Fluctuation estimates for last-passage percolation

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Back then University of Wisconsin - Madison)

Joint work with

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and

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(University of Wisconsin - Madison)

Oberwolfach, October 12, 2007

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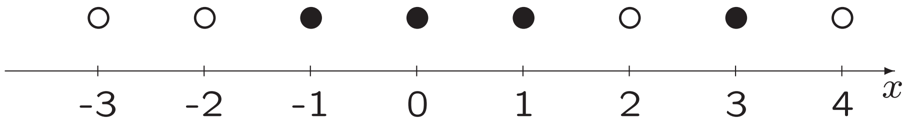

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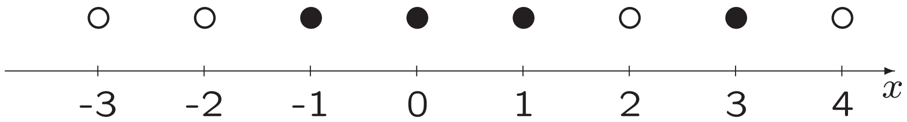

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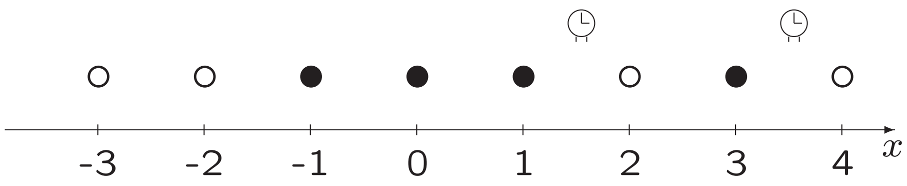

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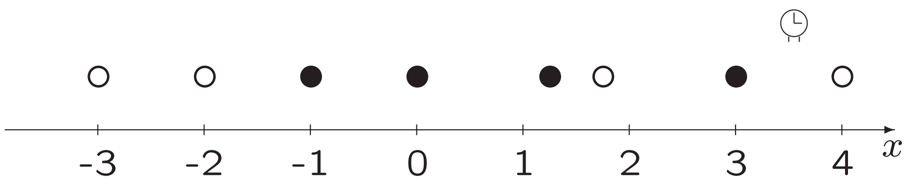

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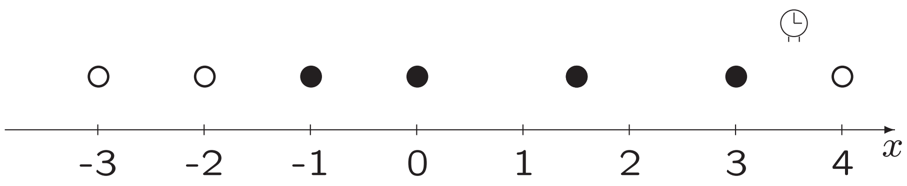

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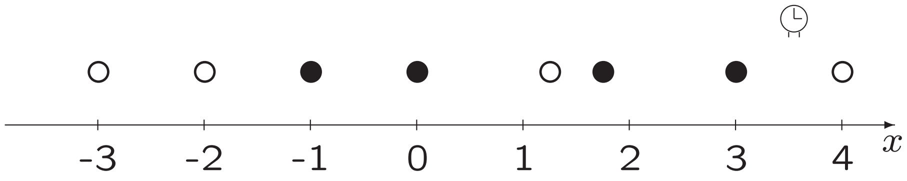

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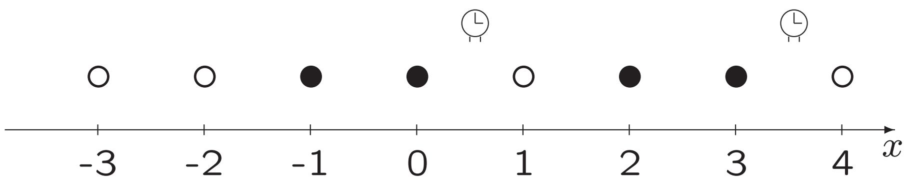

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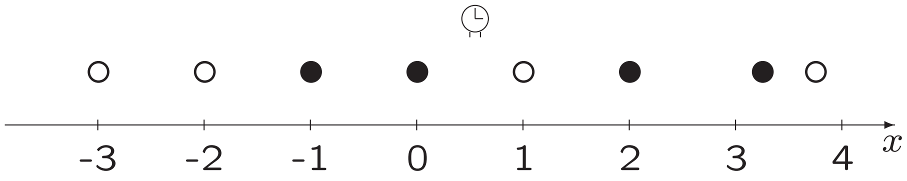

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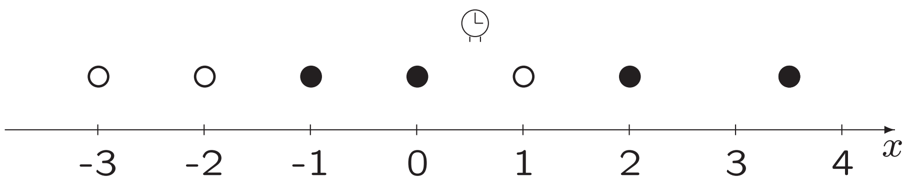

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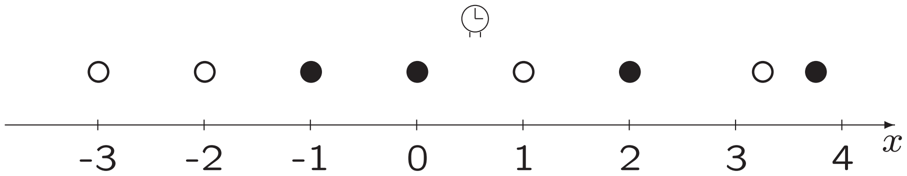

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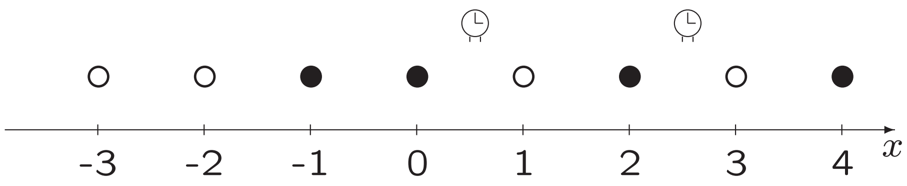

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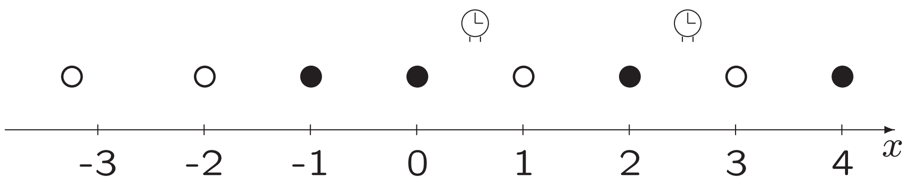

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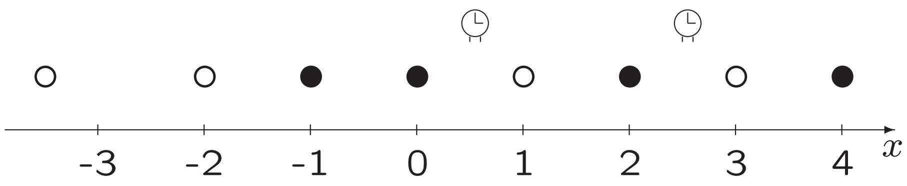

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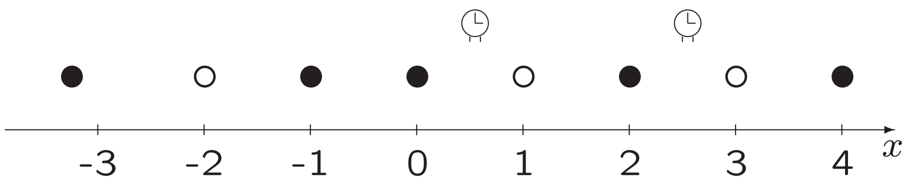

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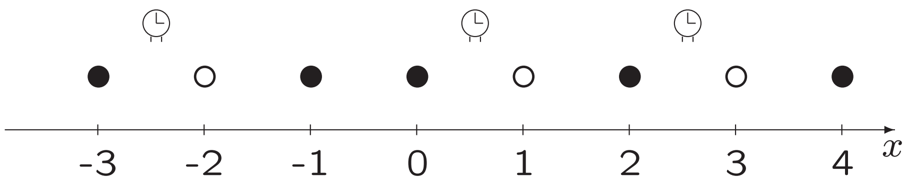

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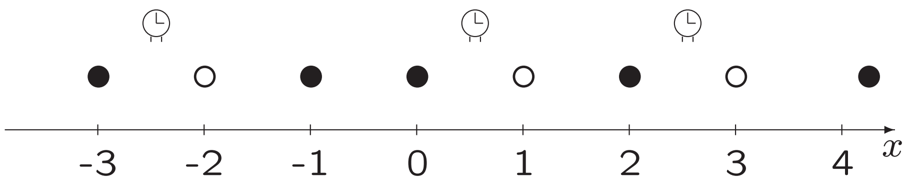

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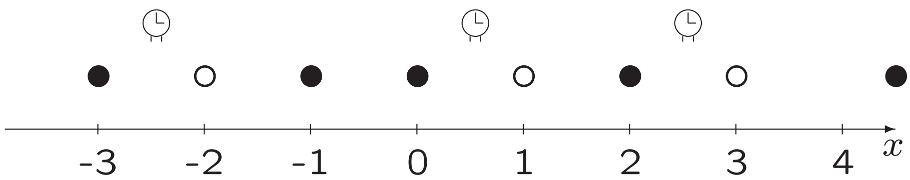

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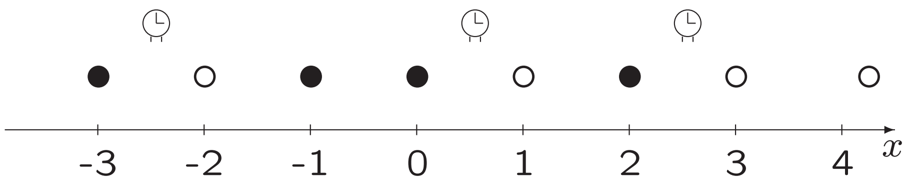

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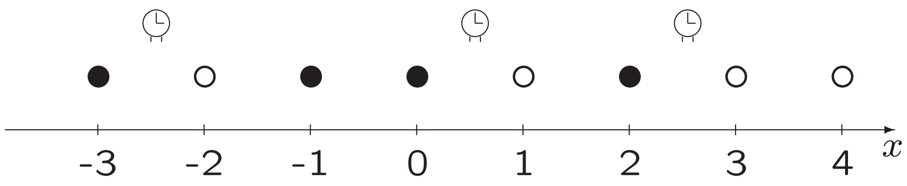

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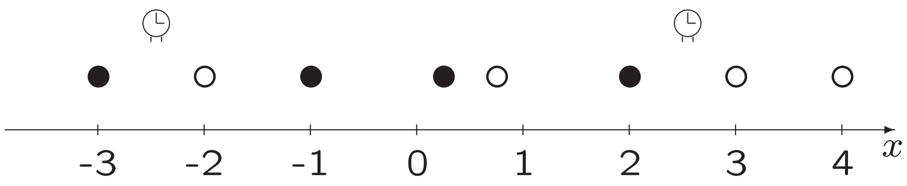

