Random Walking Shocks in Interacting Particle Systems

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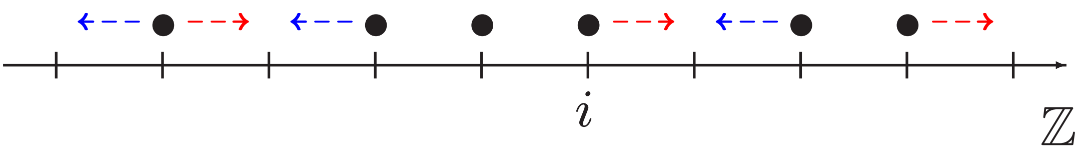
1. The Simple Exclusion Process (SE)

Consider the asymmetric simple exclusion process:



$$\eta_i = \begin{cases} 0, & \text{no particle at } i \\ 1, & \text{particle at } i, \end{cases}$$

$$\begin{aligned} (\eta_i,\, \eta_{i+1}) : (1,\, 0) &\dashrightarrow (0,\, 1) \text{ with rate } p > \tfrac{1}{2}, \\ (0,\, 1) &\dashrightarrow (1,\, 0) \text{ with rate } q = 1 - p. \end{aligned}$$

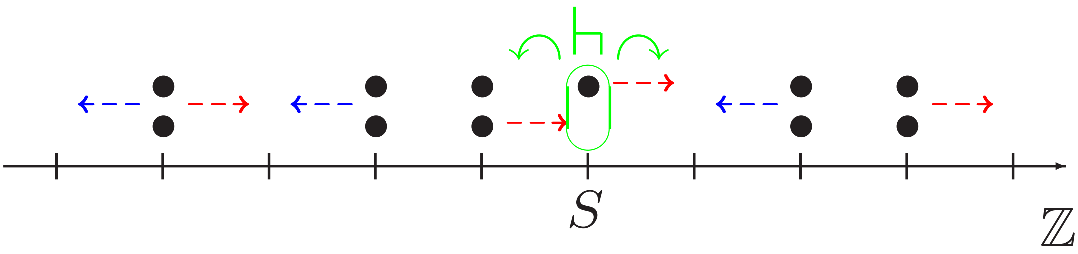
The Bernoulli-measure with $\mathbf{P}\{\eta_i = 1\} = \varrho$ is stationary for the process.

Hydrodynamic limit: $\partial_t \varrho + (p-q)\partial_x[\varrho(1-\varrho)] = 0$ (Inviscid Burgers) $\rightsquigarrow$ discontinuous shock solutions.

How do they look like at the level of particles?

2. Shock as seen by the second class particle Derrida, Lebowitz and Speer (1997)

Sit in the position of the *second class particle*:



Then the measure, stationary as seen from this position, having any $\varrho_l < \varrho_r$ left and right asymptotic densities, is found. (Matrix products)

Special case: If $\frac{p}{q} = \frac{\varrho_r(1-\varrho_l)}{\varrho_l(1-\varrho_r)}$, then this measure happens to be the Bernoulli-measure with respective parameters ϱ_l and ϱ_r on the left- and right-hand sides of the second class particle.

Questions:

- How does this measure look like from outside? (The second class particle is a complicated object.)
- How can we describe the case of multiple shocks? Ferrari, Fontes, Vares (2000)

3. Diffusion and scattering of shocks

Belitsky and Schütz (2002)

One shock: Given a vector $\underline{\varrho}$, define $\underline{\nu}$ to be the Bernoulli-measure with parameter ϱ_i at site i:

