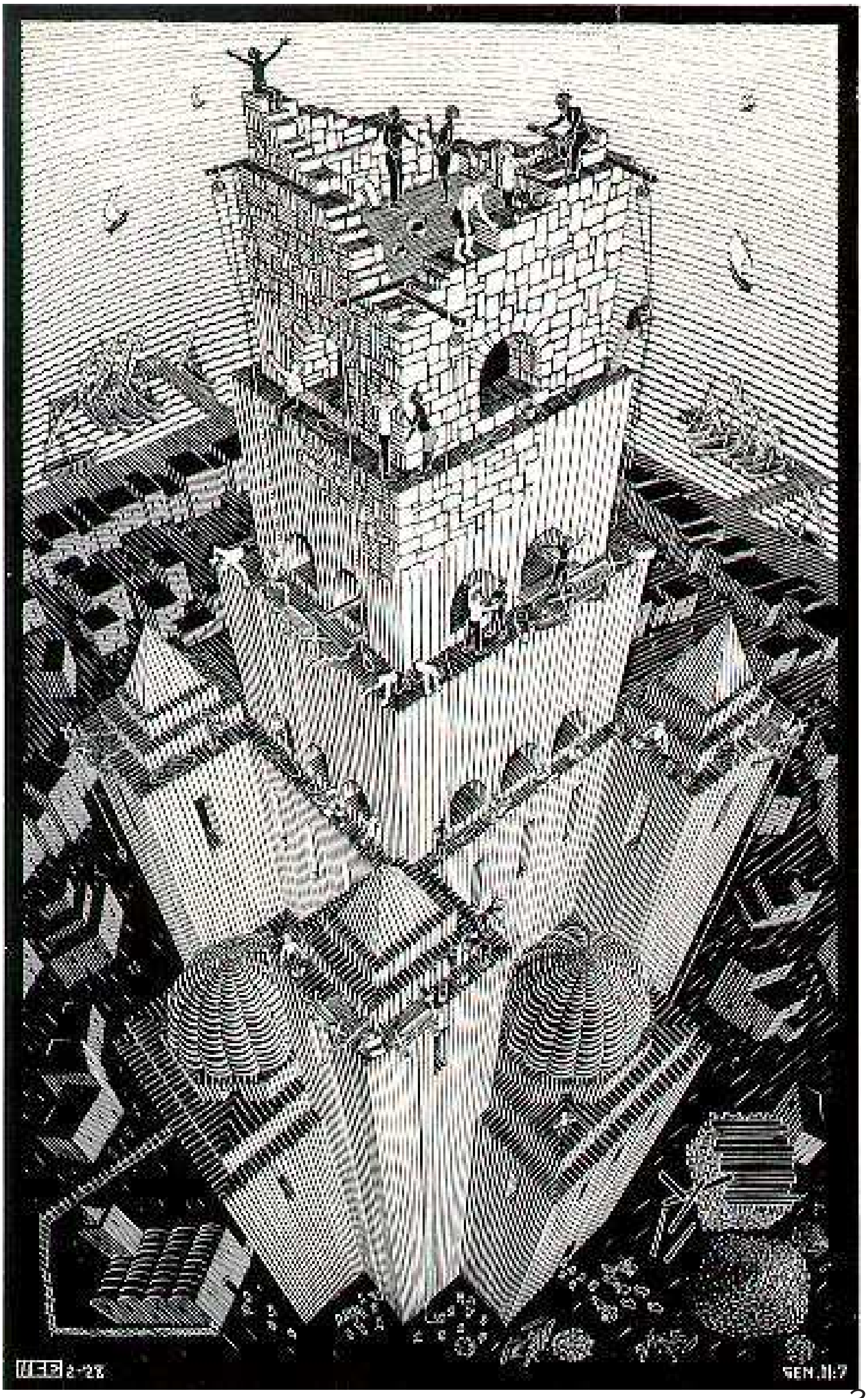
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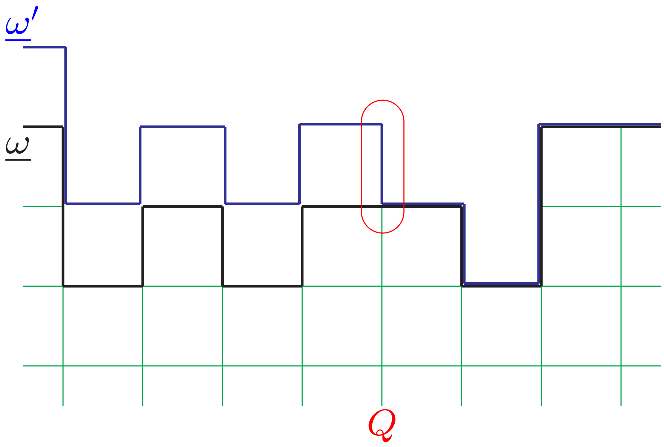
3. The second class particle

Two configurations only differ by one at site Q.



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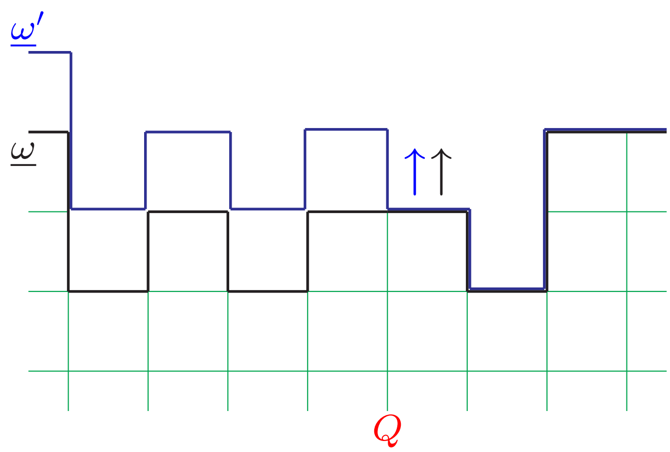

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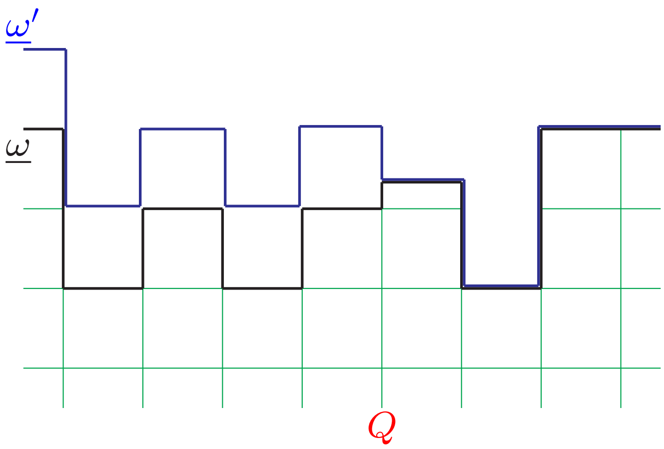

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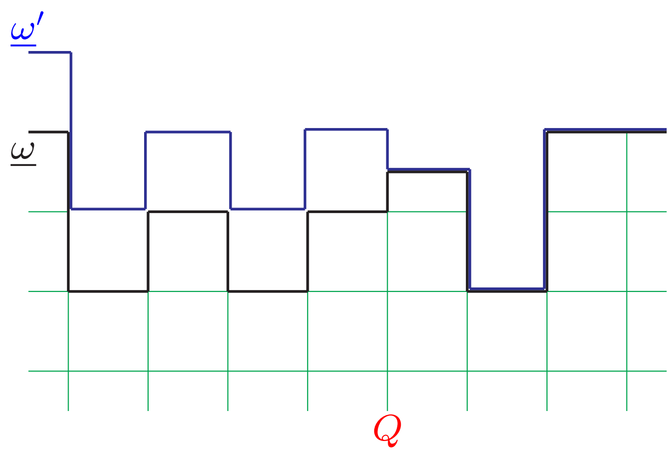

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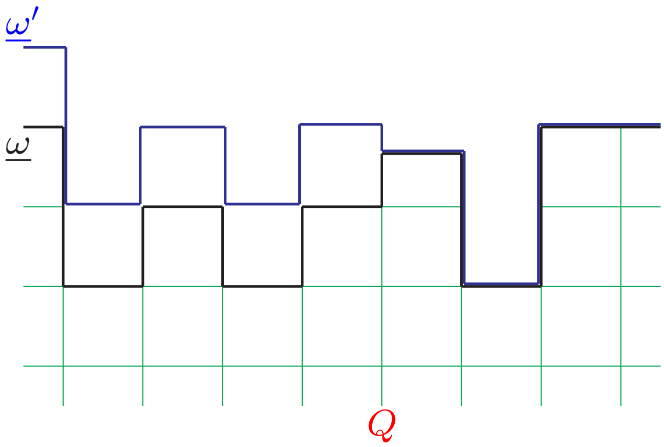

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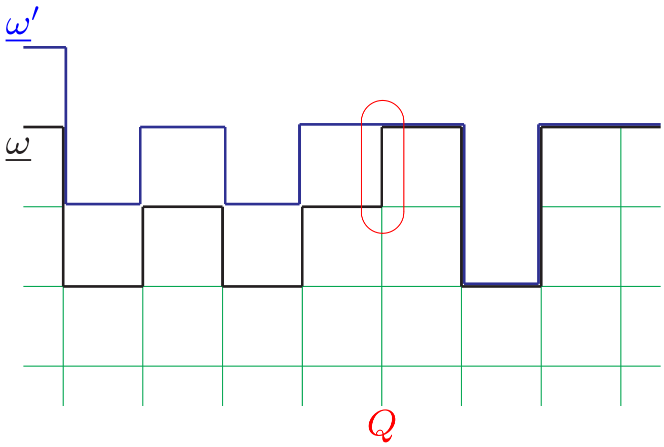

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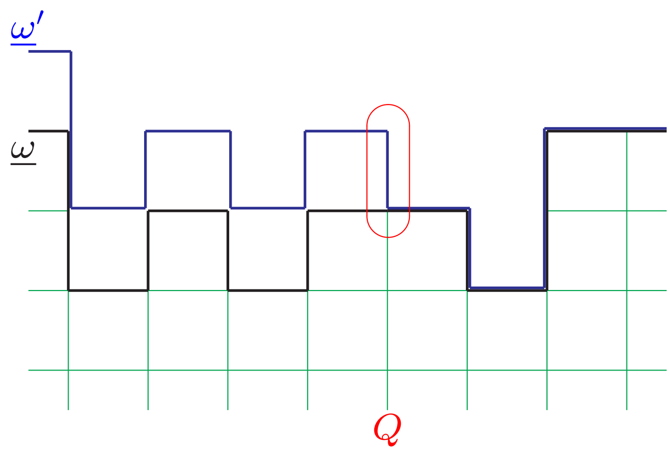

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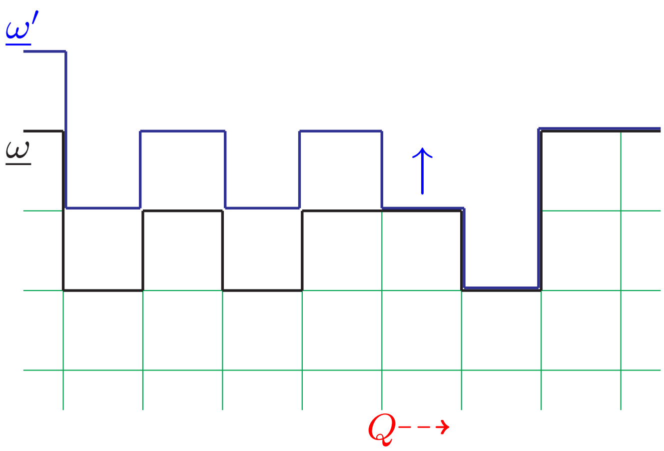

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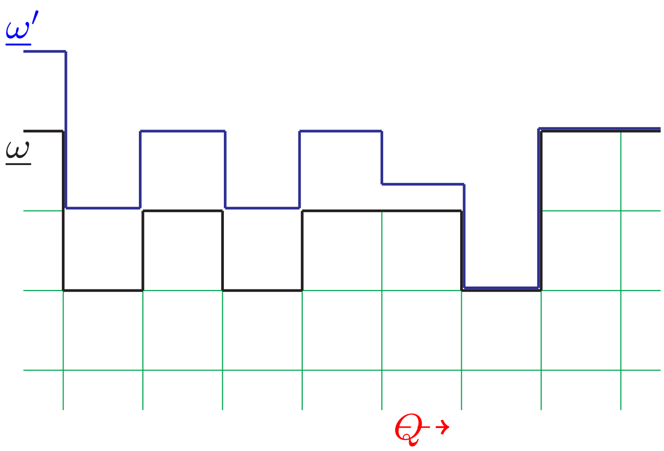