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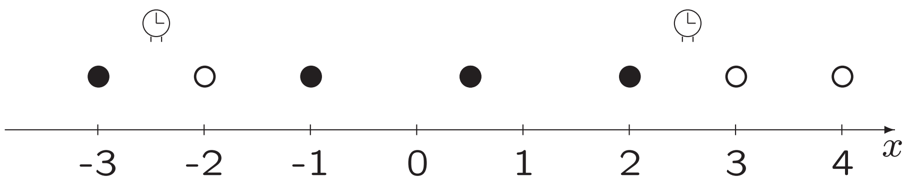

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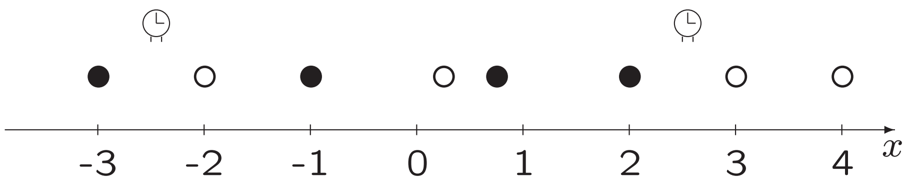

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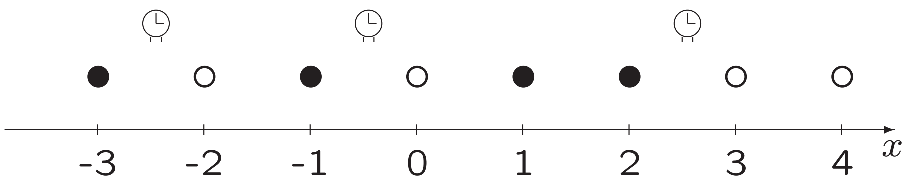

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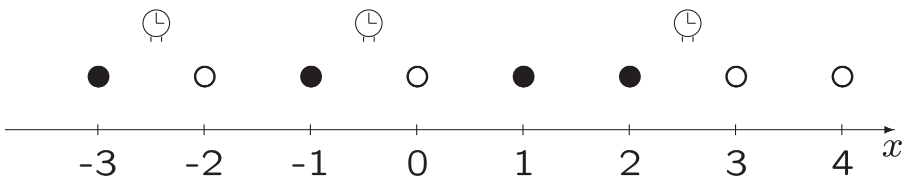

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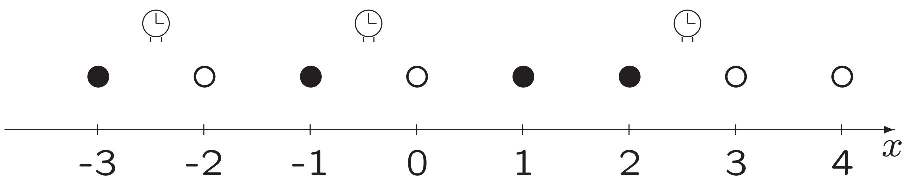

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The Bernoulli(ϱ) distribution is time-stationary for any $(0 \le \varrho \le 1)$. Any translation-invariant stationary distribution is a mixture of Bernoullis.

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$\rightsquigarrow$ The characteristic speed $C(\varrho) := 1 - 2\varrho$.
(ϱ is constant along $\dot{X}(T) = C(\varrho)$.)

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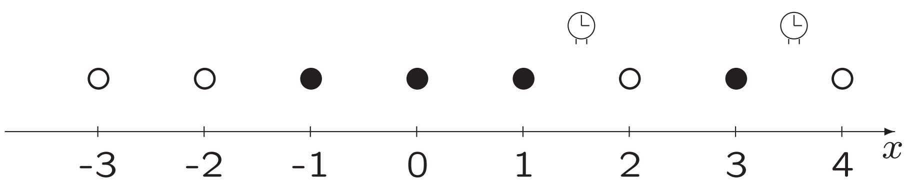

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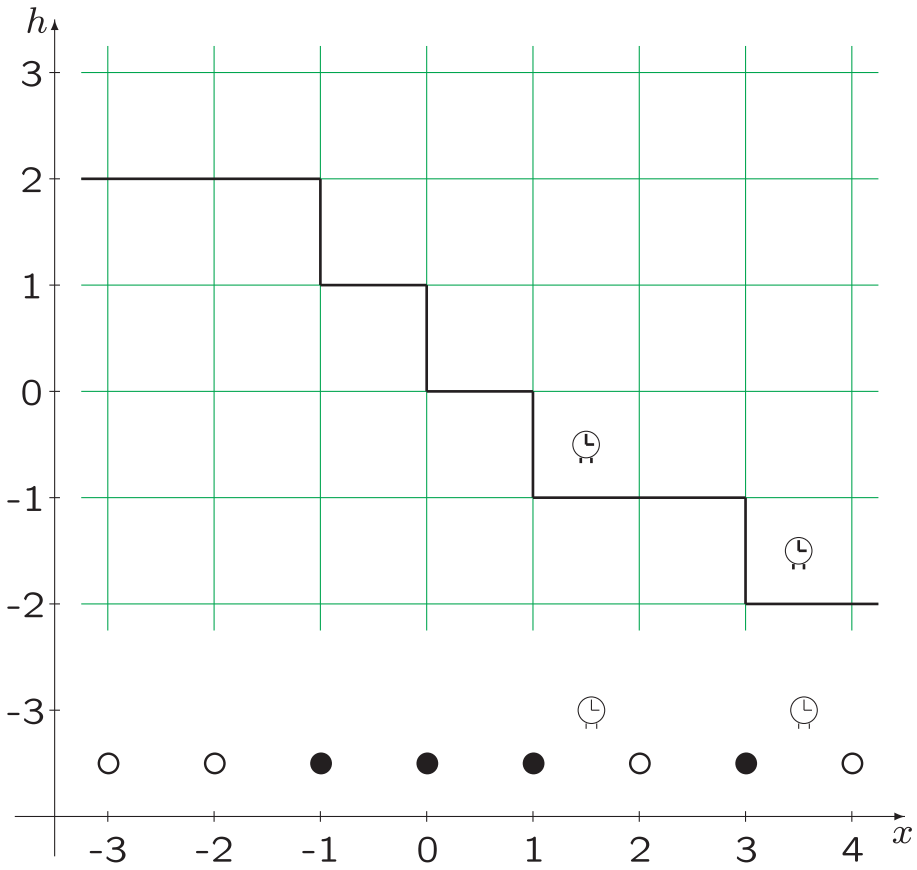

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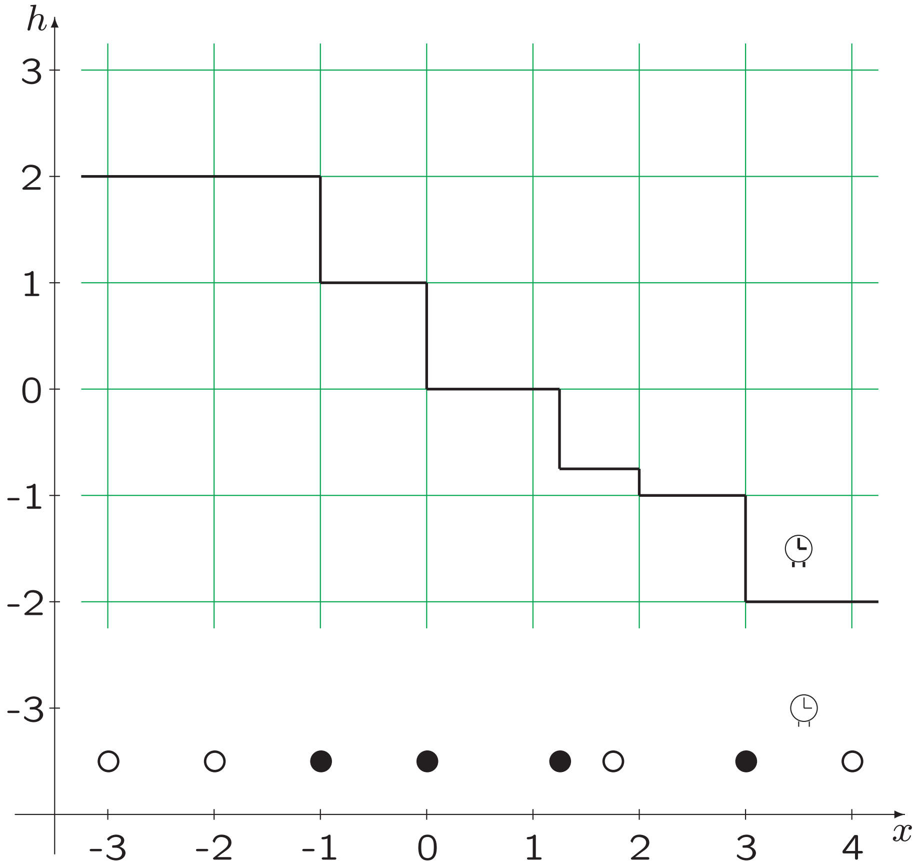

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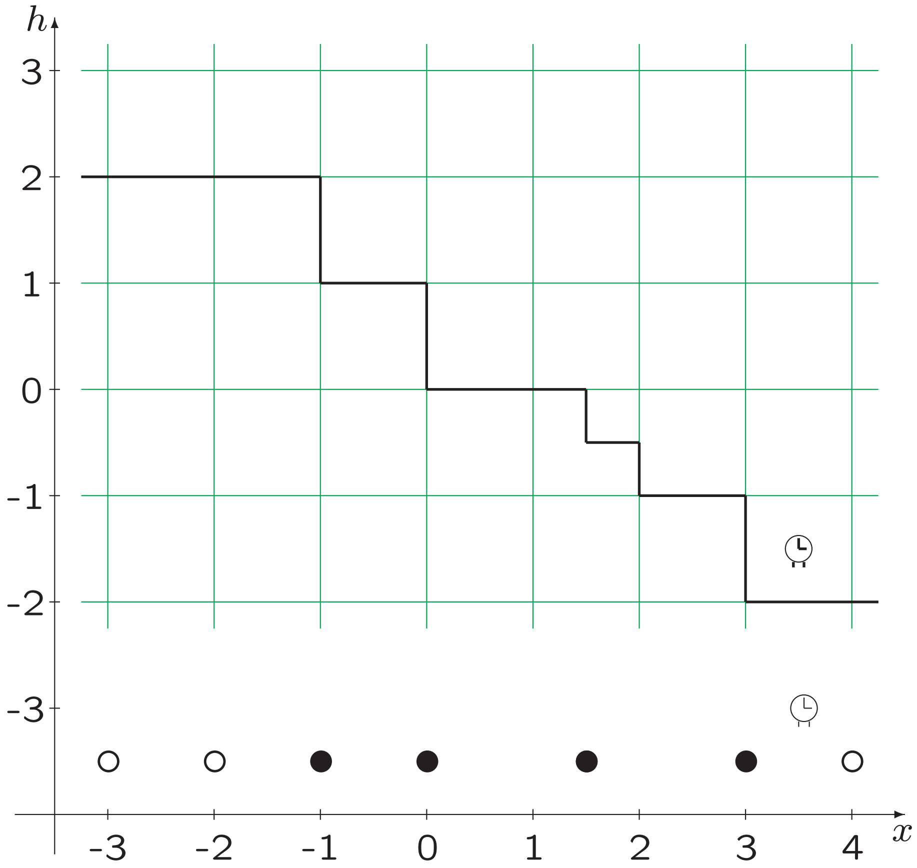