$$\underline{\nu}\{\eta_i = 1\} = \varrho_i \qquad (i \in \mathbb{Z}).$$

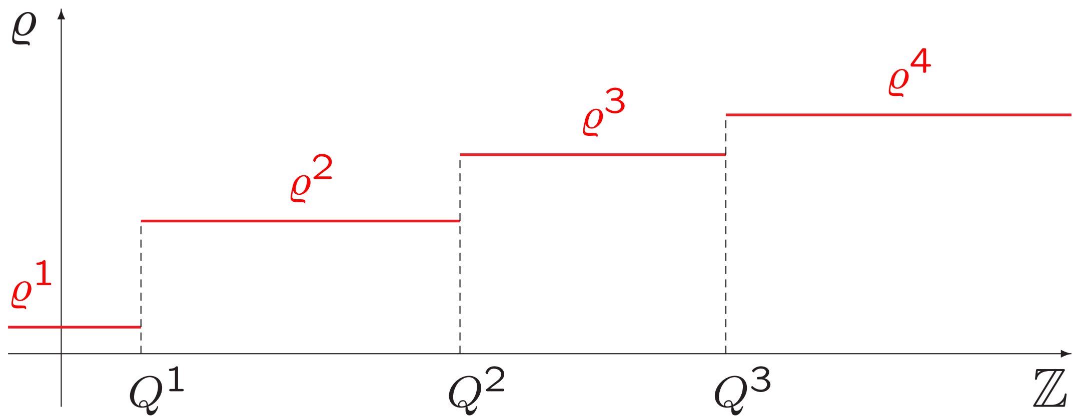
With two densities $\varrho_l < \varrho_r$, let the vector $\underline{\varrho}^{(Q)}$ have components ϱ_l left to and at the site Q, and ϱ_r right to Q. Then $\underline{\nu}^{(Q)}$ is a shock-measure.

Assume $\frac{p}{q} = \frac{\varrho_r(1-\varrho_l)}{\varrho_l(1-\varrho_r)}$ (Familiar?). Then at a later time t,

$$\underline{\nu}(t) = \sum_{i=-\infty}^{\infty} p_t(i|Q)\underline{\nu}^{(i)},$$

where $p_t(i|Q)$ is the transition probability of a continuous-time SRW from Q to i in time t. The jump rates of this SRW are $p\frac{\varrho_l}{\varrho_r}$ to the left, and $q\frac{\varrho_r}{\varrho_l}$ to the right. Method: quantum algebra.

Multiple shocks: Let us now define $\underline{\varrho}^{(Q^1,\dots,Q^n)}$ as the density vector having components ϱ^1 left to site Q^1, ϱ^2 between sites Q^1 and Q^2, etc., ϱ^{n+1} right to Q^n. For increasing values of $\varrho^1, \dots, \varrho^{n+1}$, $\underline{\nu}$ is then a multiple shock-measure with shocks located at $Q^1, \dots, Q^n$.



If the density values satisfy $\frac{p}{q} = \frac{\varrho^{m+1}(1-\varrho^m)}{\varrho^m(1-\varrho^{m+1})}$ (same condition), then the evolution of this multiple shock-measure can be described (in the above sense) as the SRW of the shocks, the m^{th} one starting from Q^m, having left jump rate $p\frac{\varrho^m}{\varrho^{m+1}}$ and right jump rate $q\frac{\varrho^{m+1}}{\varrho^m}$. The shocks interact by the exclusion rule.

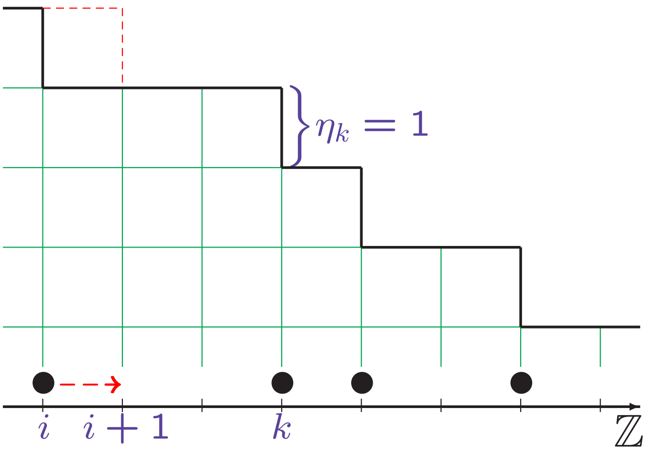
Remark: Left jump rates of shocks increase, right jump rates decrease as we go “shock by shock” from left to right. They stay in stochastically bounded distance to each other. $\rightsquigarrow$ They form one shock at any kind of spatial scaling.