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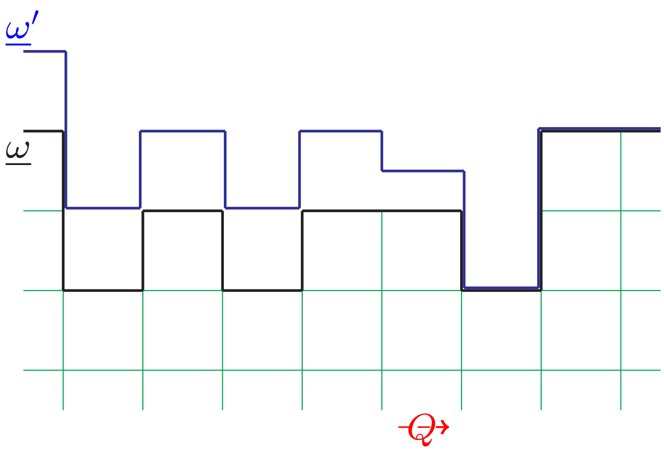

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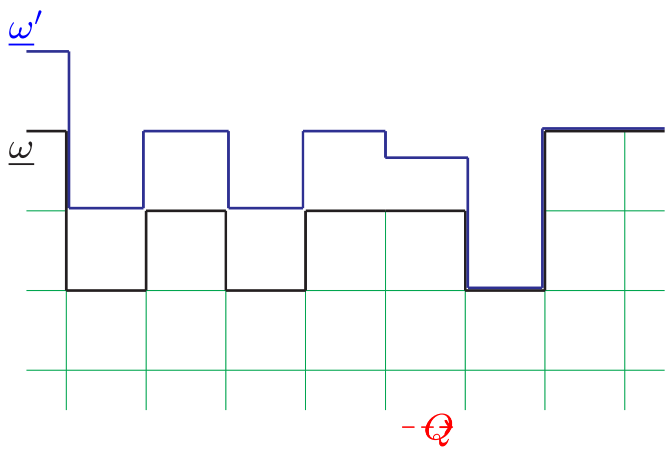

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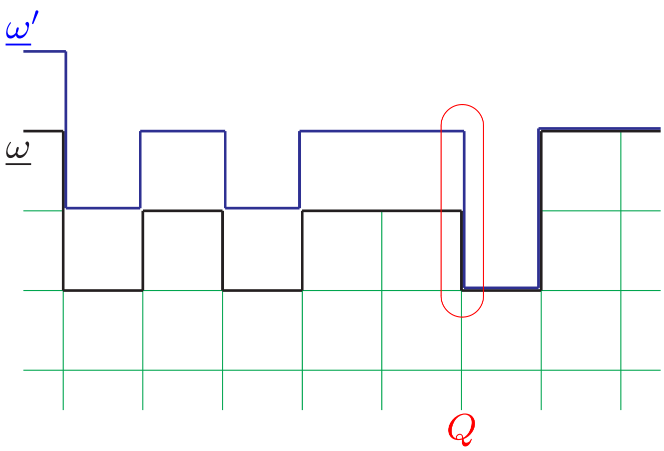

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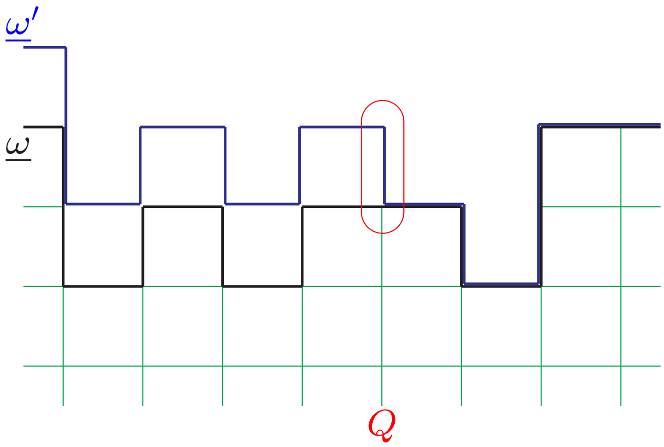

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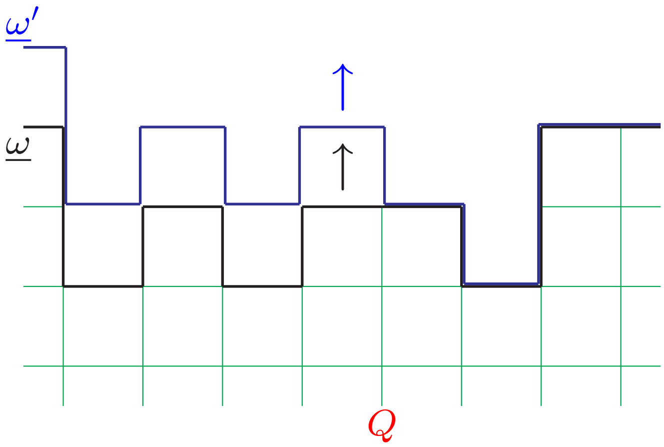

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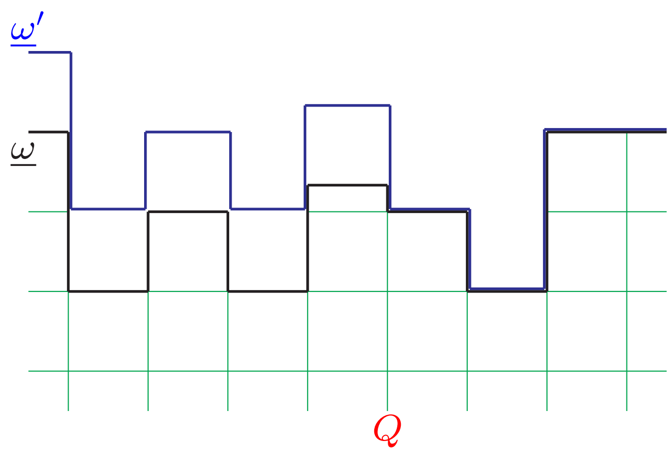

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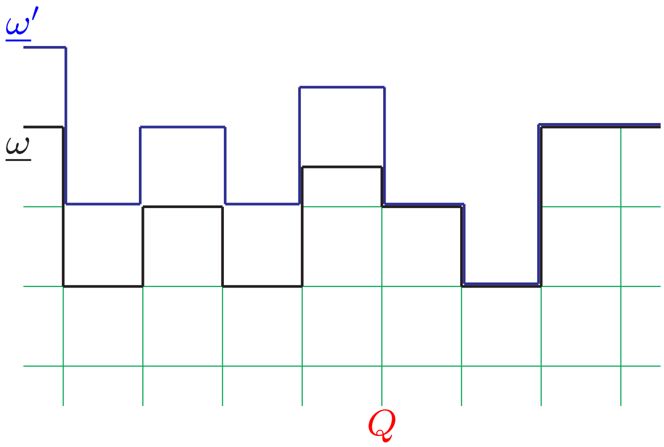

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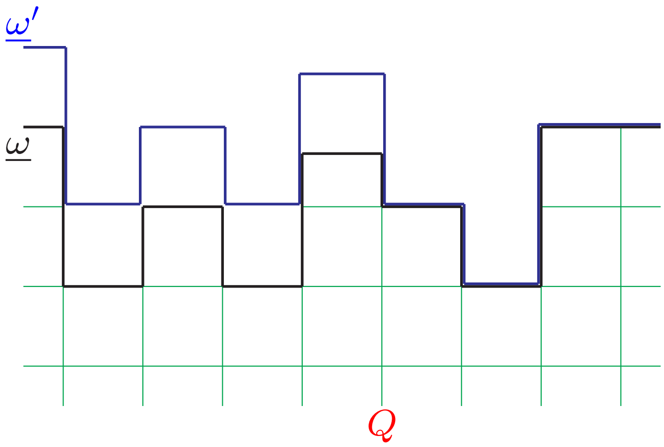

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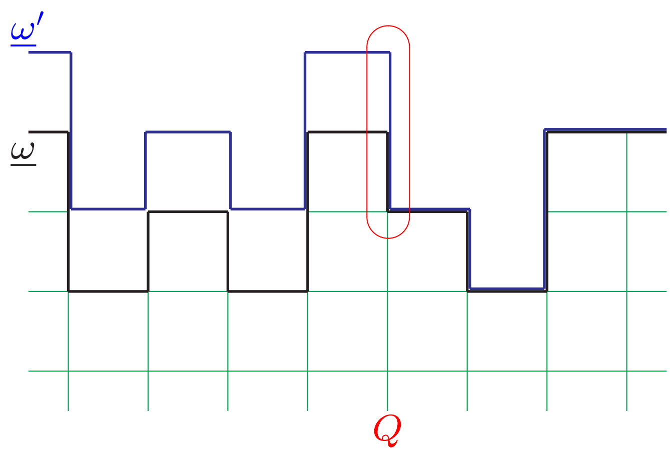

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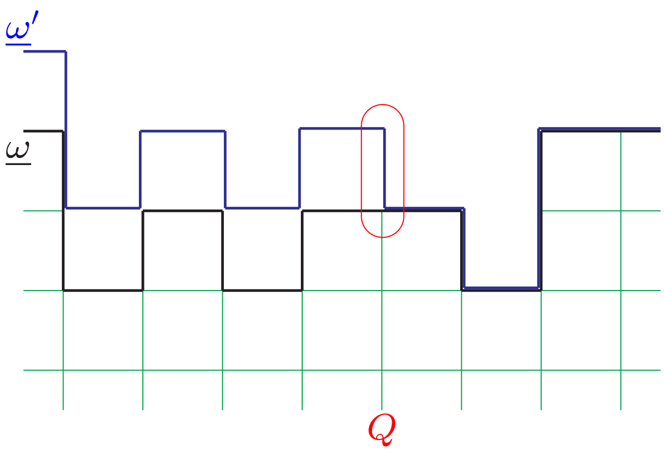

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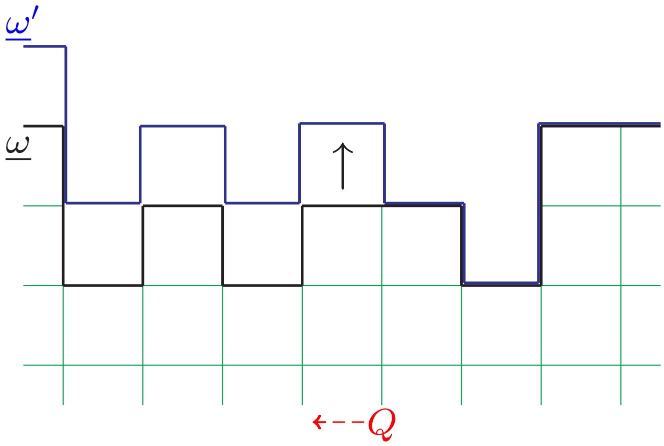

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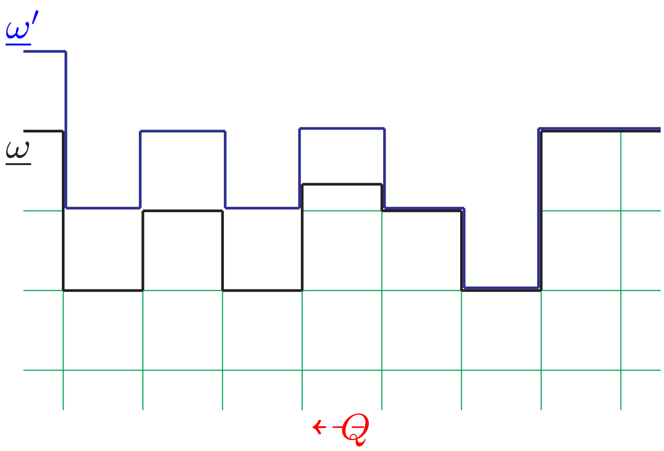

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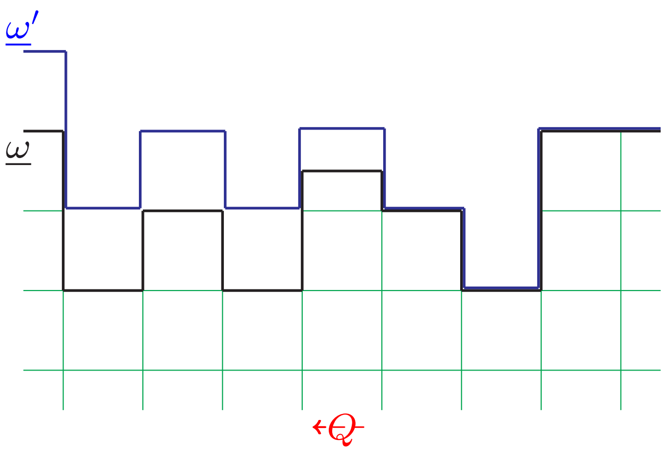

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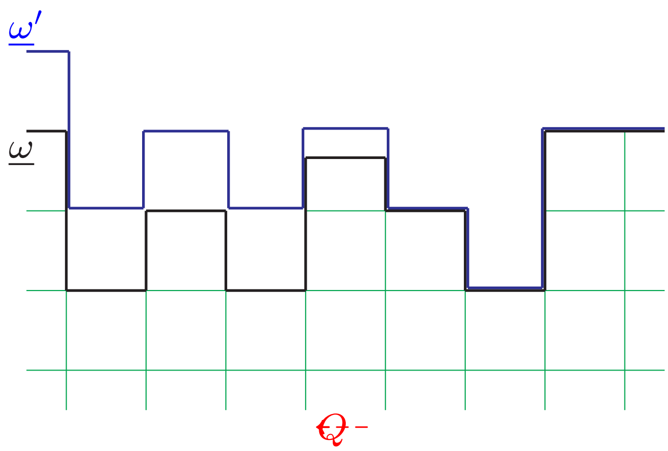

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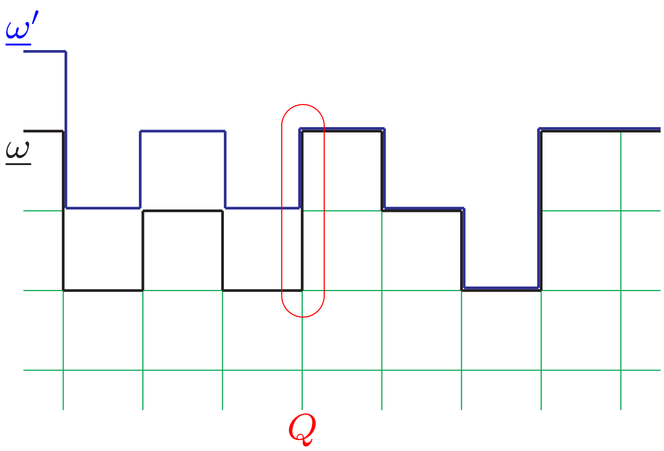

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difference of column growths above $[i, i+1]$
$=$ algebraic number of second class particles passed.