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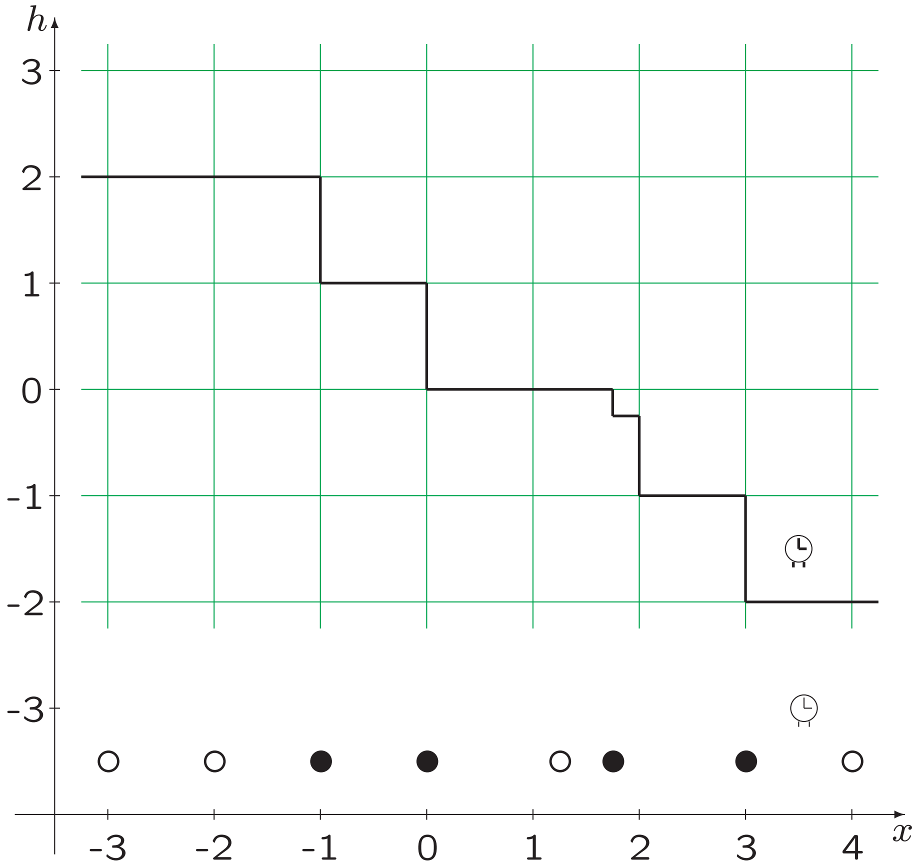

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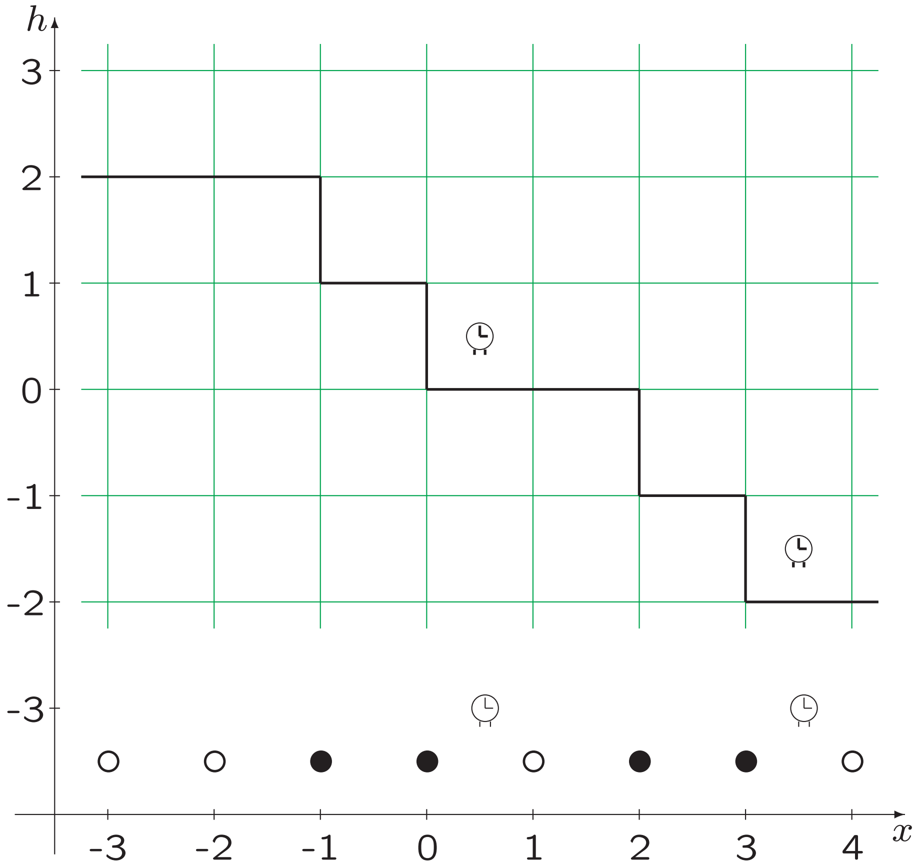

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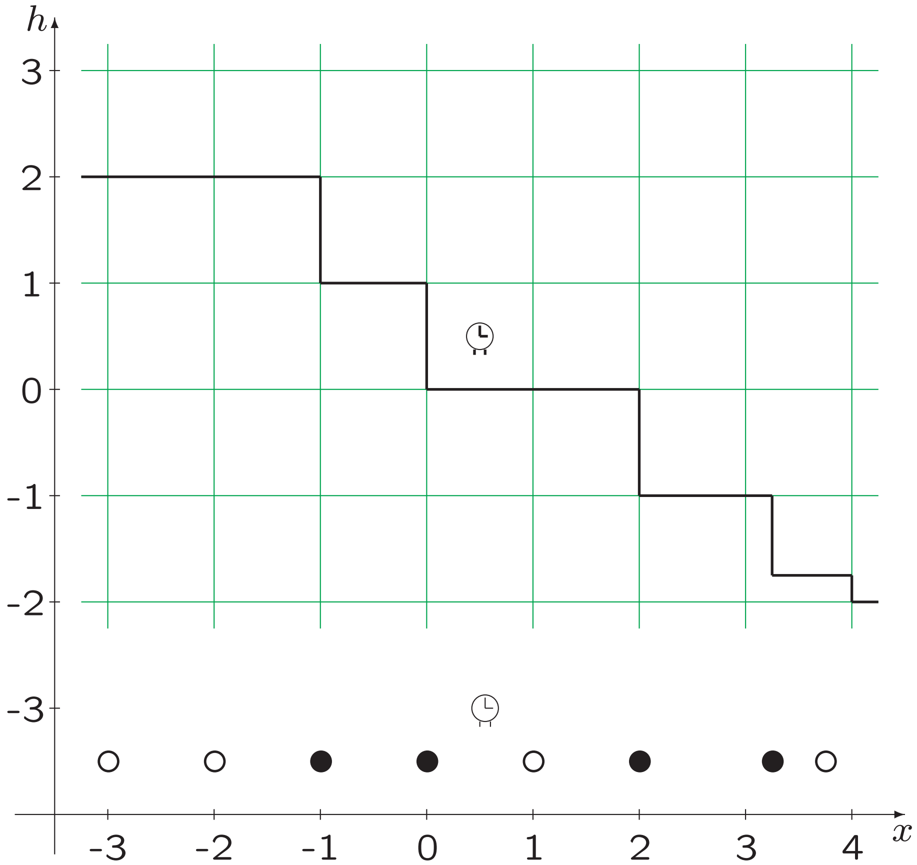

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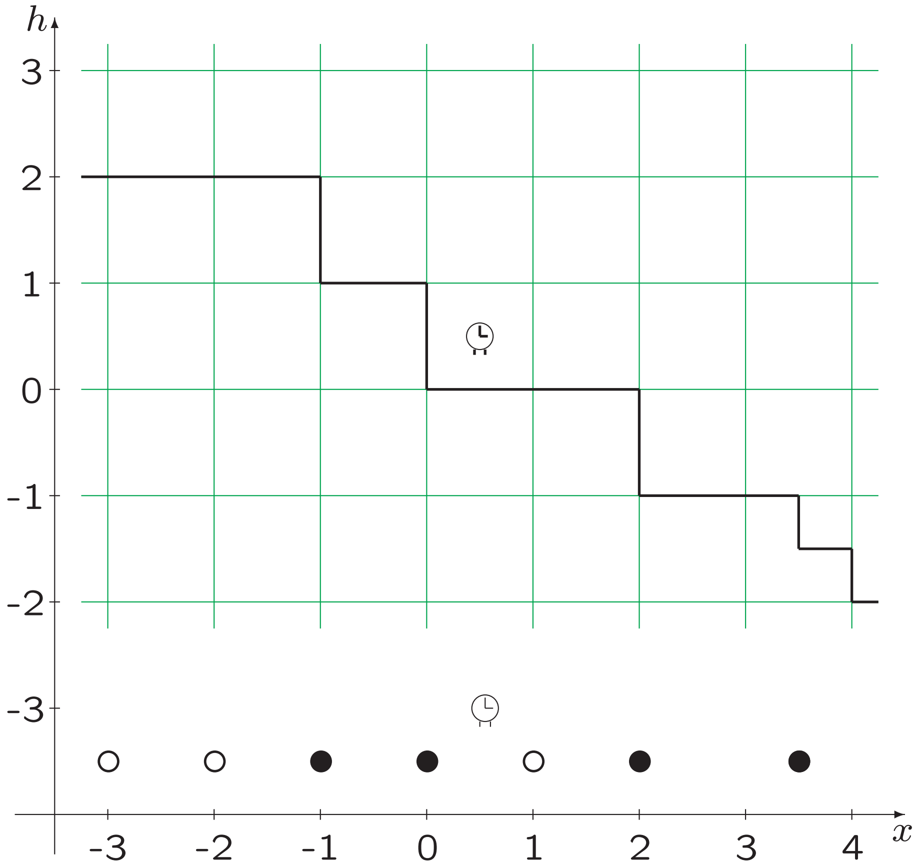

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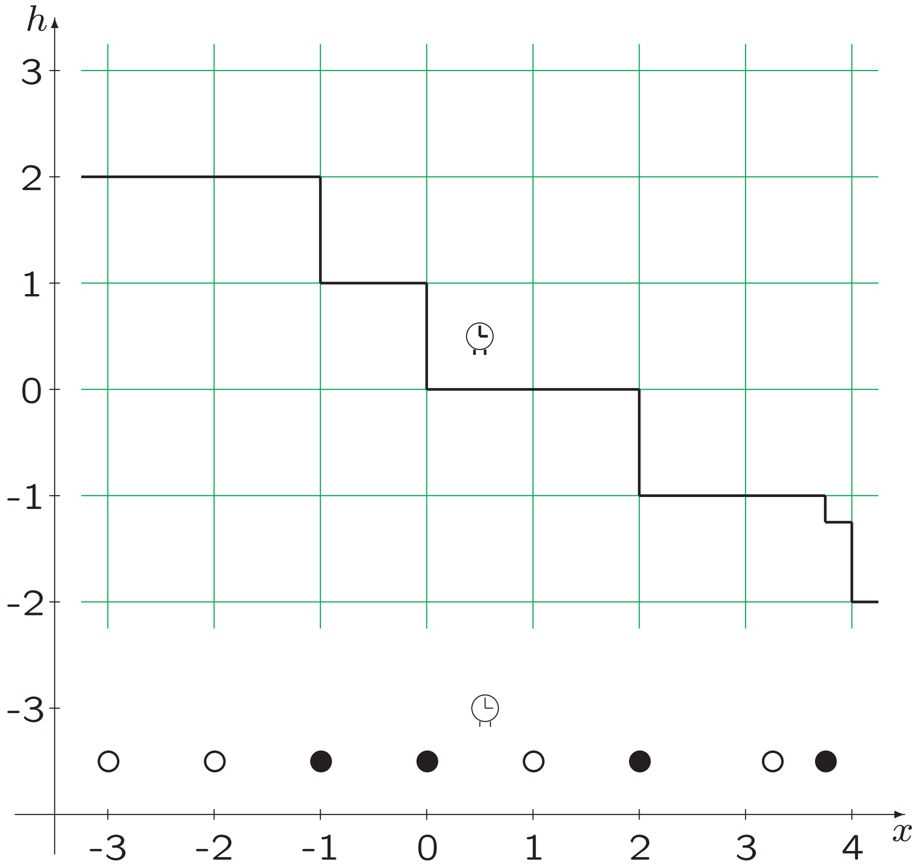