Remark: These random walkers are *not* the second class particles.

Remark: The jump rates of a single random walker agree with the *expected* jump rates of the second class particle in the same shock-measure. The speed of our walker is therefore the one predicted macroscopically for the shock (Rankine-Hugoniot) (don’t know if checked for multiple walkers).

4. The bricklayers' process (BL) (Tóth)

Surface representation of the TASE:

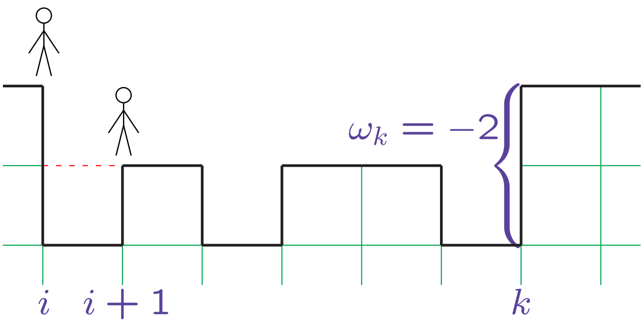


$$(\eta_i, \eta_{i+1}) \dashrightarrow (\eta_i - 1, \eta_{i+1} + 1)$$

$$\underline{\eta} \dashrightarrow \underline{\eta}^{(i,\, i+1)}$$

with rate $\eta_i(1 - \eta_{i+1})$

The bricklayers' process (BL):



$$(\omega_i, \omega_{i+1}) \dashrightarrow (\omega_i - 1, \omega_{i+1} + 1)$$

$$\underline{\omega} \dashrightarrow \underline{\omega}^{(i,\, i+1)}$$

with rate $r(\omega_i) + r(-\omega_{i+1})$

Infinitesimal generator:

$$(L\varphi)\,(\underline{\omega}) = \sum_i \left[r(\omega_i) + r(-\omega_{i+1})\right] \cdot \left[\varphi(\underline{\omega}^{(i,\, i+1)}) - \varphi(\underline{\omega})\right]$$

$\longrightarrow$ The rate function r is monotone increasing.
$\rightsquigarrow$ The process is *attractive*, second class particles can be introduced.

$\longrightarrow$ For all $z \in \mathbb{Z}$, $r(z) \cdot r(1-z) = \text{const.}$
$\rightsquigarrow$ The product measure $\underline{\mu}$ with marginals

$$\mu(z) = \frac{1}{Z(\theta)} \cdot \frac{\mathrm{e}^{\theta z}}{r(z)!} \qquad (z \in \mathbb{Z})$$

is stationary for the process, where $\theta \in \mathbb{R}$ is a parameter setting the average slope of the wall, and $r(z)!$ is defined as a product of $r(1)$, $r(2), \ldots, r(z)$ (extended naturally to $z \leq 0$). If the constant is zero $\longrightarrow$ zero range (ZR). If it's one $\longrightarrow$ bricklayers' (BL).

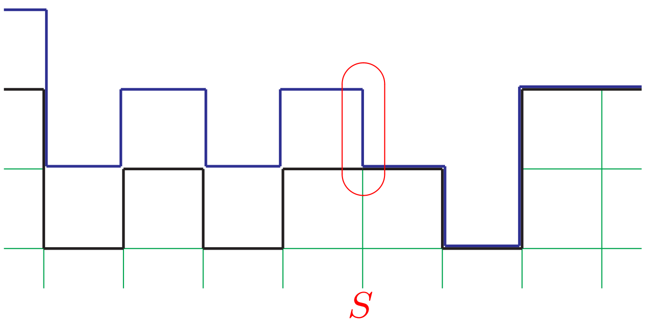
The hydrodynamic limit is of Burgers-type, with decreasing shocks (if $r(z)$ is convex).