4. Growth/current fluctuations

Let $h_i(t)$ be the height of the column above $[i, i+1]$ at time t. Fix a velocity value $V \in \mathbb{R}$. Define

$$J^{(V)}(t) := h_{\lfloor Vt \rfloor}(t) - h_0(0).$$

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Assume that $\underline{\omega}$ is started from equilibrium with parameter θ, and assume also the Law of Large Numbers

$$\frac{Q(t)}{t} \xrightarrow[t\to\infty]{L^2} C(\theta)$$

for the second class particle.

$\rightsquigarrow$ I.e. the second class particle has a *speed*.

Then for the whole class of models: (B. 2003)

LLN:

$$\frac{J^{(V)}(t)}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \mathbf{E}^{(\theta)}[r(\omega_i, \omega_{i+1})] - V \cdot \mathbf{E}^{(\theta)}(\omega_i)$$

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CLT:

$$\frac{J^{(V)}(t) - \mathbf{E}^{(\theta)} J^{(V)}(t)}{\sqrt{t}} \xrightarrow[t\to\infty]{\text{D}} \mathcal{N},$$

a normal random variable with the above variance.

Simple consequence of the variance formula; fluctuations of the initial state are transported.

Ferrari - Fontes 1994 for SE.

Remarks:

$\rightsquigarrow$ The fluctuations are Gaussian (of order $t^{1/2}$) if $V \neq C(\theta)$. In this scale, basically fluctuations coming from the initial state are observed. For $V = C(\theta)$, these fluctuations disappear, and only the dynamical noise remains. The latter is expected to appear on the $t^{1/3}$ time-scale for most systems, this is one of the greatest open questions in the field. T. Seppäläinen showed the limit on the $t^{1/4}$ scale for independent random walks, and we are currently working on a similar result for the so-called *random average process*.

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For ZR and BL: Assume the rate $f(z)$ is convex. Then

$$\frac{Q(t)}{t} \xrightarrow[t\to\infty]{L^n} C(\theta)$$

for any n. B. 2003.

$C(\theta)$ is the *characteristic speed* in hydrodynamics.

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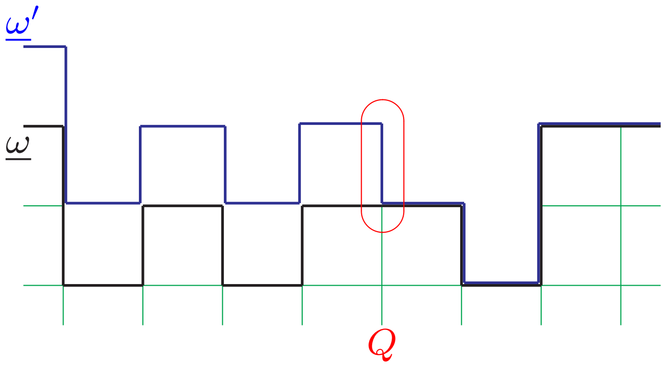
$\longrightarrow$ The non-trivial term is

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{n}{t} \mathbf{Cov}(\omega_n(t), \omega_0(0)).$$



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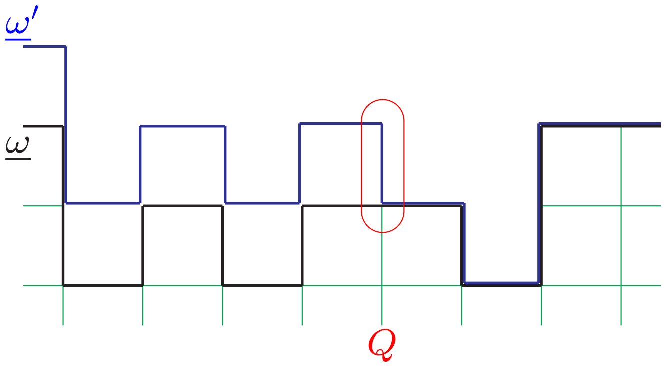
$$\omega'_n(t) = \omega_n(t) + \mathbf{1}\{Q(t) = n\}$$



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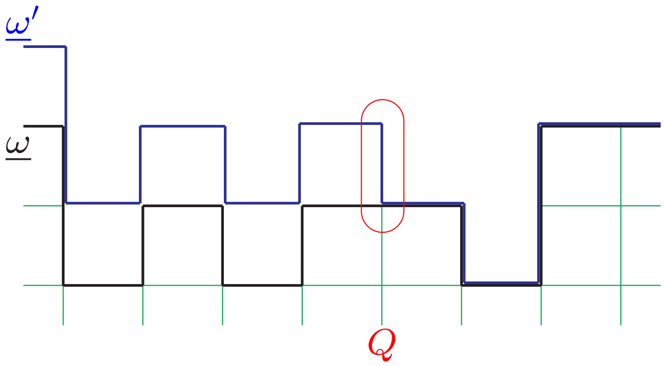

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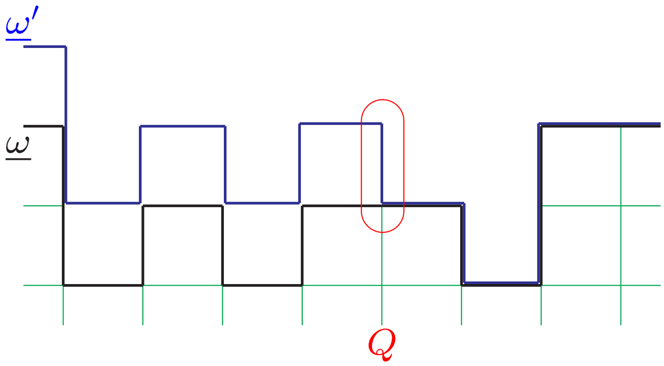

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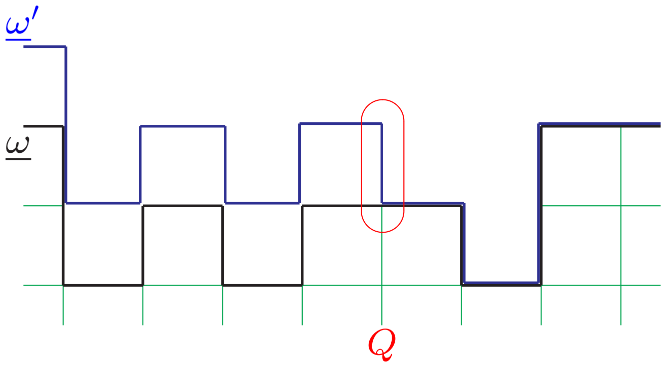

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Compare the two sides, build the covariance step by step.

The argument and the LLN for the second class particle shows that for $V_1 \neq C(\theta) \neq V_2$,

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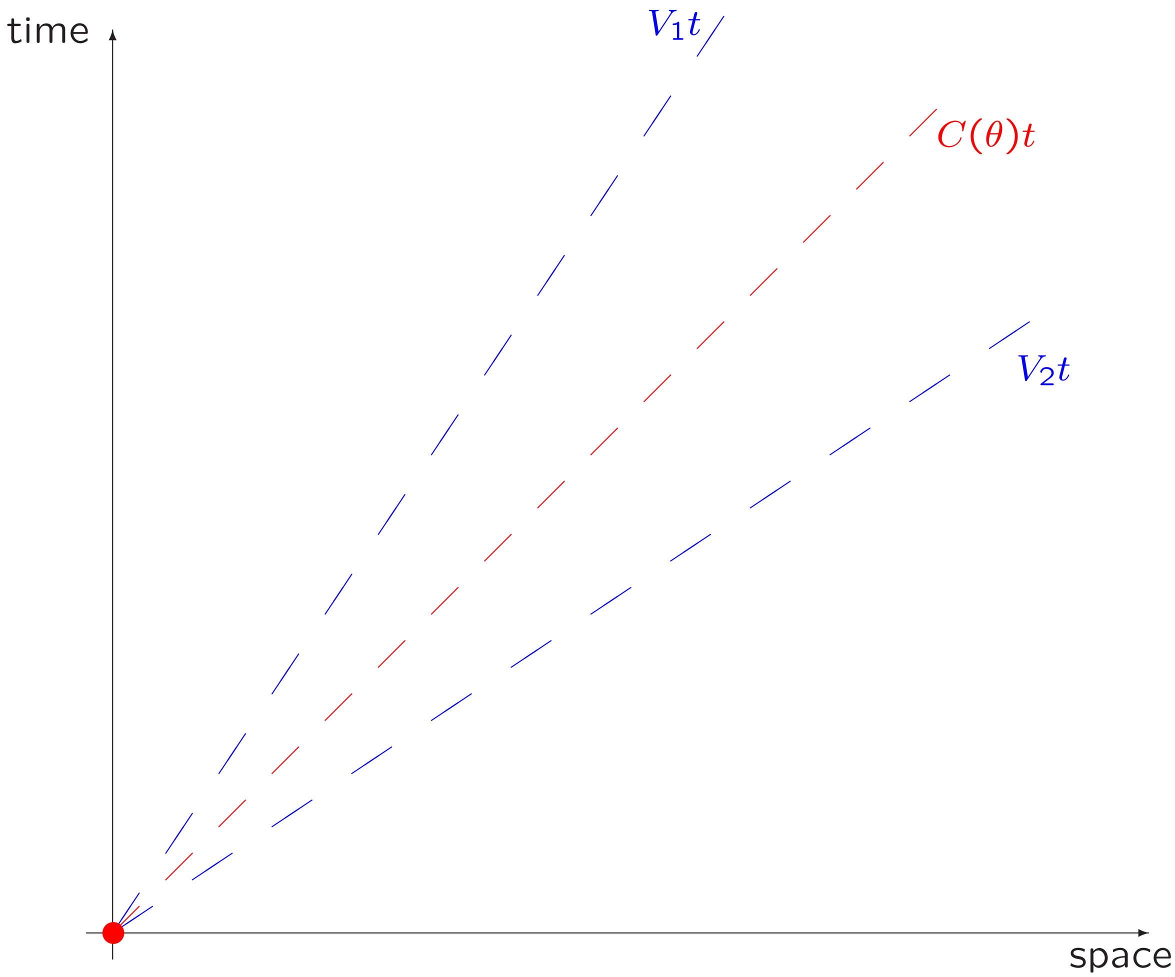