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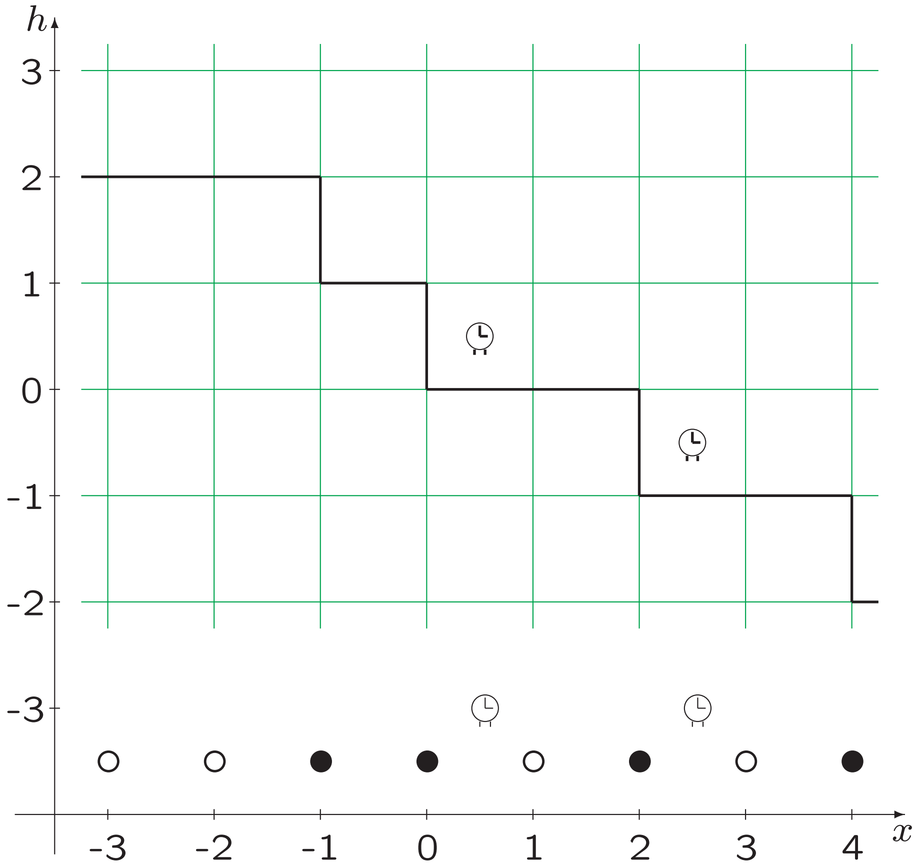

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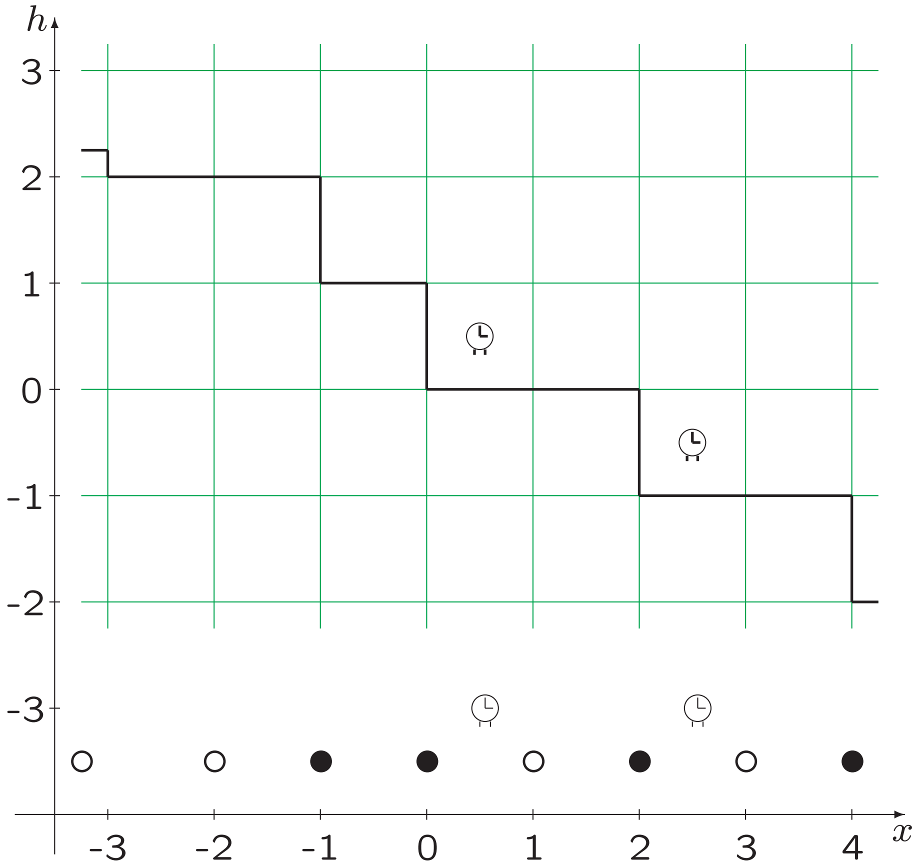

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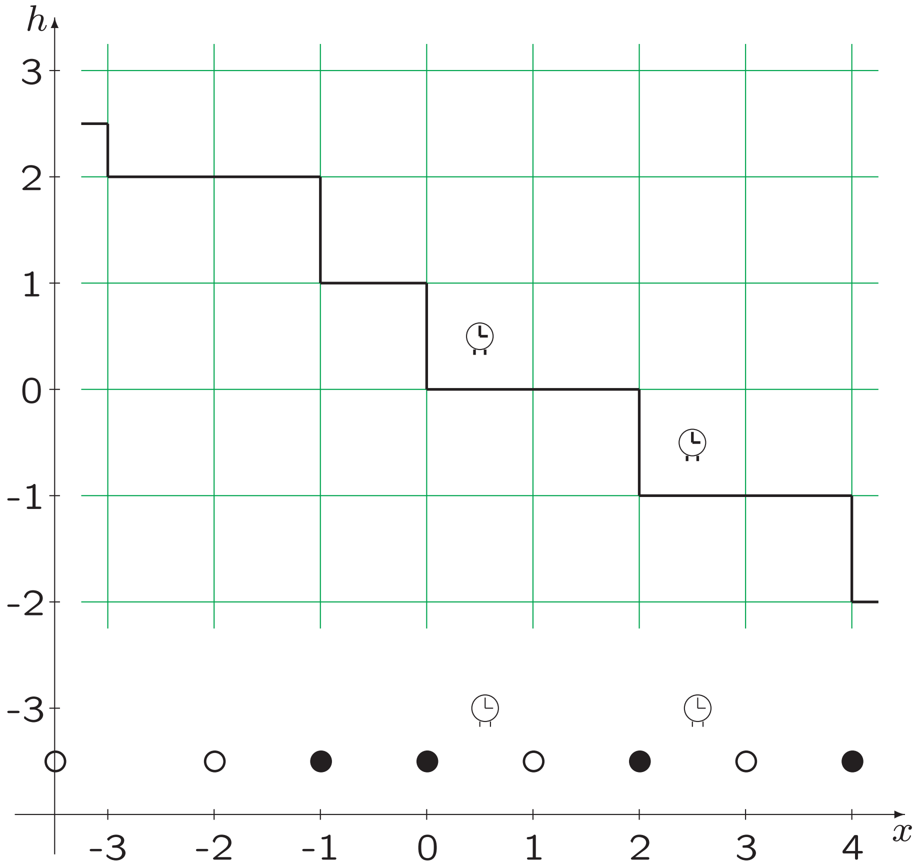

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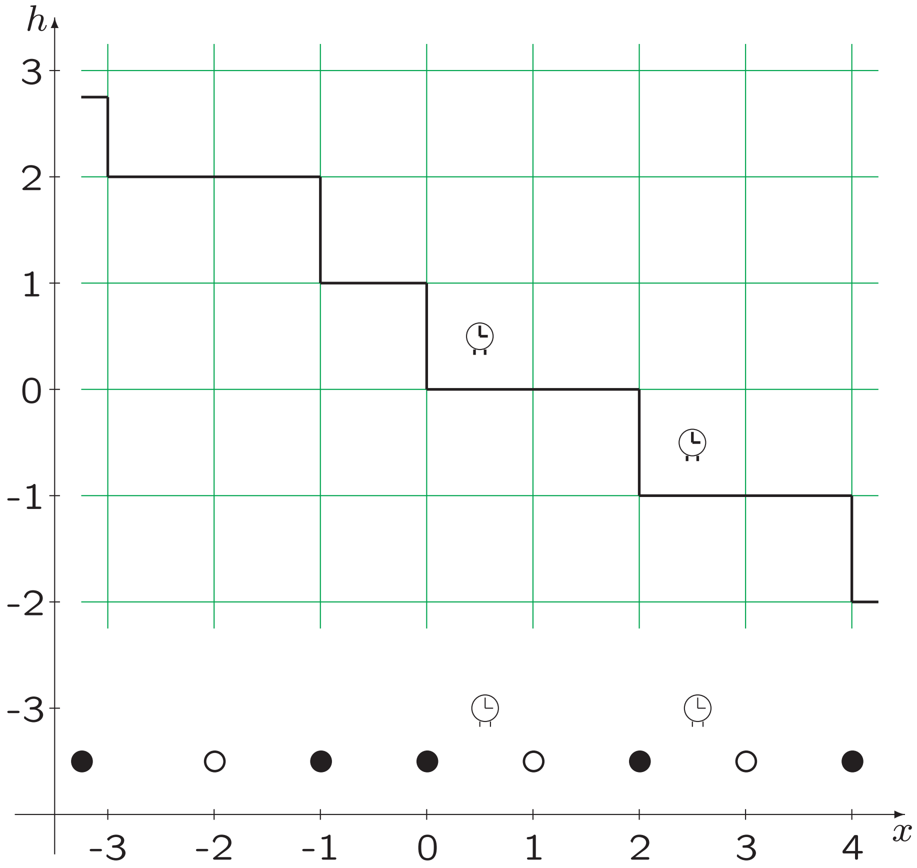