Example: The exponential bricklayers' process (EBL) has rate function

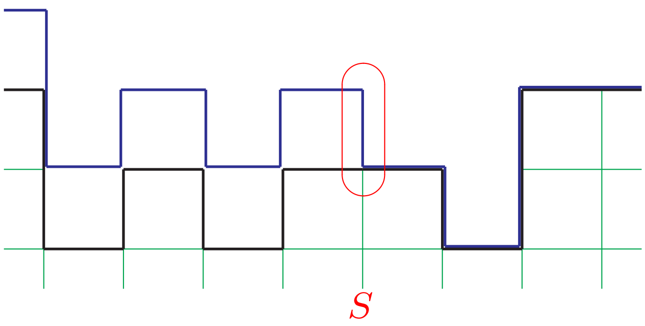
$$r(z) = \mathrm{e}^{-\frac{\beta}{2}} \cdot \mathrm{e}^{\beta z}.$$

with a real parameter $\beta > 0$.

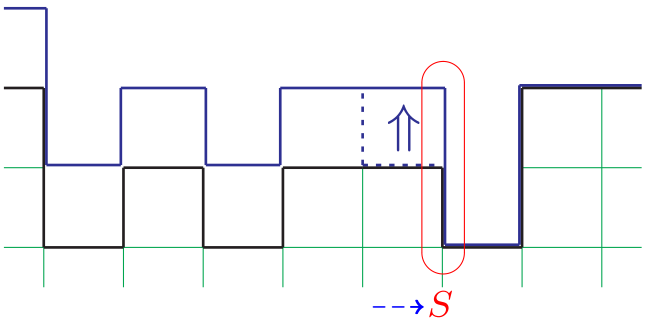
The second class particle



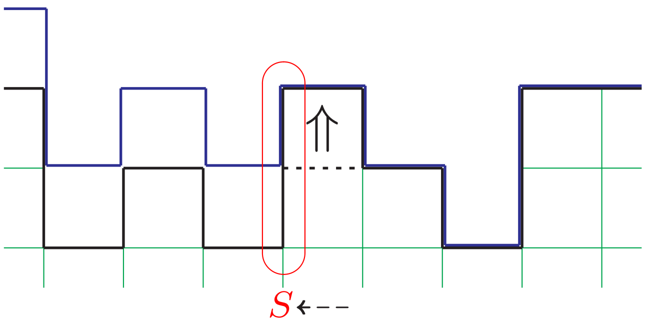
The two configurations only differ by one at site S.



The second class particle



The two configurations only differ by one at site S.



5. Shock as seen by the second class particle (B. 2001)

As before, consider a parameter-vector $\underline{\theta}$, and the product measure $\underline{\mu}$ with these parameters:

$$\mu\{\omega_i = z\} = \frac{1}{Z(\theta_i)} \cdot \frac{\mathrm{e}^{\theta_i z}}{r(z)!} \qquad (z \in \mathbb{Z})$$

Theorem: $\underline{\mu}$ is stationary for the process as seen by the second class particle, if and only if

$\longrightarrow$ The model is the *exponential* one (EBL), i.e. $r(z) = \mathrm{e}^{-\frac{\beta}{2}} \cdot \mathrm{e}^{\beta z}$, and

$\longrightarrow$ The parameters $\underline{\theta}$ have value θ_l left to the second class particle, $\theta_r = \theta_l - \beta$ at and right to the position of the second class particle.

Remark: Then $\underline{\mu}$ is a shock product-measure, the shock having a jump of size one *precisely*.

Method: Direct computation.

6. Shock as seen from outside (B. 2004)

Given $\underline{\theta}$, the product measure $\underline{\mu}$ has parameters

$$\underline{\theta} = (\dots,\, \theta_{i-1},\, \theta_i,\, \theta_{i+1},\, \dots).$$

With a $\beta > 0$, define the product measure $\underline{\mu}^{(i,\,\pm)}$, having parameters

$$\underline{\theta}^{(i,\,\pm)} := (\dots,\, \theta_{i-1},\, \theta_i \pm \beta,\, \theta_{i+1},\, \dots).$$

Theorem: For the EBL model, and no other bricklayers' model, $\underline{\mu}$ evolves into a linear combination of similar measures, with different parameters. The evolution is described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\mu} = \sum_i \left[\mathrm{e}^{\theta_i} - \mathrm{e}^{\theta_{i+1}}\right] \cdot \left[\underline{\mu}^{(i+1,\,+)} - \underline{\mu}\right] + \\ + \sum_i \left[\mathrm{e}^{-\theta_i} - \mathrm{e}^{-\theta_{i-1}}\right] \cdot \left[\underline{\mu}^{(i-1,\,-)} - \underline{\mu}\right].$$