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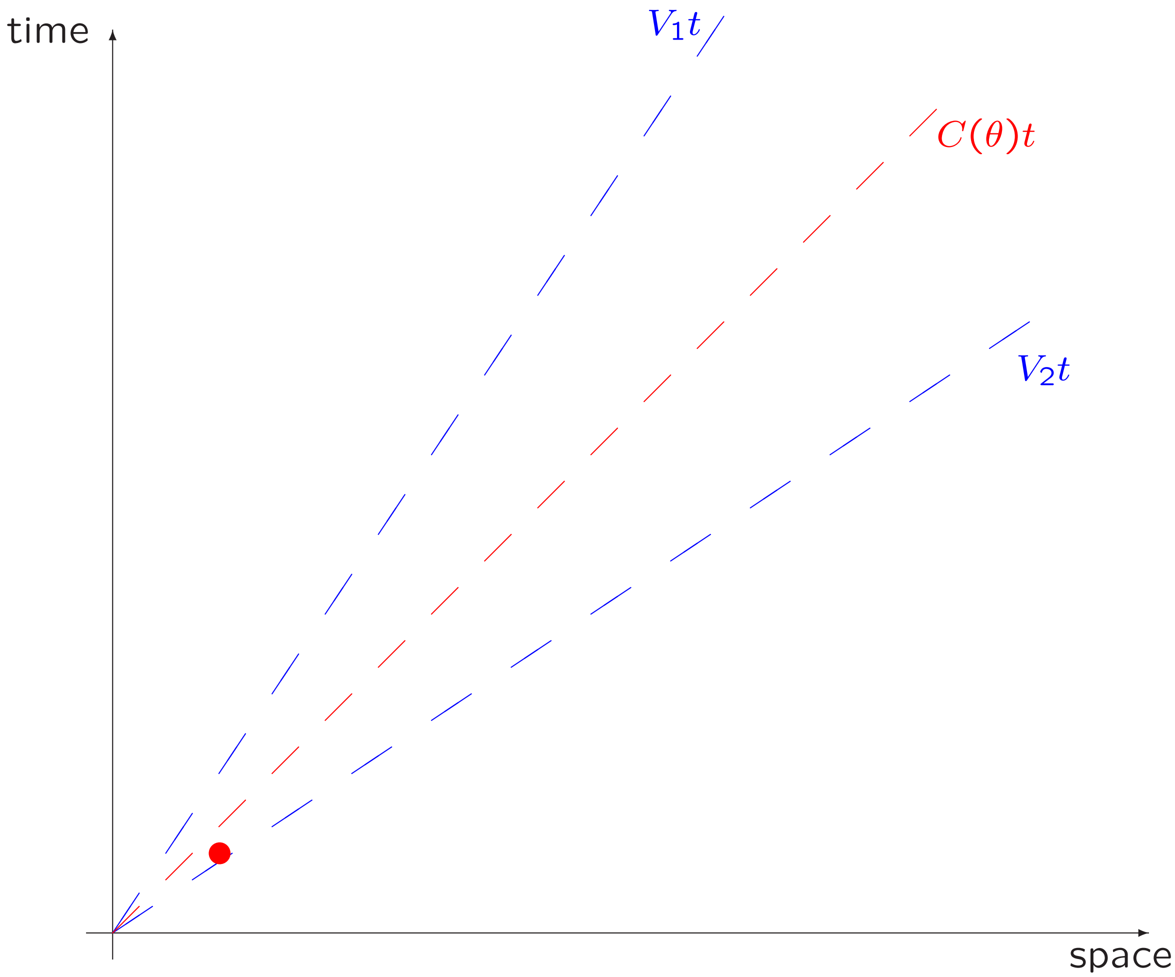

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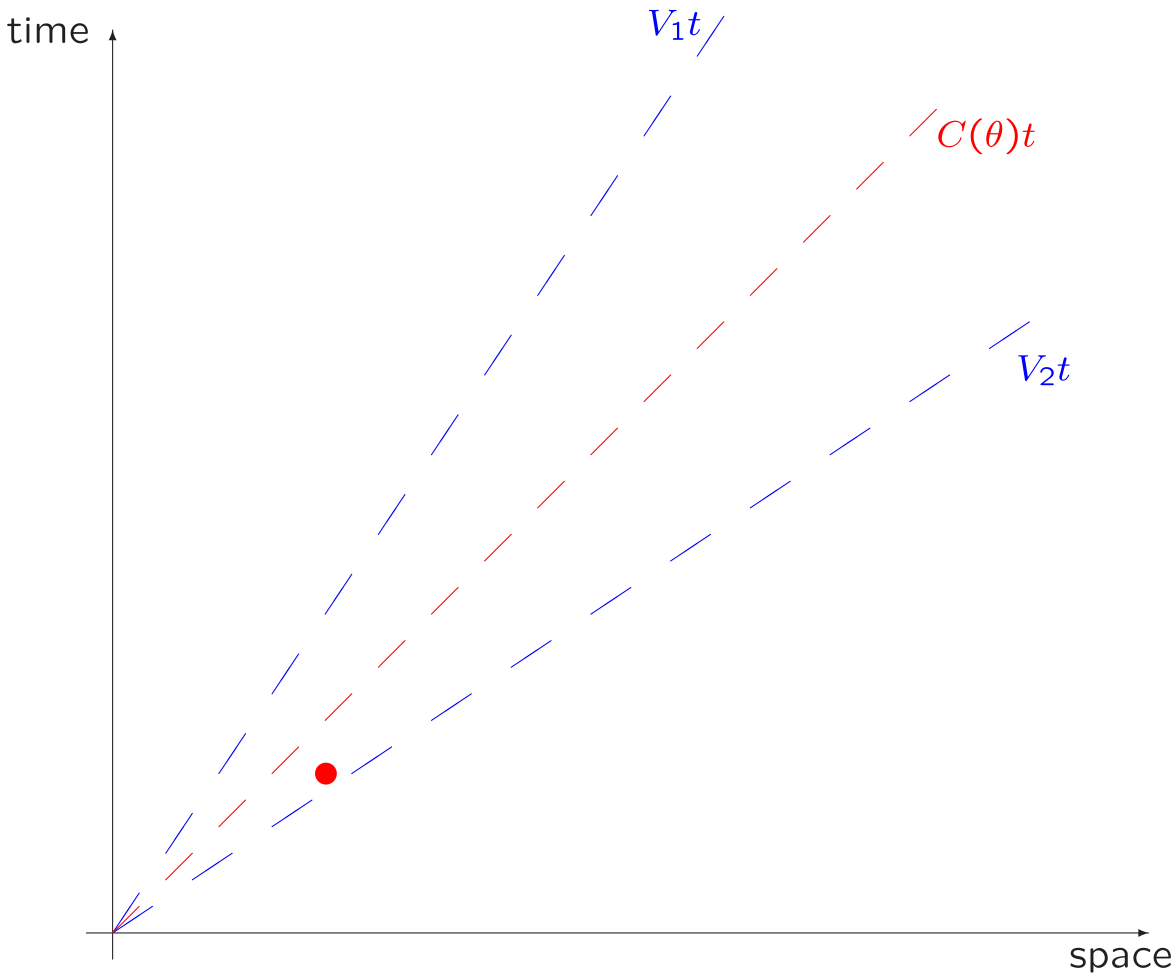

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$$\lim_{t\to\infty} \sum_{n=V_1 t}^{V_2 t} \cdot \frac{n}{t} \cdot \ \mathbf{Cov}(\omega_n(t), \omega_0(0))$$
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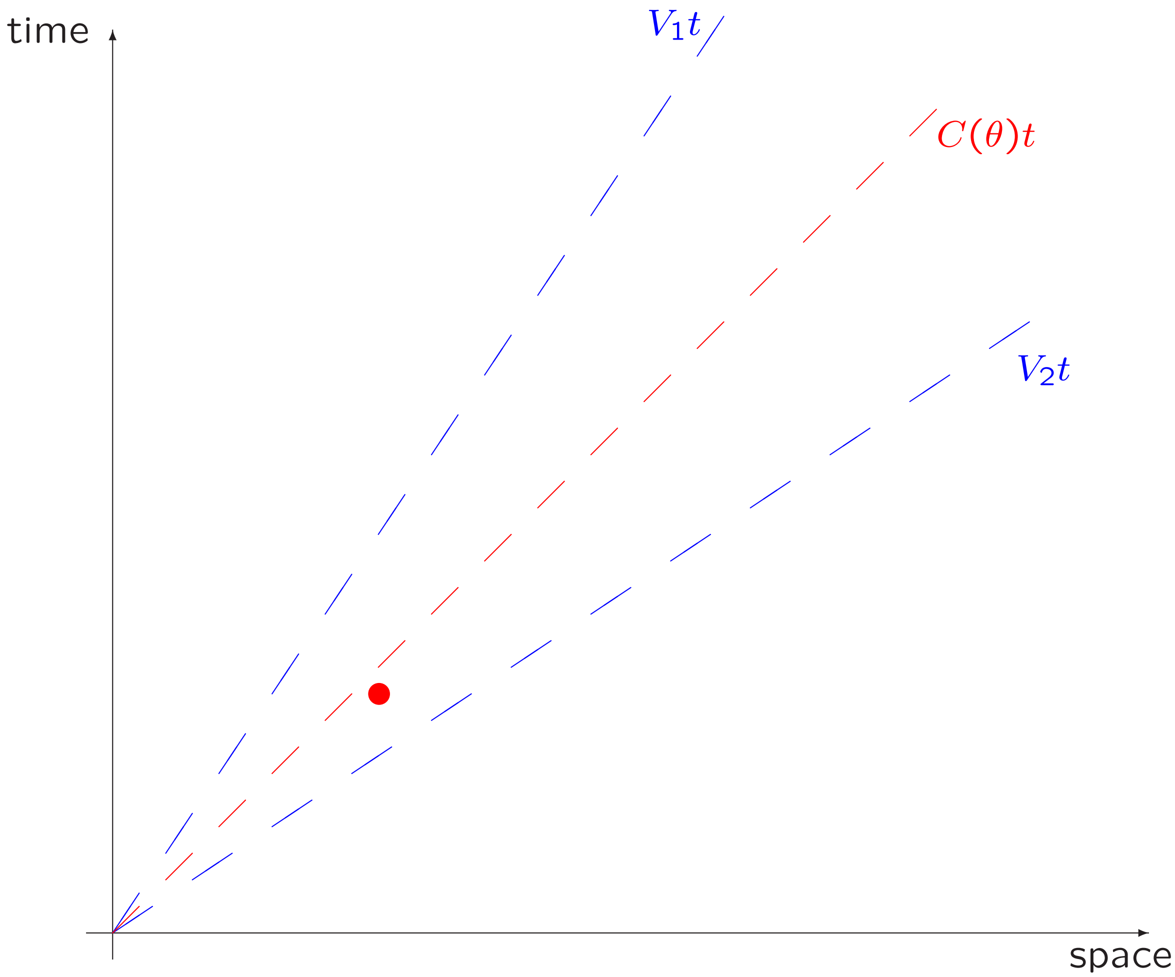

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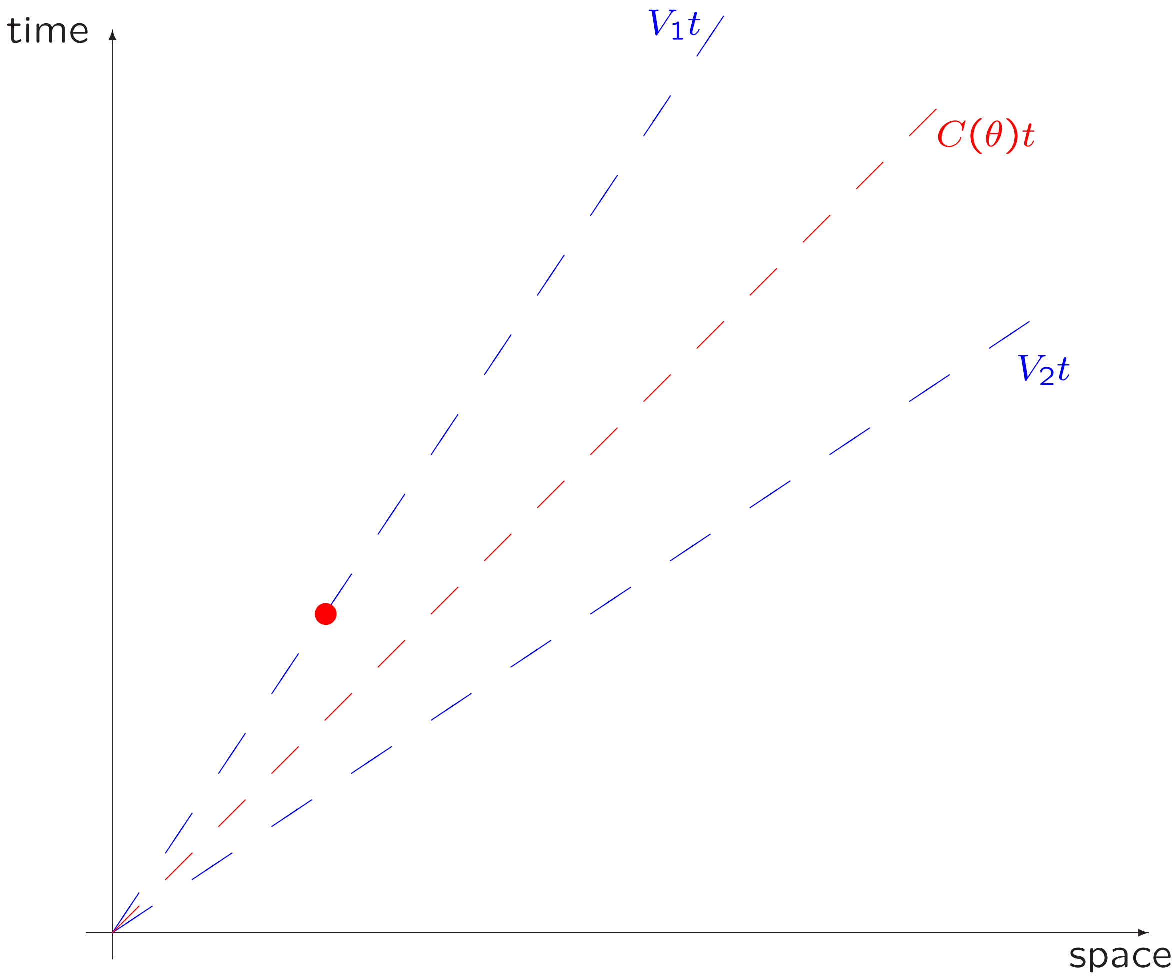