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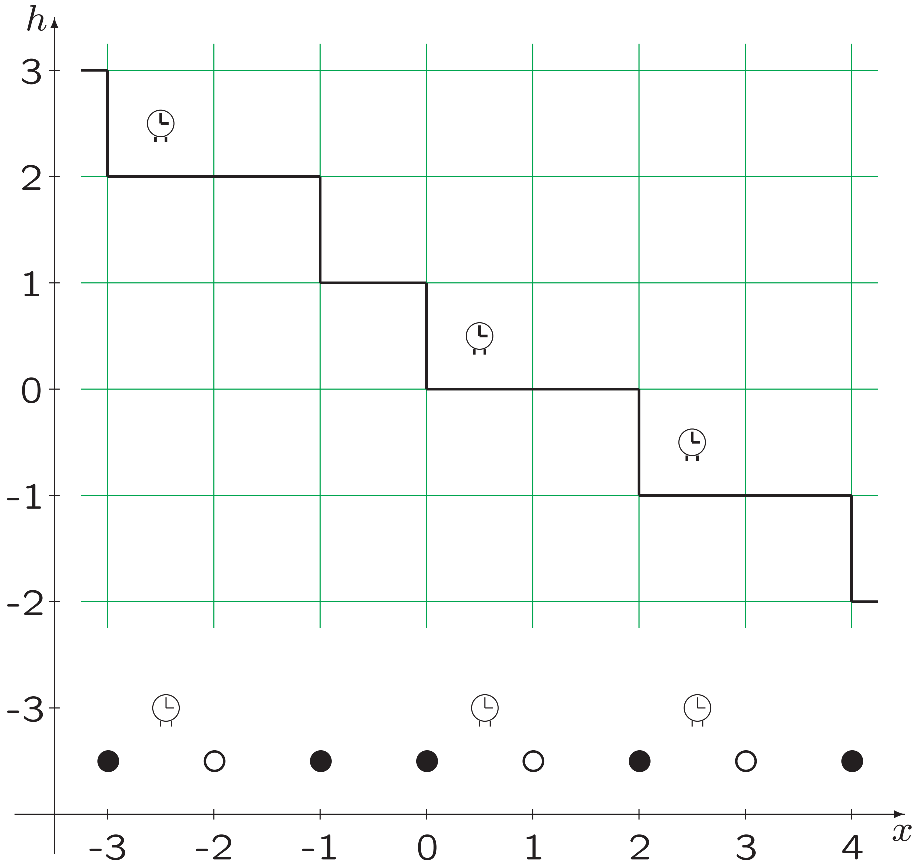

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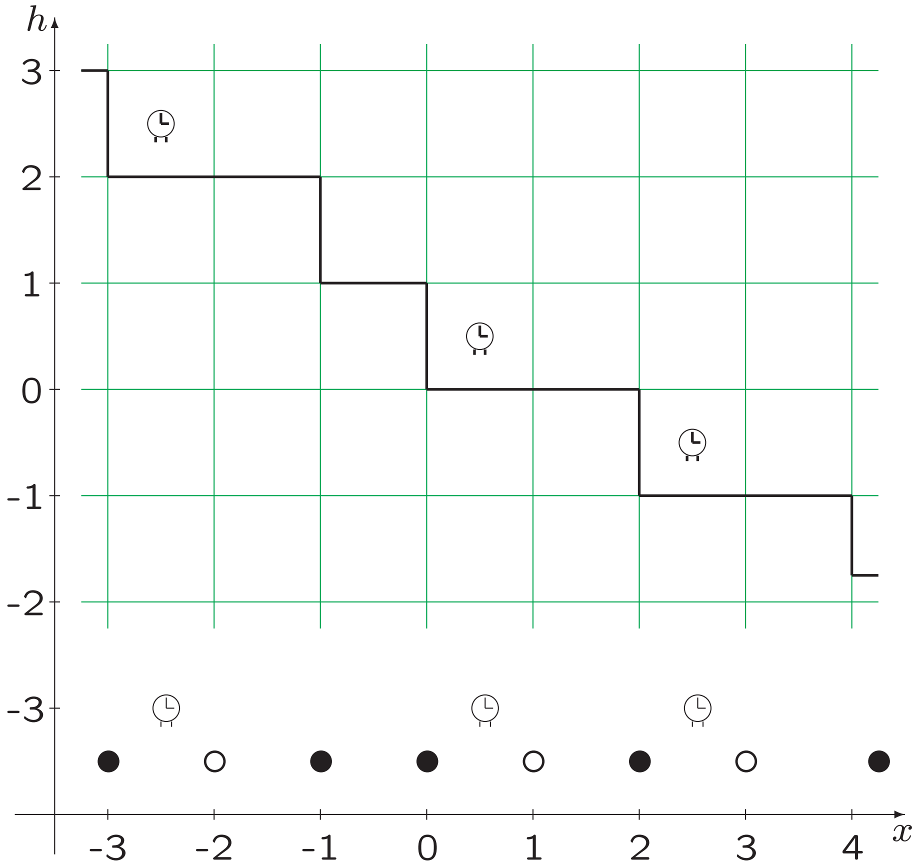

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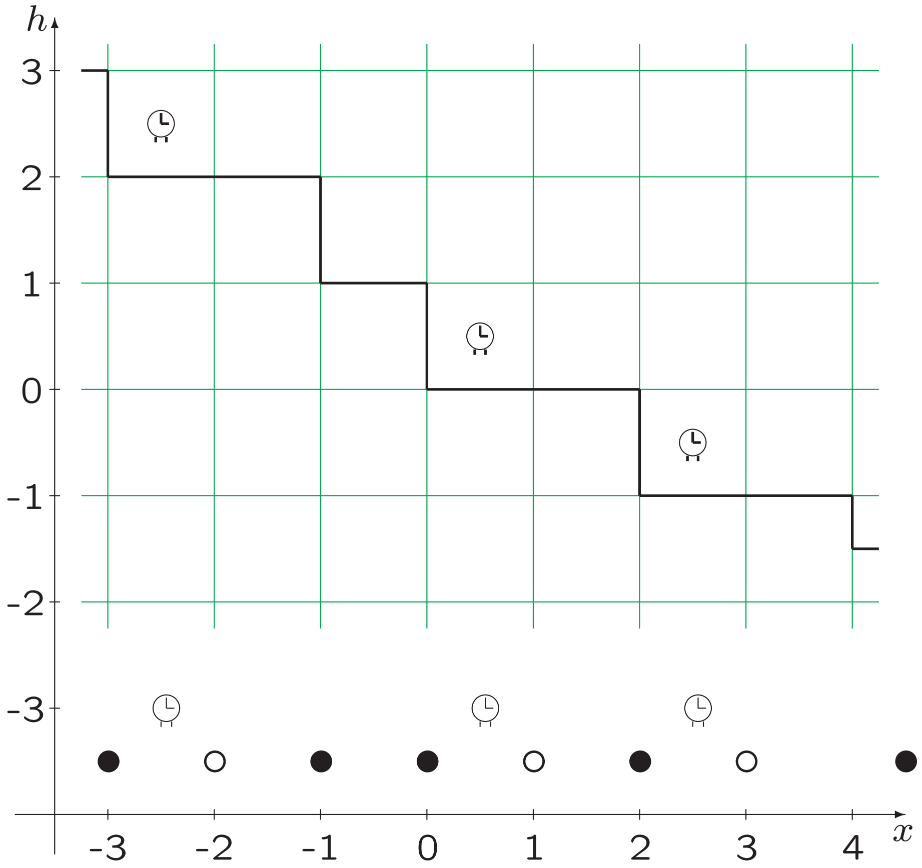

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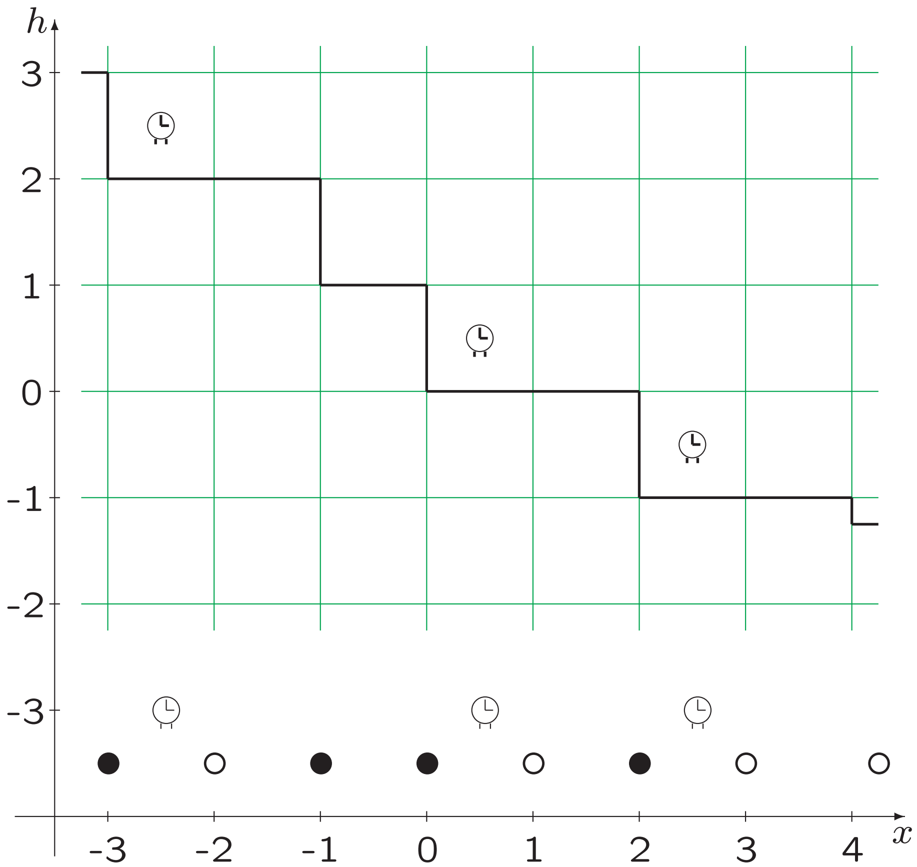

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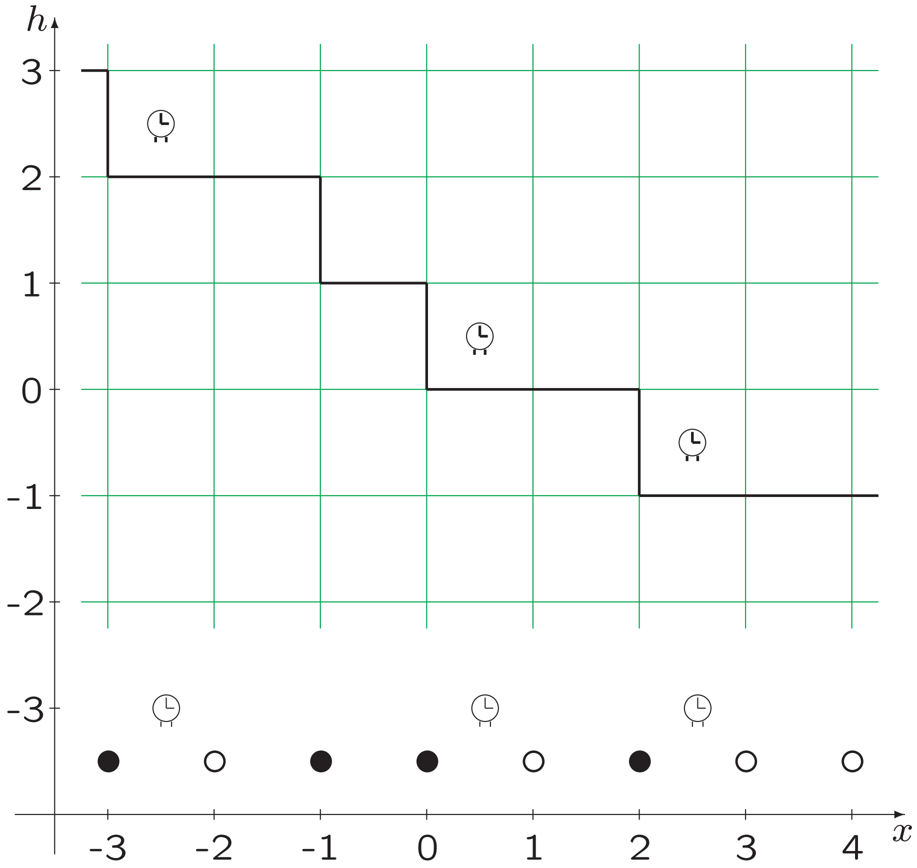