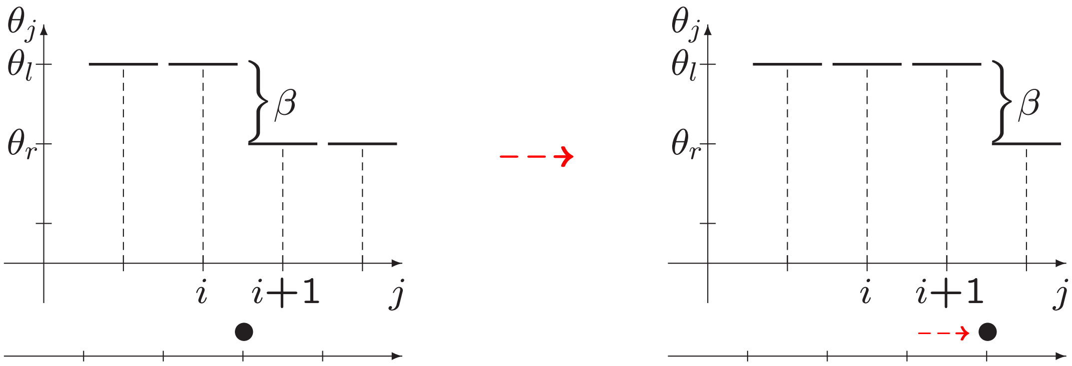
Remarks: $\underline{\theta}$ only changes

→ next to i where $\theta_i \neq \theta_{i+1}$,

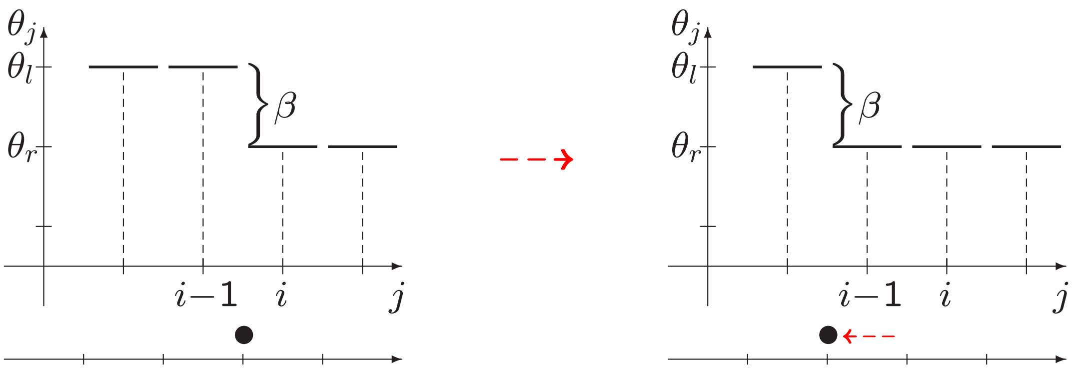
→ in multiples of β.

→ The eq. looks like a Kolmogorov-eq., provided the "rate part" is positive.

<u>One shock</u>:



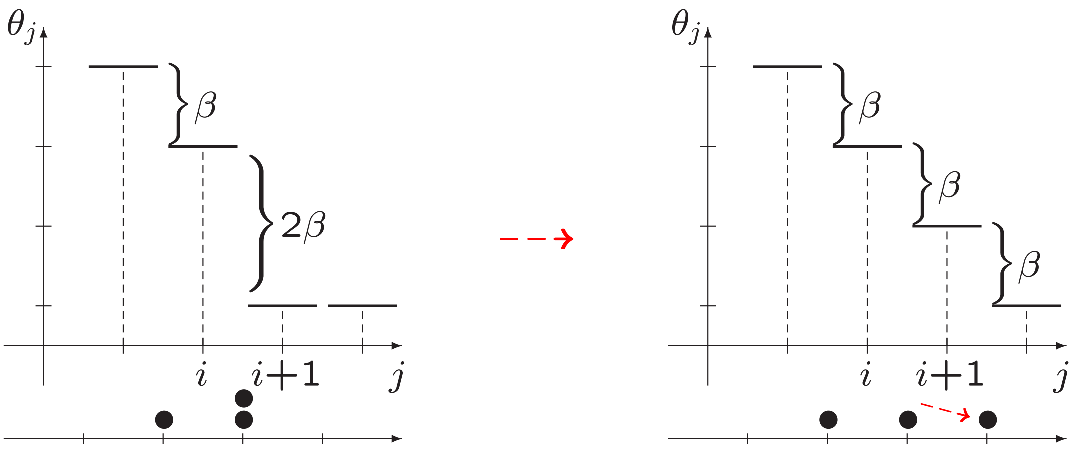
Move according to the first term



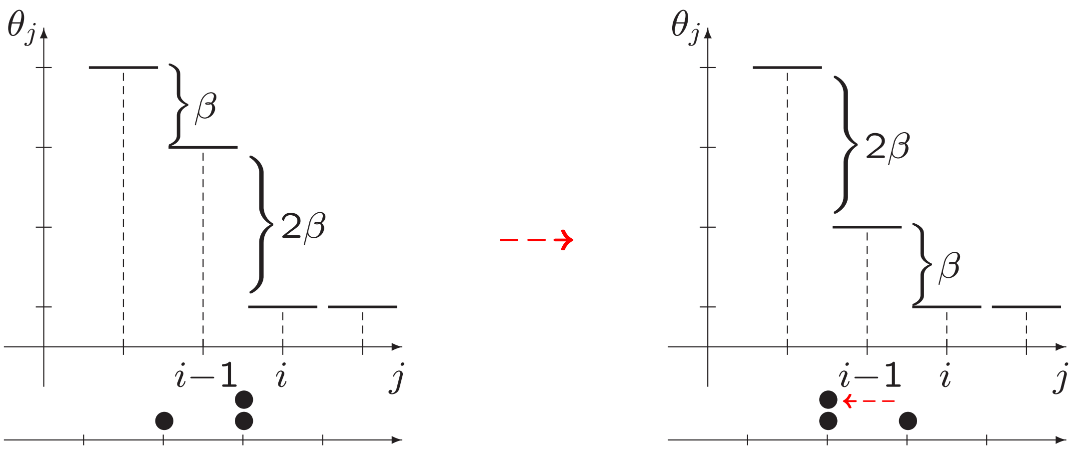
Move according to the second term

EBL model, shocks of jump-size one: precisely the same conditions as needed for the stationarity as seen by the second class particle.

Multiple shocks:



Move according to the first term



Move according to the second term

So, if θ_i as function of i is decreasing finitely many times and in multiples of β, then the corresponding multiple-shock measure describes finitely many shocks of integer jump sizes. In this case, we have the representation with random walking unit-sized shocks. These random walkers are *not* the second class particles.

Properties of the random walker shocks:

$\longrightarrow$ The Theorem and thus the random walking shocks representation only works for the EBL model. No other bricklayers' process and no zero range process has this property.

$\longrightarrow$ The jump rates of a single random walker agree with the *expected* jump rates of the second class particle in the same shock-measure. The speed of our walker is therefore the one predicted macroscopically for the shock (Rankine-Hugoniot).

$\longrightarrow$ A larger integer sized shock is only an accidental meeting of unit-sized ones. Larger shocks don't move, only unit-sized shocks separate from them on the microscopic level. (This is new compared to SE, where there was no space for more than one shock at the same position.)

$\longrightarrow$ Left jump rates increase, right jump rates decrease as we go "shock by shock" from left to right; the rates only depend on the order of the shocks. $\rightsquigarrow$ The shocks stay in stochastically bounded distance to each other, they form one large shock in any spatial scaling.

$\longrightarrow$ There is no interaction of shocks in a strict sense (no exclusion rule). They only attract each other due to their jump rates (see the previous point).