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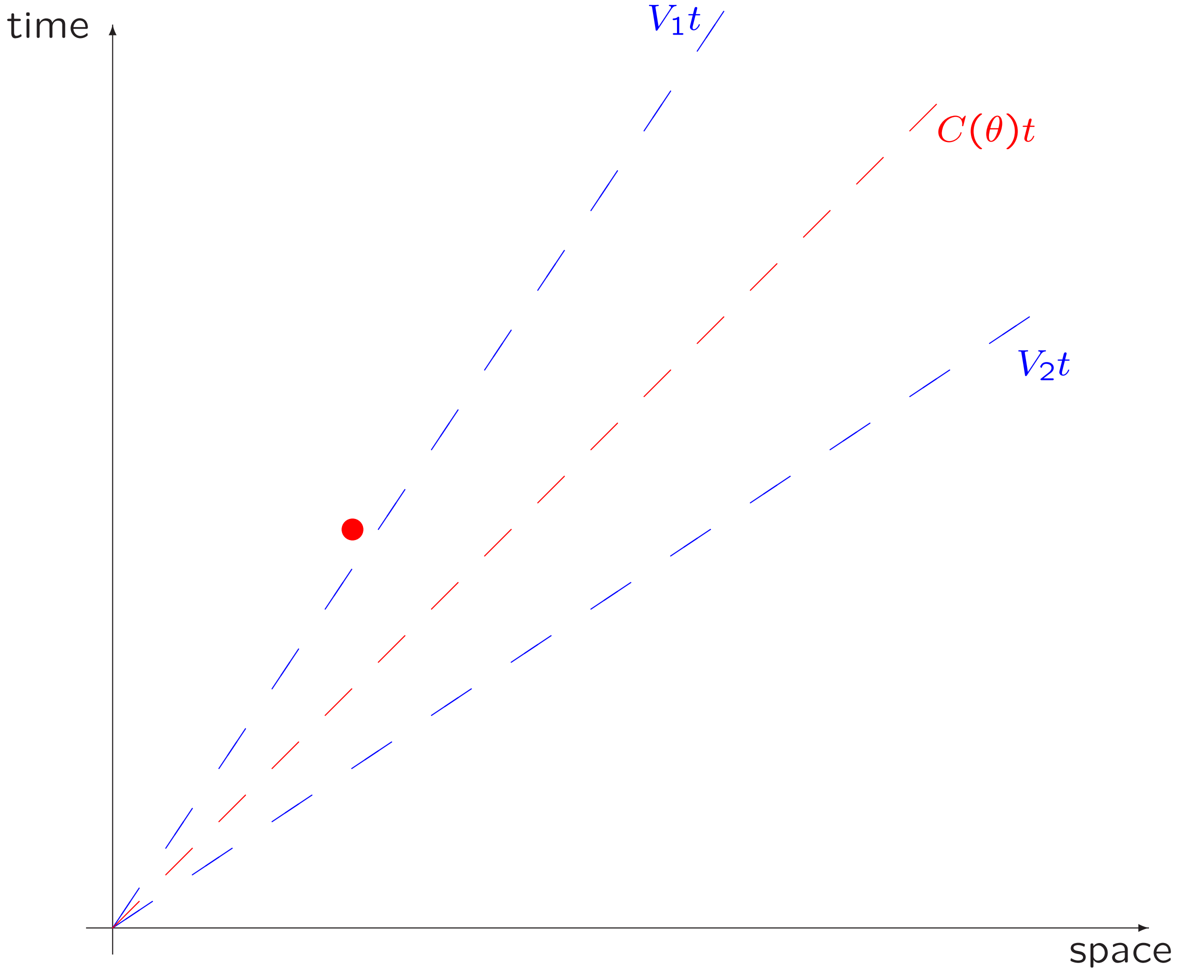

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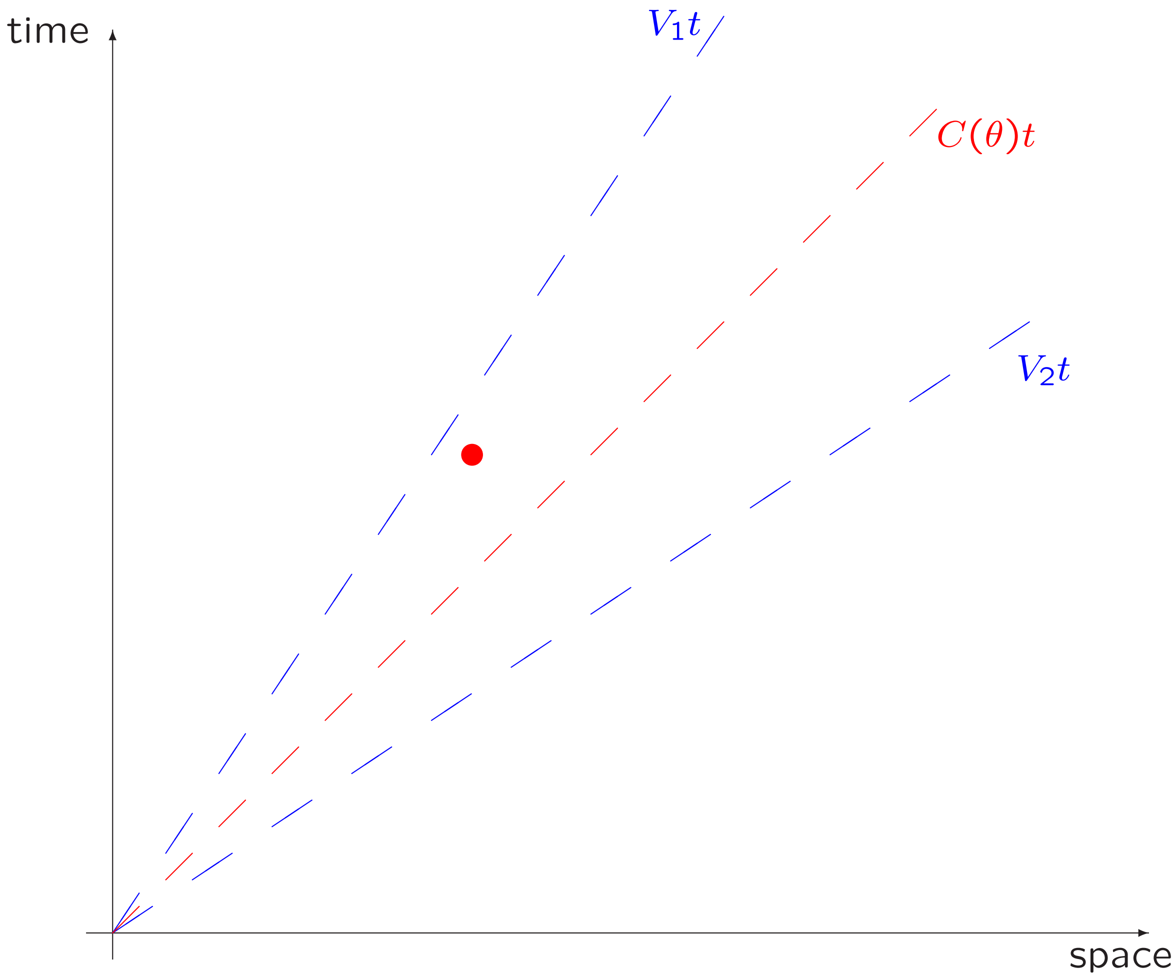

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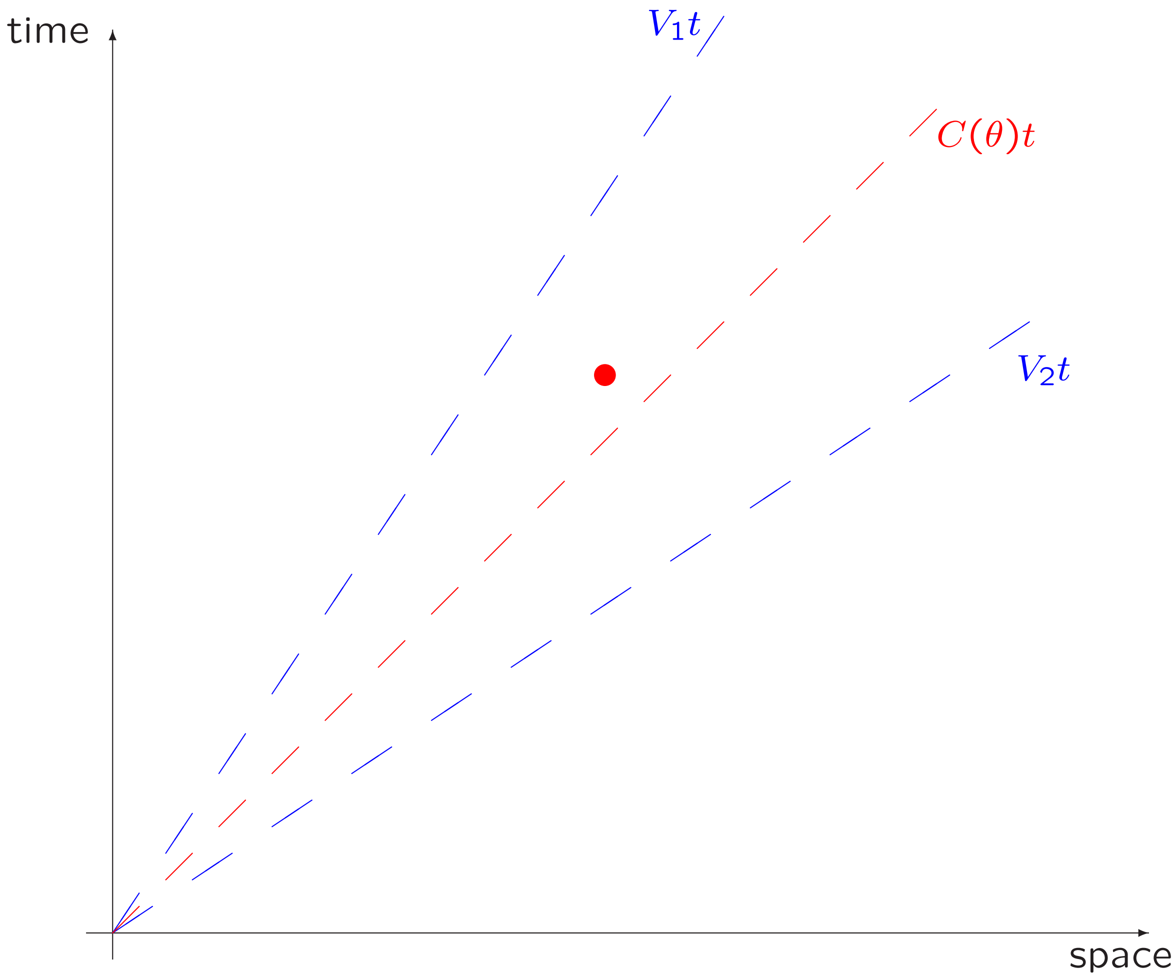

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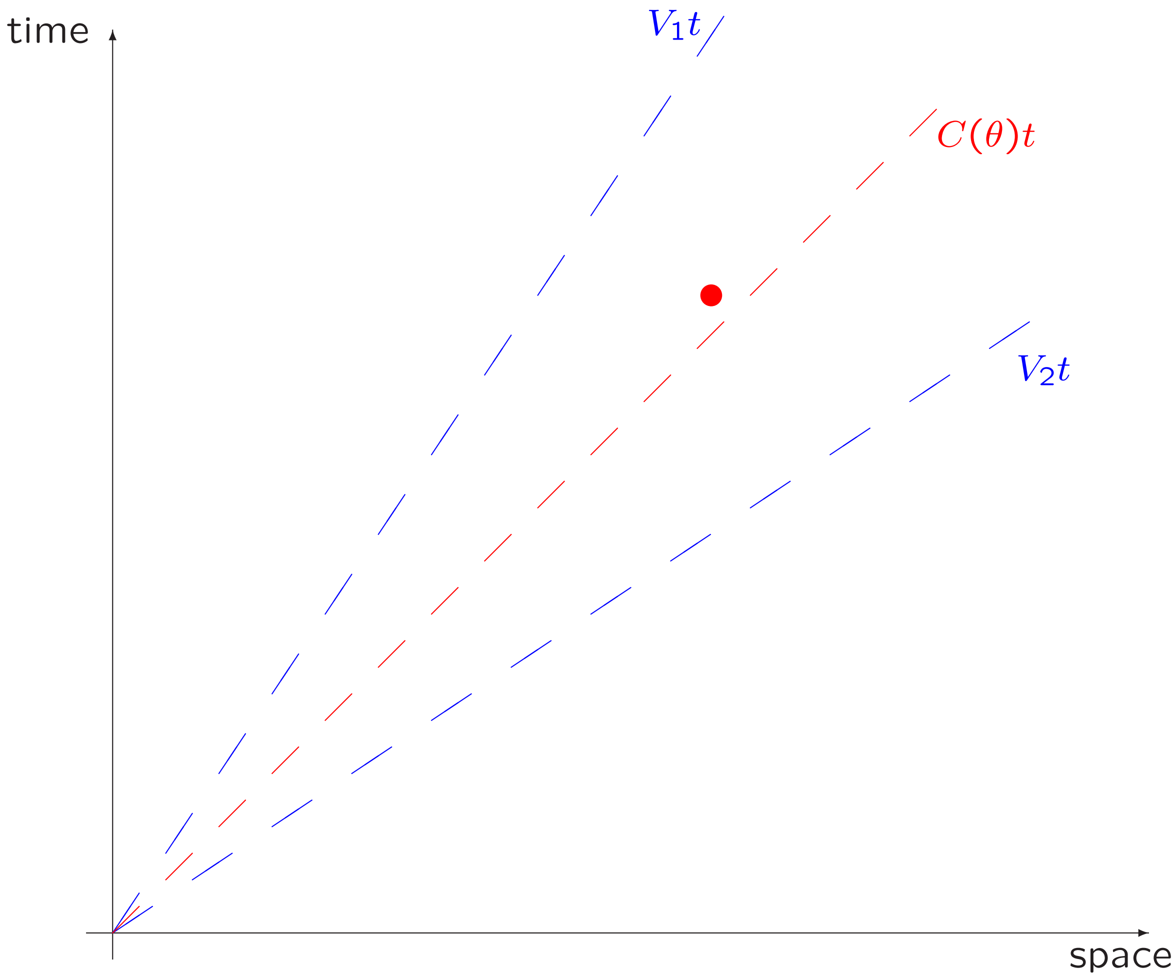