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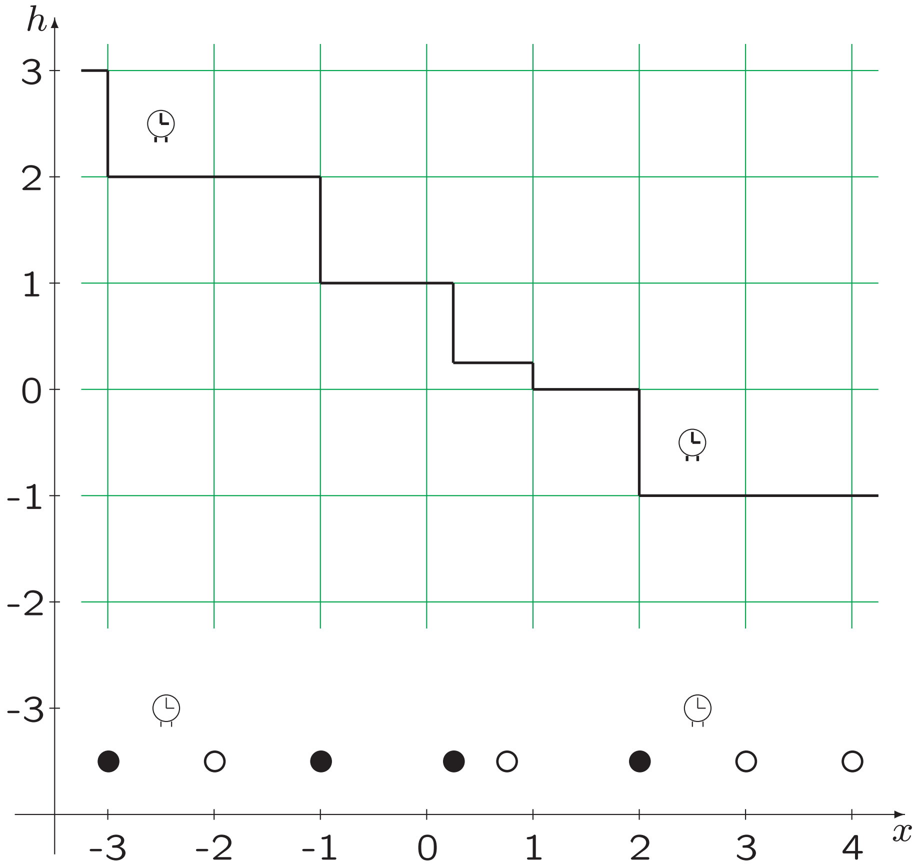

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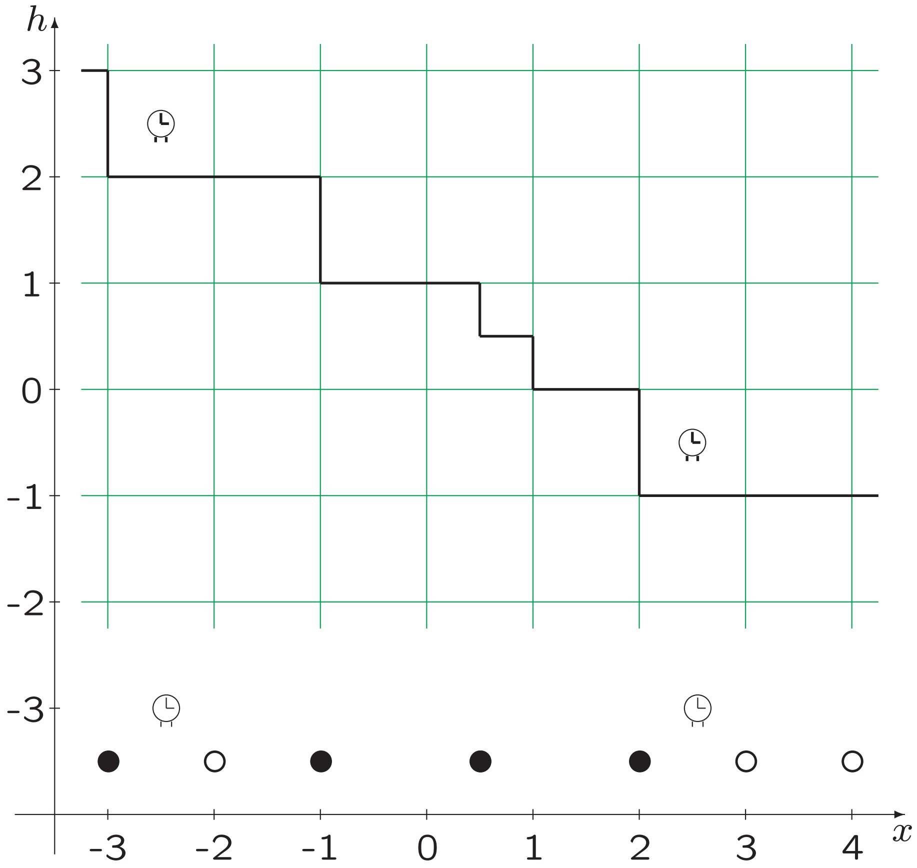

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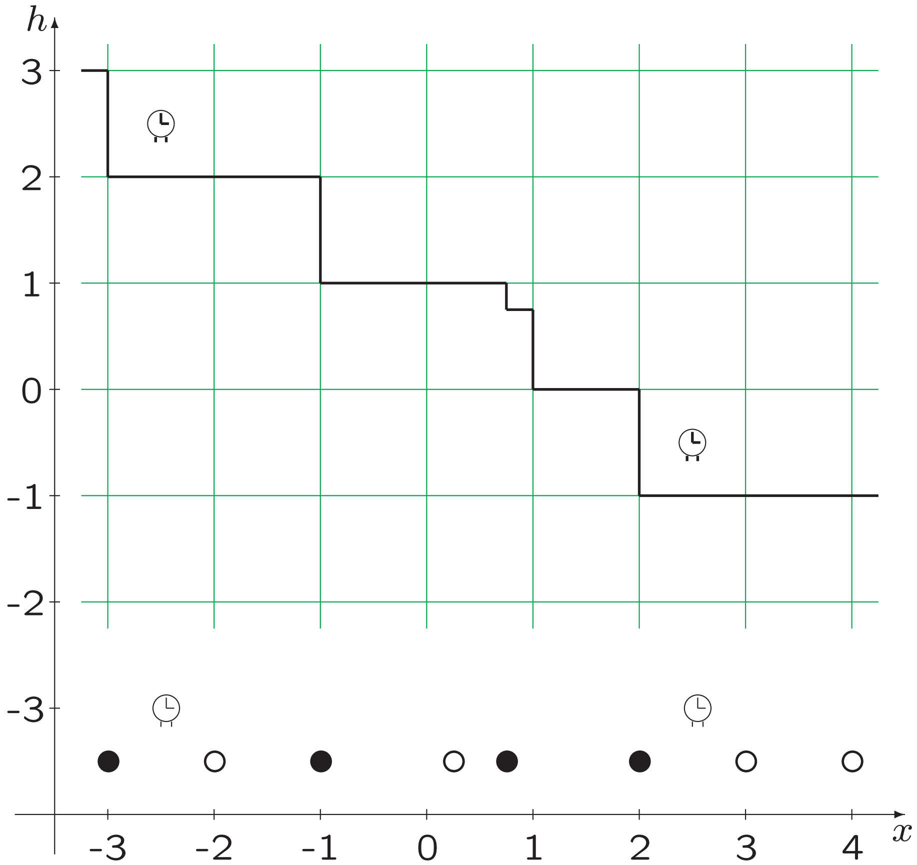
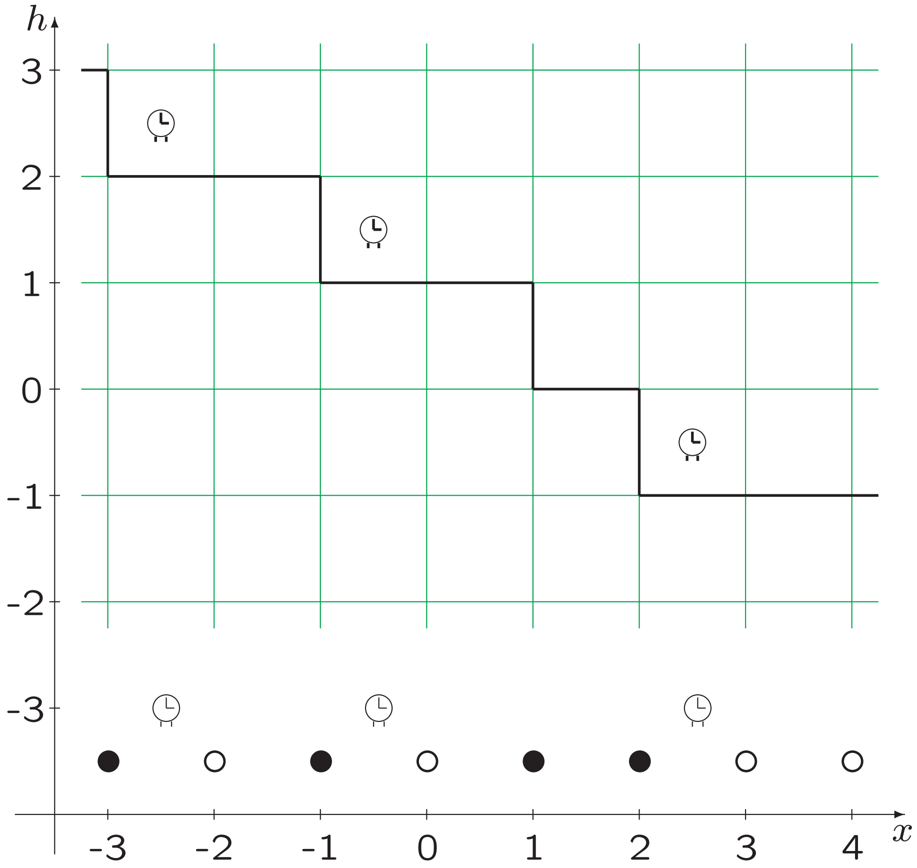