$\longrightarrow$ If we have n unit-sized shocks, then the average position

$$\frac{1}{n}\sum_{m=1}^{n} X_m$$

of the corresponding n walkers $X_1, \ldots, X_n$ performs a continuous-time SRW. It has a speed in accordance with the speed of the macroscopic shock of jump size n, formed by the n unit-sized shocks (Rankine-Hugoniot).

The Theorem holds, but the random walk representation fails whenever θ_i increases in i (unstable discontinuity). In this case, jump rates of our walkers become negative. This also happens eventually, if the initial $\underline{\theta}$ decreases other than in multiples of β: separating one-sized shocks from a non-integer sized one eventually leaves us with a "negative shock" i.e., increasing $\underline{\theta}$.

7. A few words on the proof of the Theorem

Given a configuration

$$\underline{\omega} = (\dots, \omega_{i-1}, \omega_i, \omega_{i+1}, \dots),$$

define

$$\underline{\omega}^{(i,\pm)} := (\dots, \omega_{i-1}, \omega_i \pm 1, \omega_{i+1}, \dots).$$

Step 1: Take a bounded cylinder function φ, a set of parameters $\underline{\theta}$, and the corresponding product-measure $\underline{\mu}$. Using the structure of $\underline{\mu}$, and some standard magic with change of variables and telescopic sums leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}^{(\underline{\theta})}\,\varphi(\underline{\omega}) &= \mathbf{E}^{(\underline{\theta})}\,(L\varphi)(\underline{\omega}) = \\ &= \sum_i \left[\mathrm{e}^{\theta_i} - \mathrm{e}^{\theta_{i+1}}\right] \cdot \left[\mathbf{E}^{(\underline{\theta})}\,\varphi(\underline{\omega}^{(i+1,+)}) - \mathbf{E}^{(\underline{\theta})}\,\varphi(\underline{\omega})\right] + \\ &\quad + \sum_i \left[\mathrm{e}^{-\theta_i} - \mathrm{e}^{-\theta_{i-1}}\right] \cdot \left[\mathbf{E}^{(\underline{\theta})}\,\varphi(\underline{\omega}^{(i-1,-)}) - \mathbf{E}^{(\underline{\theta})}\,\varphi(\underline{\omega})\right]\end{aligned}$$

for *all* bricklayers' processes. (A similar formula is even true for all nearest-neighbor zero range processes.)

The trick is to make $\underline{\theta}^{(i,\pm)}$ out of $\underline{\omega}^{(i,\pm)}$.

Step 2:

$$\mathbf{E}^{(\underline{\theta})}\,\varphi(\underline{\omega}^{(i,\,\pm)}) = \mathbf{E}^{(\theta^{(i,\,\pm)})}\varphi(\underline{\omega}) \quad \text{or, equivalently,}$$
$$\mu^{(\theta)}(z \mp 1) = \mu^{(\theta\pm\beta)}(z)$$

if and only if the model is the EBL model with parameter β. Other bricklayers' or zero range models do not follow the previous eq.

The reason is the form $\mu^{(\theta)}(z) = \frac{1}{Z(\theta)} \cdot \frac{\mathrm{e}^{\theta z}}{r(z)!}$ of the measures (valid for BL and ZR processes). Substituting it into the previous eq. immediately eliminates any non-exponential model. On the other hand, the exponential rates imply μ to be of discrete Gaussian type, having precisely the shifting-property required above.

8. Some open questions

→ How does the quantum-algebra behind the BL and ZR models look like? These models have locally infinite state space, in contrary to SE. Where is the special symmetry in the algebra of EBL?

→ What is the connection of our random walkers to the second class particles? The random walkers are defined without any particular state of the model, while the motion of the second class particle depends strongly on the model's configuration. How can we compare these two? Is it possible that the law of the second class particle, integrated out w.r.t. the initial shock-measure, agrees with the law of our simple random walker (and hence is Markovian)?

→ The EBL model deserves a rigorous construction. These kind of processes are constructed by Andjel (1982) *only* if the rate function has bounded increments. We have a regularity statement for attractive models with faster-growing rate functions. Unfortunately, no full construction yet.