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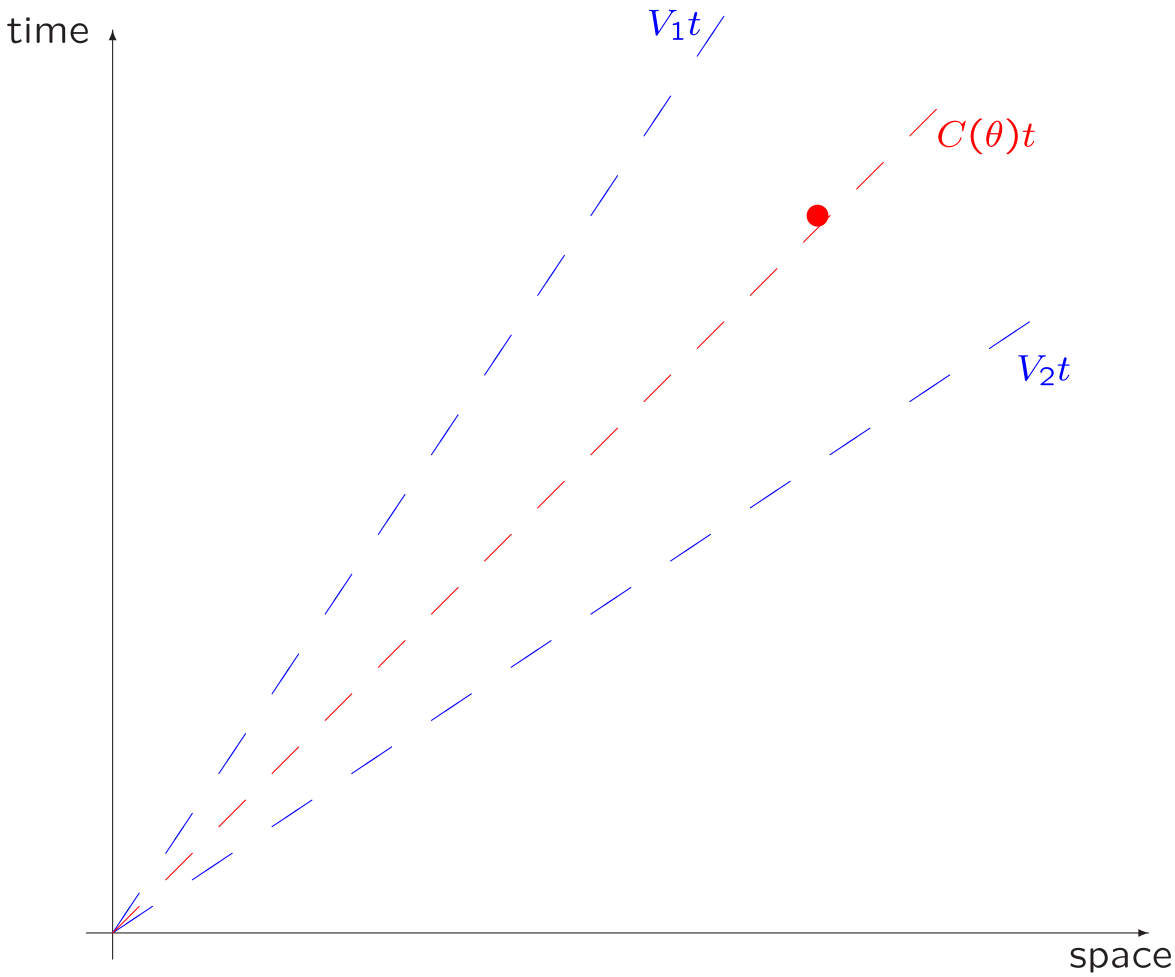

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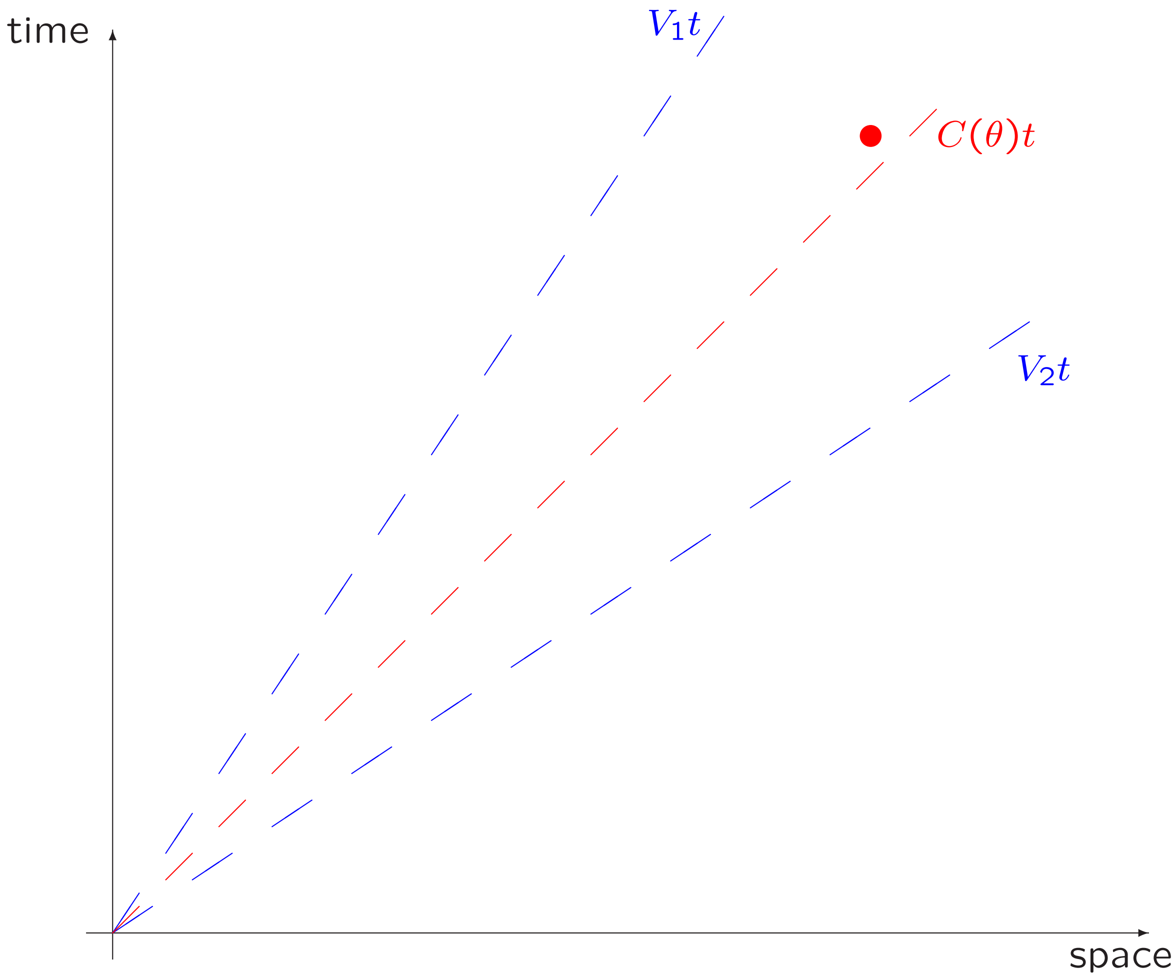

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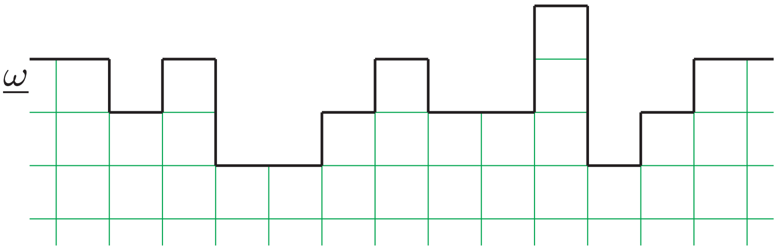
$\rightsquigarrow$ We used that the second class particle has the characteristic speed of the hydrodynamics.

$\longrightarrow$ We need to prove this, i.e. LLN for the second class particle.

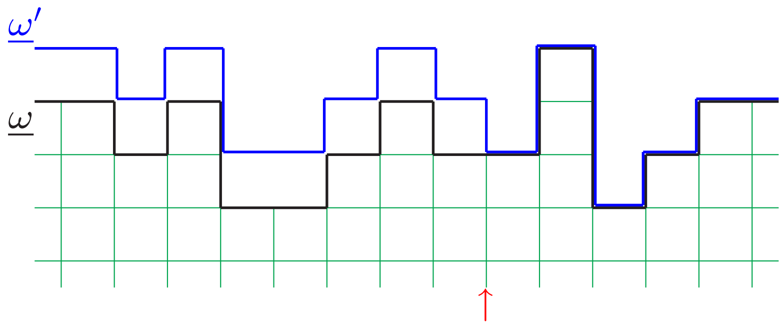
$\rightsquigarrow$ Once it's done, we see that the second class particle transports disturbances both in the microscopic and the hydrodynamic picture.

6. The speed of the second class particle



$\underline{\omega}$ is in equilibrium (θ).

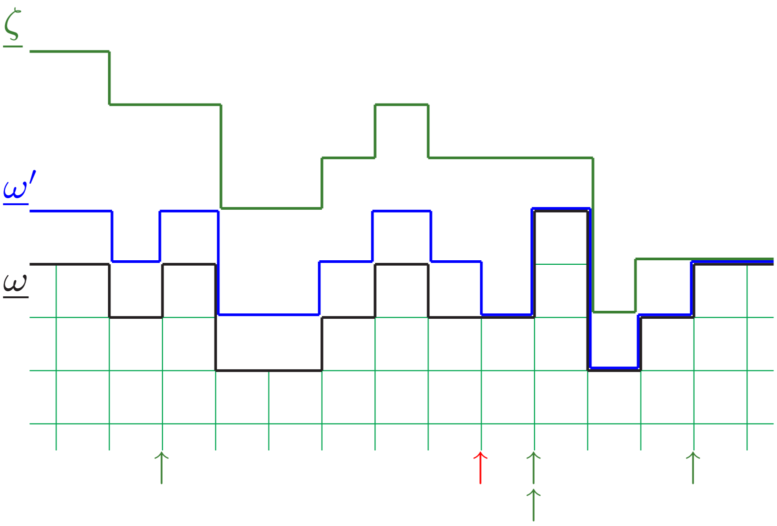
6. The speed of the second class particle



$\underline{\omega}$ is in equilibrium (θ).

$\omega'_i = \omega_i + \mathbf{1}\{Q = i\}$, not in equilibrium,
one single second class particle ↑.

6. The speed of the second class particle



$\underline{\omega}$ is in equilibrium (θ).