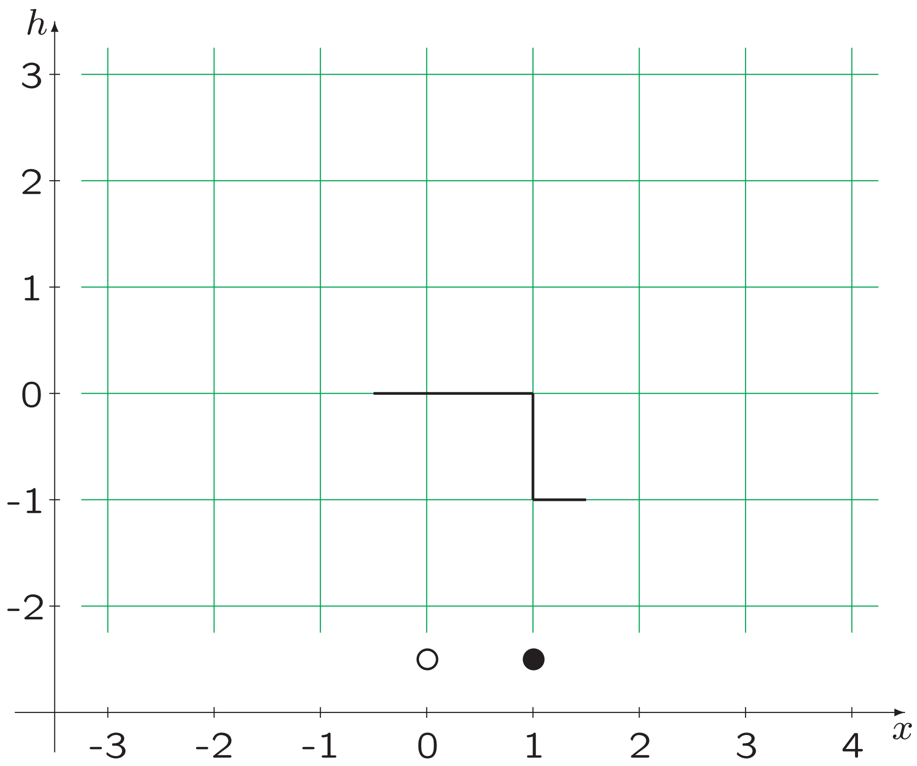
Bernoulli(ϱ) distribution

TASEP: Surface growth

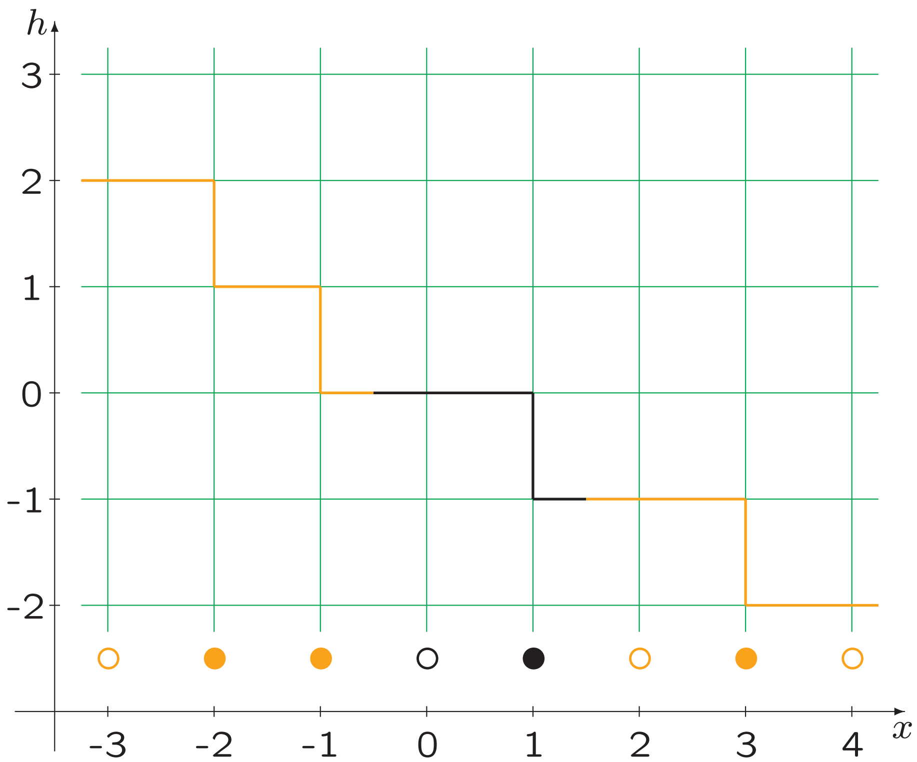


Bernoulli(ϱ) distribution

TASEP: Last passage percolation

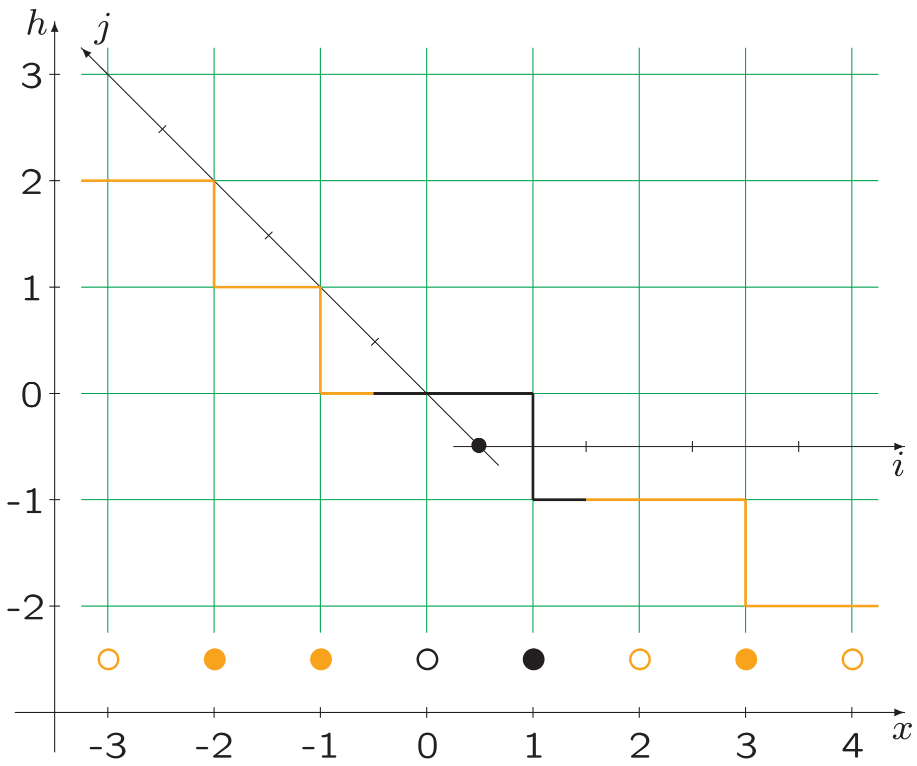


TASEP: Last passage percolation



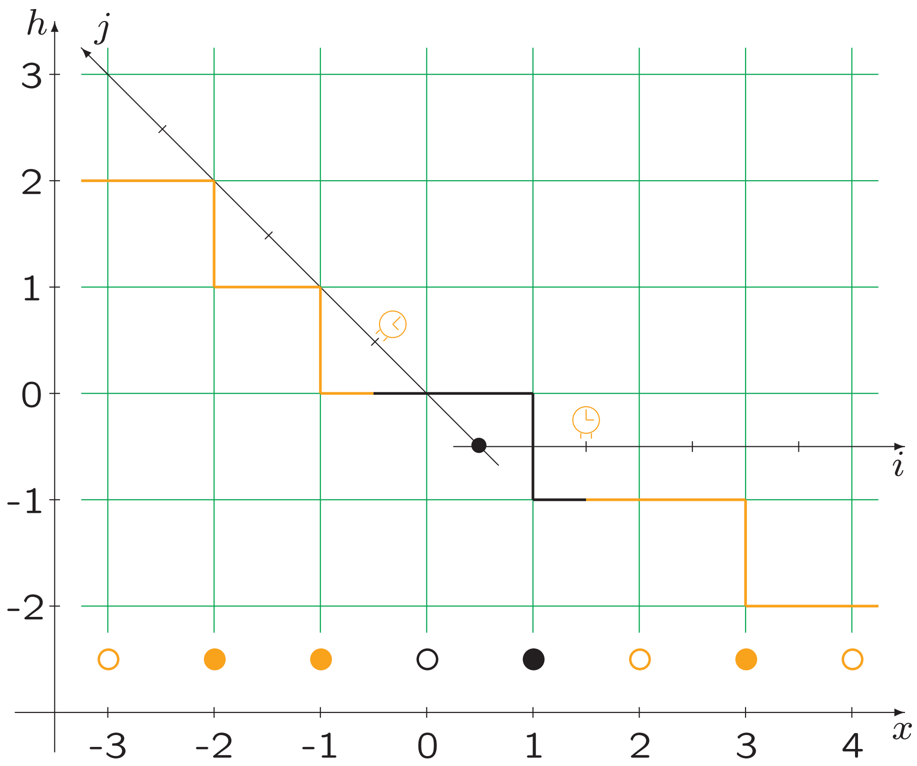
Bernoulli(ϱ) distribution

TASEP: Last passage percolation

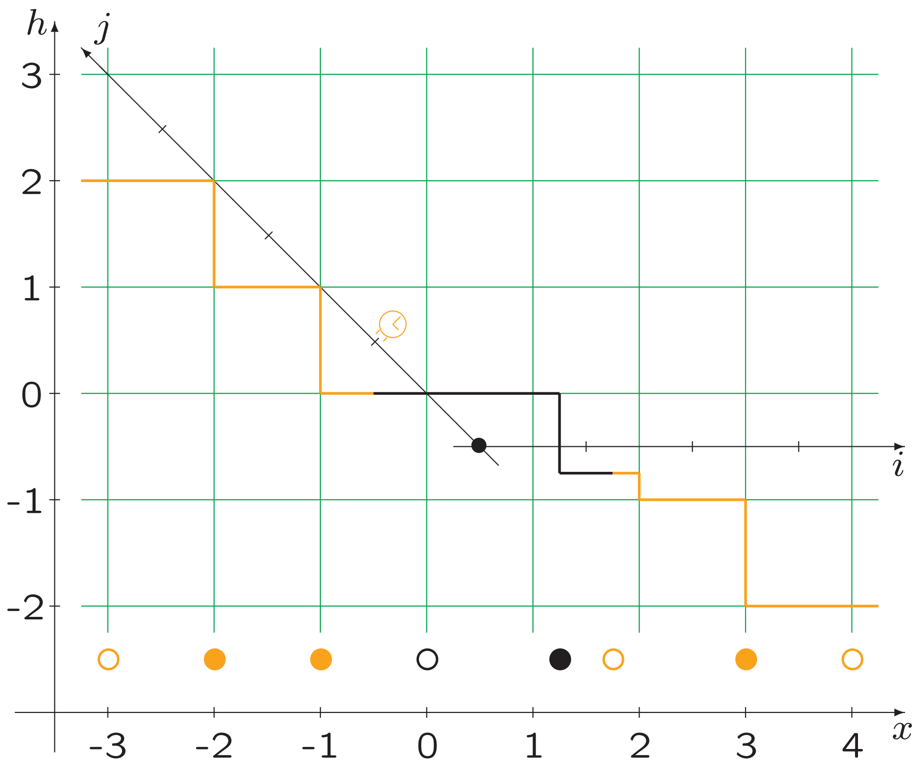


Bernoulli(ϱ) distribution

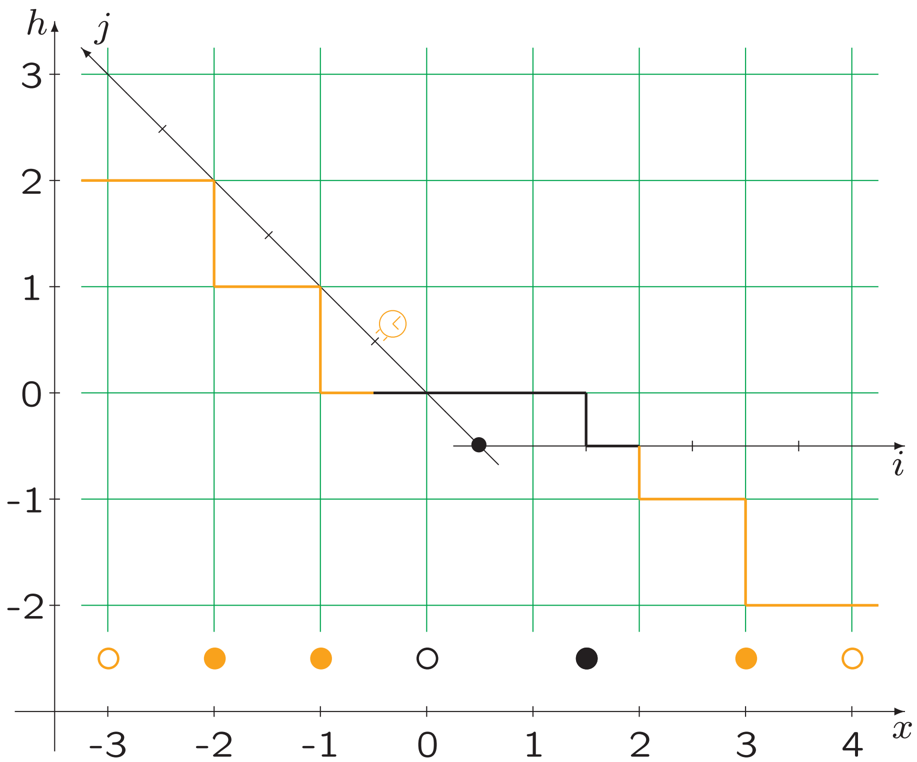