$\omega'_i = \omega_i + \mathbf{1}\{Q = i\}$, not in equilibrium,
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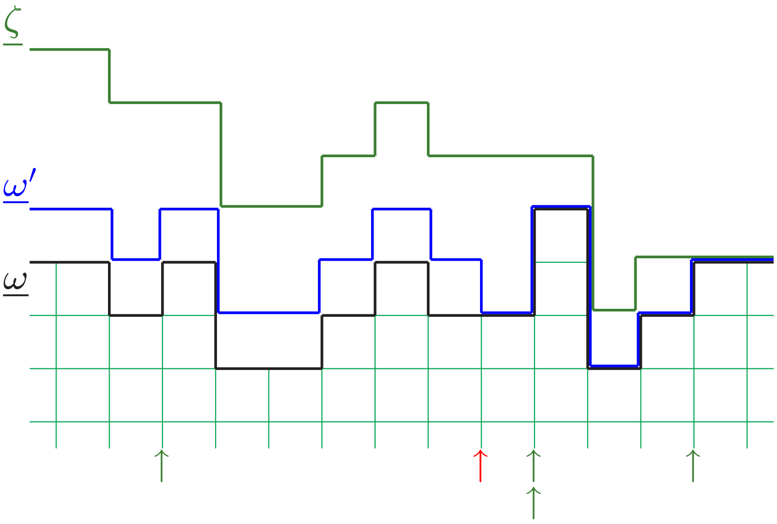

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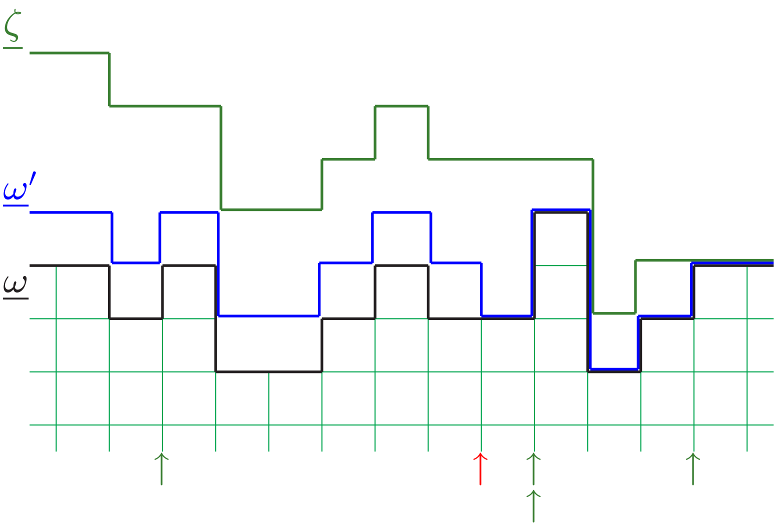
a density of second class particles $\uparrow$.

$\rightsquigarrow$ *Initially*, the $\uparrow$'s are product-distributed.
Not at later times, the stationary distribution for TASE with second class particles was discovered by Derrida, Janowsky, Lebowitz, Speer 1993.



$\underline{\omega}$ is in equilibrium (θ).

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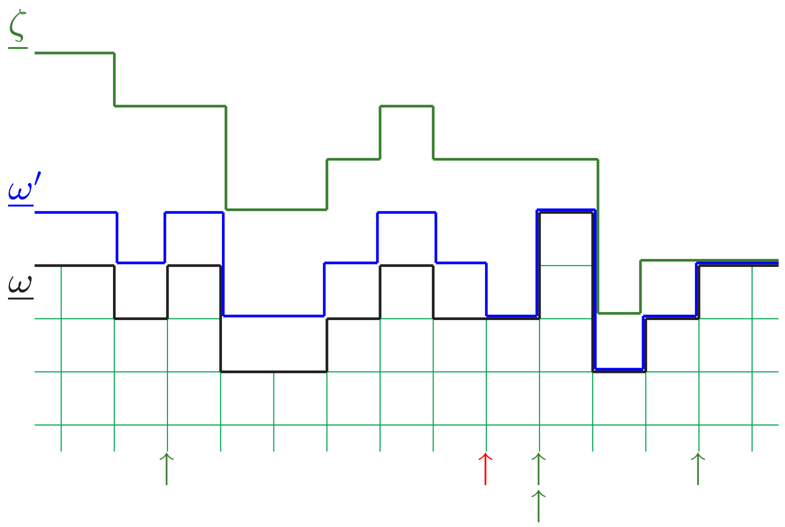
$\rightsquigarrow$ LLN for their columns' growth, and thus for the current of the $\uparrow$'s. (Current of second class particles $\sim$ difference of columns' growth.)



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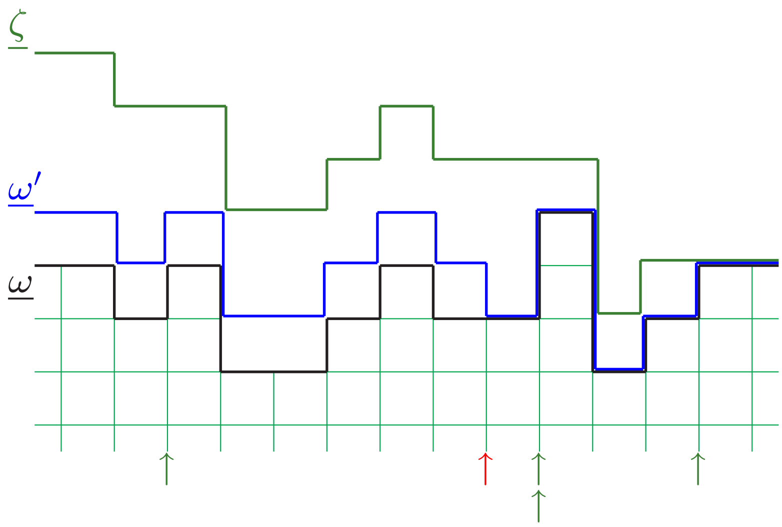
$\rightsquigarrow$ We know much about the $\uparrow$'s. Let's compare the $\uparrow$'s with the single $\uparrow$.



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Initially, the $\uparrow$'s are product-distributed.

$\rightsquigarrow$ We know much about the $\uparrow$'s. Let's compare the $\uparrow$'s with the single $\uparrow$.

It works fine.

Until a point.

Until a point when more $\uparrow$'s meet with $\uparrow$:

$$\uparrow$$

$$\begin{array}{c}\uparrow\\ \uparrow\\ \uparrow\\ \uparrow\end{array}$$

Until a point when more $\uparrow$'s meet with $\uparrow$:

$\uparrow$

$\uparrow$
$\uparrow$
$\uparrow$
$\uparrow$

Which one to couple $\uparrow$ to?

Until a point when more $\uparrow$'s meet with $\uparrow$:

$\uparrow$

$\uparrow$6.
$\uparrow$5.
$\uparrow$4.
$\uparrow$3.

Which one to couple $\uparrow$ to?

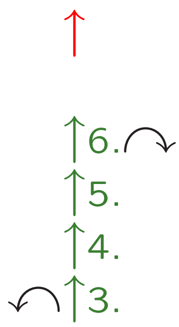
Let's label the $\uparrow$'s in order.

Until a point when more $\uparrow$'s meet with $\uparrow$:

Which one to couple $\uparrow$ to?

Let's label the $\uparrow$'s in order. Then
3. has a (too) large rate to jump to the left,

Until a point when more $\uparrow$'s meet with $\uparrow$:



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Let's label the $\uparrow$'s in order. Then
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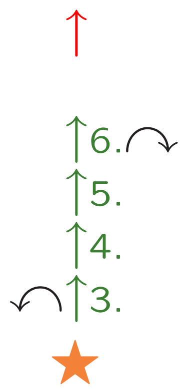
Until a point when more ↑'s meet with ↑:



Which one to couple ↑ to?

Let's label the ↑'s in order. Then
3. has a (too) large rate to jump to the left,
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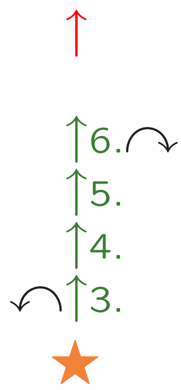
Introduce the ★ particle. With probability 1/4, it follows the ↑ to jump.

Until a point when more $\uparrow$'s meet with $\uparrow$:



With prob. $1/4$

Which one to couple $\uparrow$ to?

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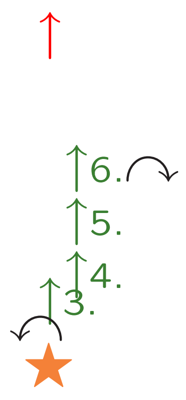
Introduce the $\bigstar$ particle. With probability $1/4$, it follows the $\uparrow$ to jump.

Until a point when more ↑'s meet with ↑:



With prob. 1/4

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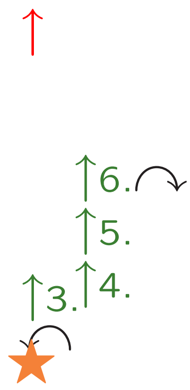
Introduce the ★ particle. With probability 1/4, it follows the ↑ to jump.

Until a point when more ↑'s meet with ↑:



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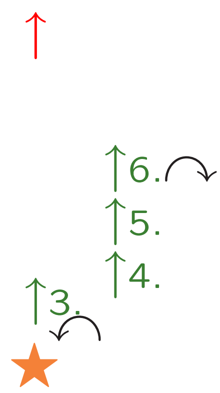
Until a point when more $\uparrow$'s meet with $\uparrow$:



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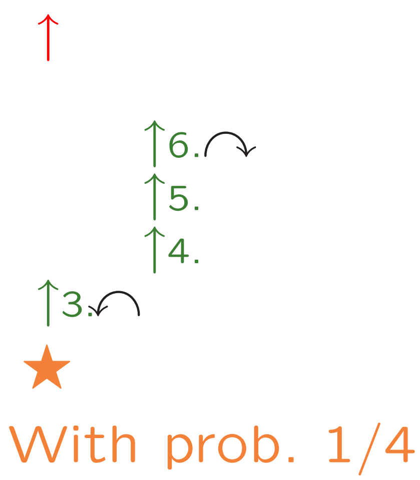

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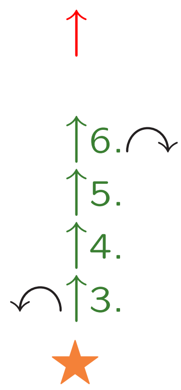

With prob. 3/4

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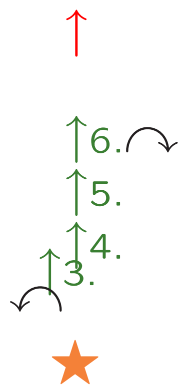
Introduce the ★ particle. With probability 1/4, it follows the $\uparrow$ to jump.

Until a point when more ↑'s meet with ↑:



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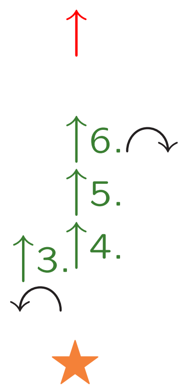

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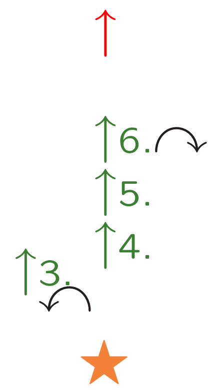

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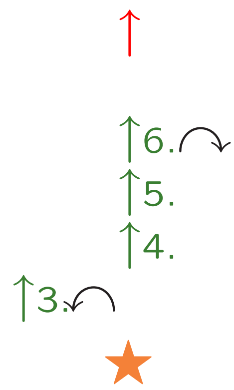

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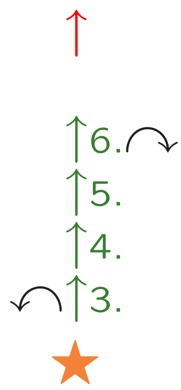


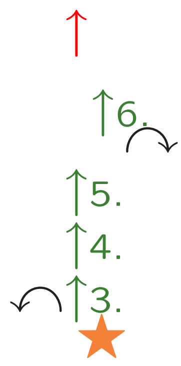
With prob. $1/4$

Which one to couple $\uparrow$ to?

Let's label the $\uparrow$'s in order. Then
3. has a (too) large rate to jump to the left,
6. has a (too) large rate to jump to the right.

Introduce the ★ particle. With probability $1/4$, it follows the $\uparrow$ to jump.

Until a point when more $\uparrow$'s meet with $\uparrow$:



With prob. 1/4

Which one to couple $\uparrow$ to?

Let's label the $\uparrow$'s in order. Then
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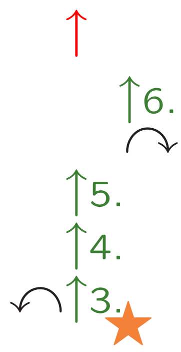
Introduce the ★ particle. With probability 1/4, it follows the $\uparrow$ to jump.

Until a point when more $\color{green}\uparrow$'s meet with $\color{red}\uparrow$:



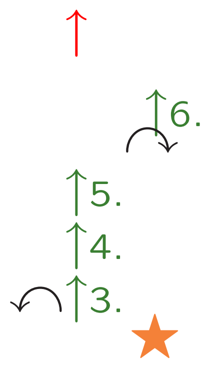
With prob. 1/4

Which one to couple $\color{red}\uparrow$ to?

Let's label the $\color{green}\uparrow$'s in order. Then
3. has a (too) large rate to jump to the left,
6. has a (too) large rate to jump to the right.

Introduce the ★ particle. With probability 1/4, it follows the $\color{green}\uparrow$ to jump.

Until a point when more ↑'s meet with ↑:



With prob. 1/4

Which one to couple ↑ to?

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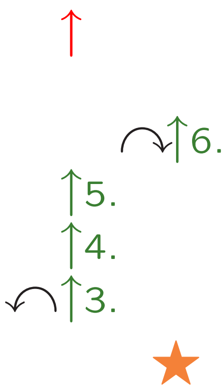
Introduce the ★ particle. With probability 1/4, it follows the ↑ to jump.

Until a point when more $\uparrow$'s meet with $\uparrow$:



With prob. 1/4

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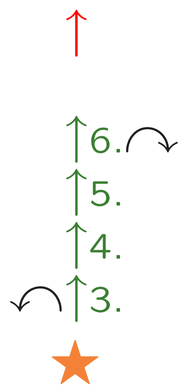
Until a point when more $\uparrow$'s meet with $\uparrow$:



With prob. 3/4

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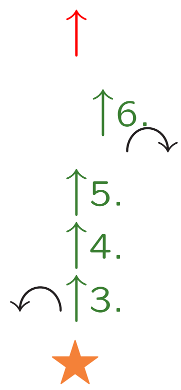
Until a point when more ↑'s meet with ↑:



With prob. 3/4

Which one to couple ↑ to?

Let's label the ↑'s in order. Then
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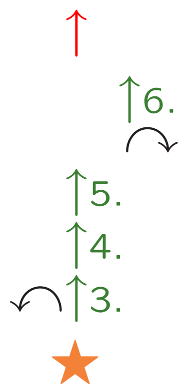
6. has a (too) large rate to jump to the right.

Introduce the ★ particle. With probability 1/4, it follows the ↑ to jump.

Until a point when more ↑'s meet with ↑:



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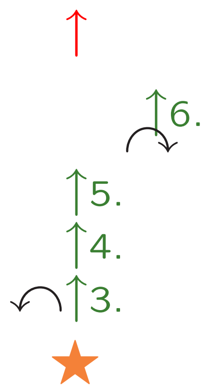
Introduce the ★ particle. With probability 1/4, it follows the ↑ to jump.

Until a point when more $\uparrow$'s meet with $\uparrow$:



With prob. 3/4

Which one to couple $\uparrow$ to?

Let's label the $\uparrow$'s in order. Then
3. has a (too) large rate to jump to the left,
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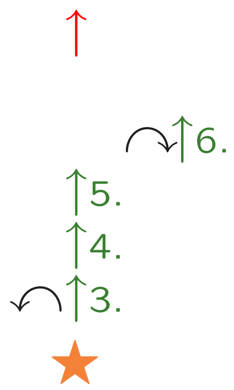

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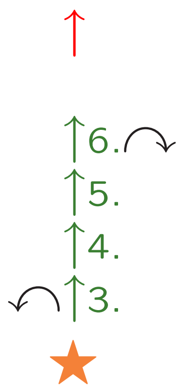

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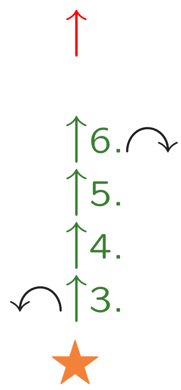

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Introduce the $\star$ particle. With probability 1/4, it follows the $\uparrow$ to jump.
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$\rightsquigarrow$ If the rate function $f(z)$ is convex, then $\uparrow$ is comparable to $\star$: $\uparrow$ is always to the left of $\star$. This $\star$ is nice enough to inherit LLN from the $\uparrow$'s.

7. A few words on hydrodynamics

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Imagine a local equilibrium with θ depending on some large-scale time and space parameters t, x. Then on this large scale $u = u(t, x)$, and

$$\partial_t u(t, x) + \partial_x H(u(t, x)) = 0 \qquad \left(\begin{array}{c}\text{conservation}\\ \text{law}\end{array}\right)$$

Rezakhanlou 1991, Tóth and Valkó 2002

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Is it convex for ZR or BL?

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So, $\dot{x}(t) = H'(u)$ is the *characteristic speed*.

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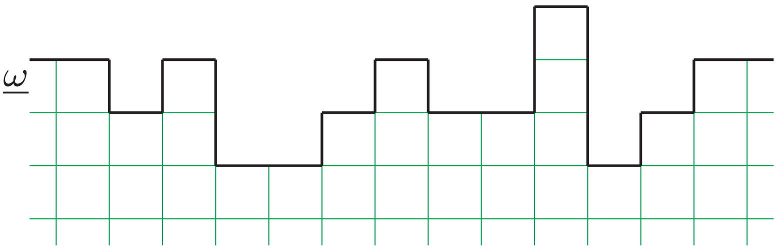
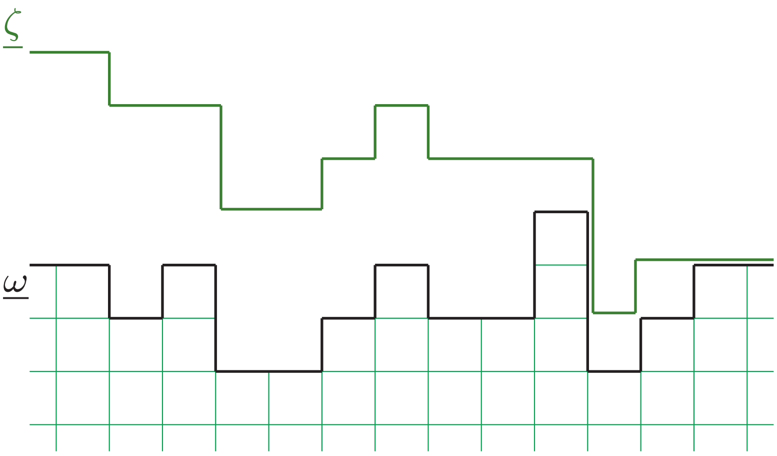
$\rightarrow$ $H(u)$ is convex, if $H'(u(\theta)) = C(\theta)$ is increasing (in either u or θ).

The way to check this is comparing second class particles.

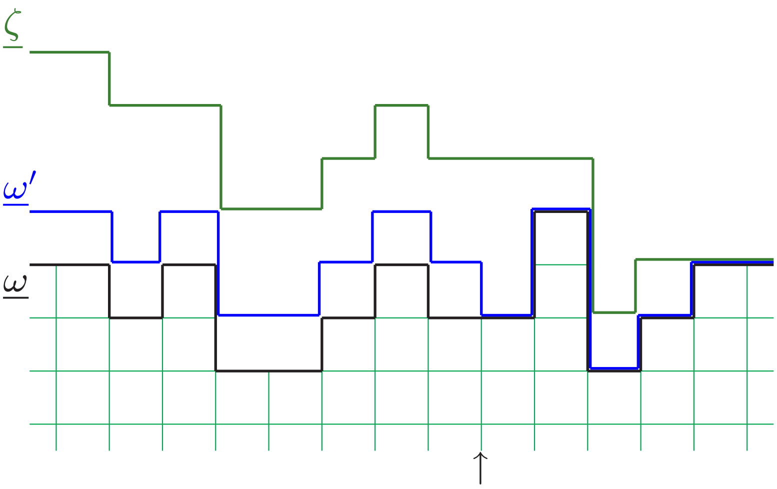


$\underline{\omega}$ is in equilibrium (θ).



$\underline{\omega}$ is in equilibrium (θ).

$\underline{\zeta}$ is in equilibrium $(\widetilde{\theta})$, with $\widetilde{\theta} > \theta$, $\zeta_i \geq \omega_i$.



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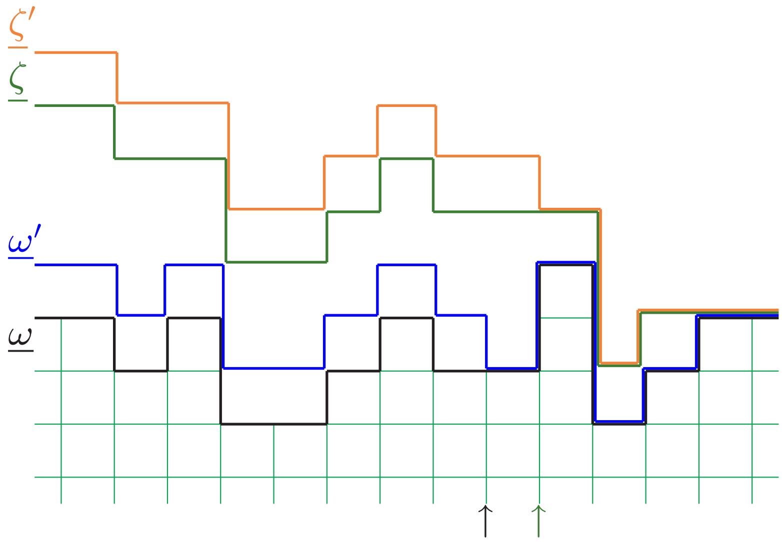

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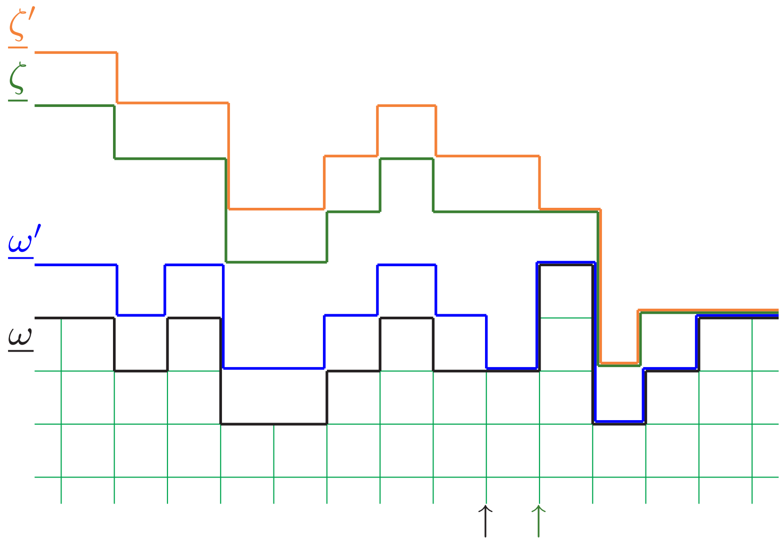

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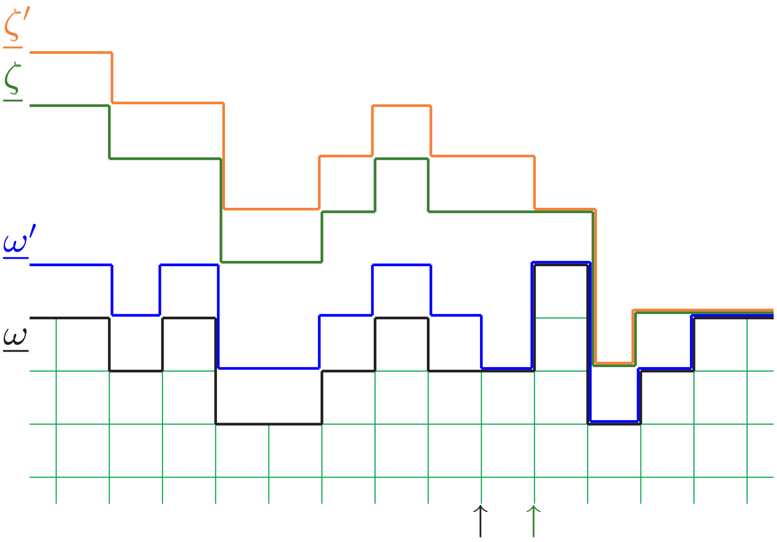

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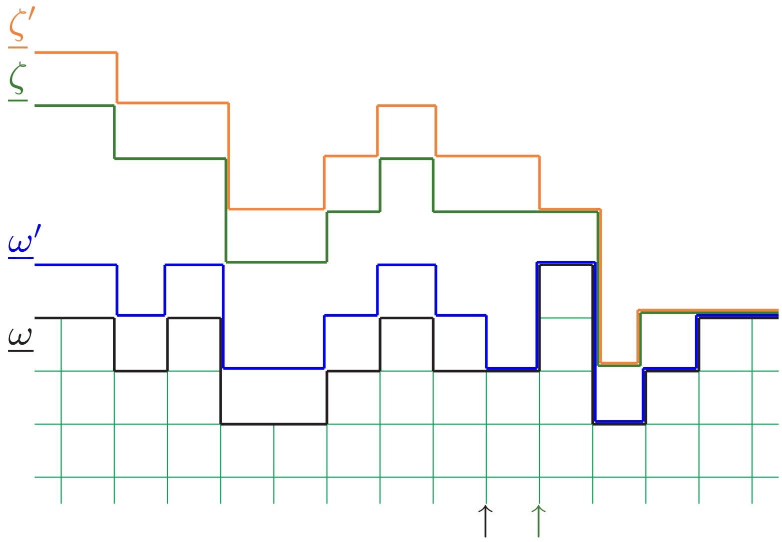

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Strict convexity also follows by analytic arguments.

Thank you.