TASEP: Last passage percolation



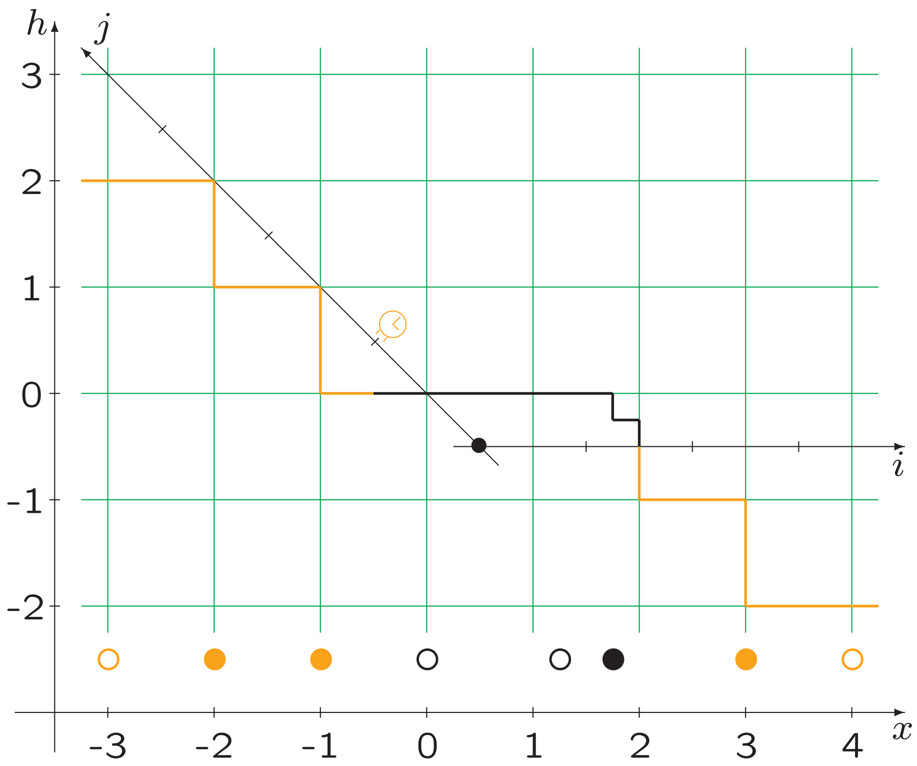
TASEP: Last passage percolation



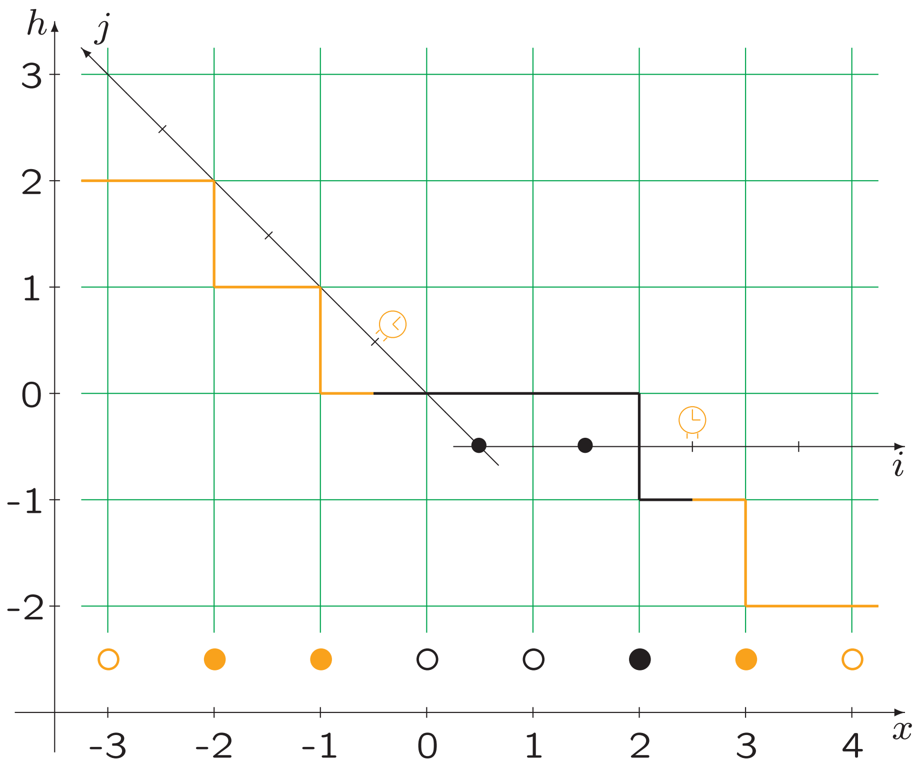
TASEP: Last passage percolation



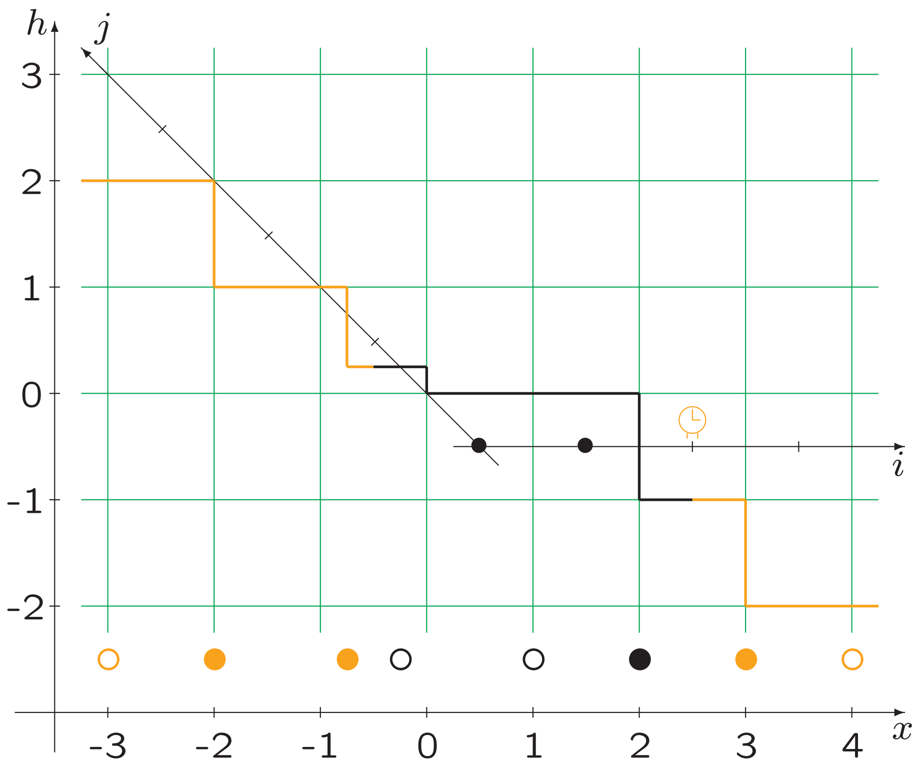
TASEP: Last passage percolation



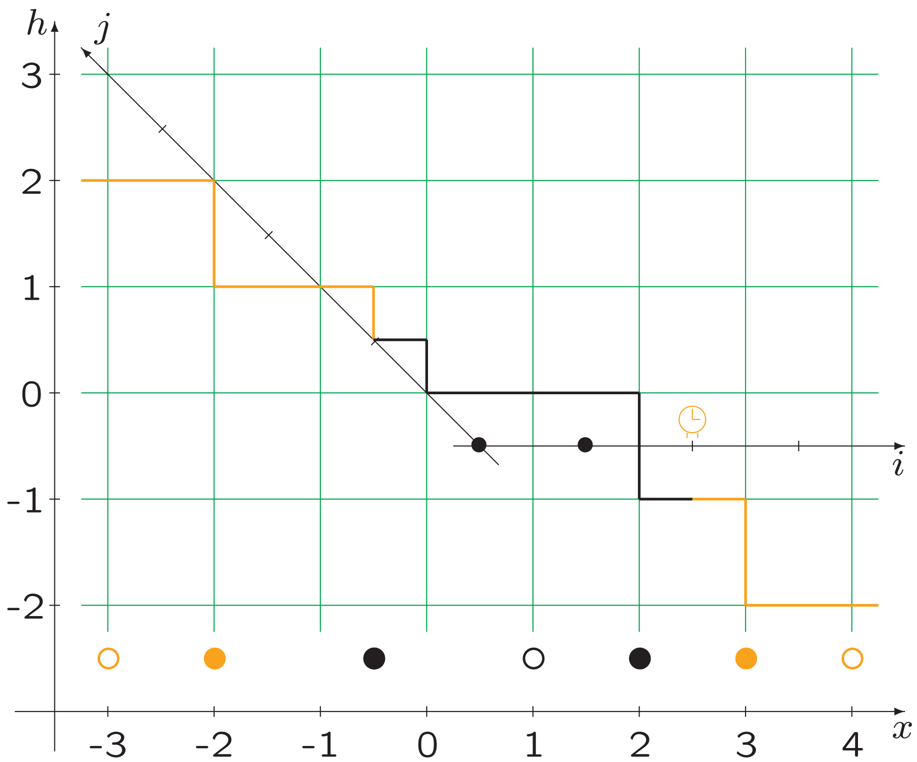
TASEP: Last passage percolation



TASEP: Last passage percolation



TASEP: Last passage percolation



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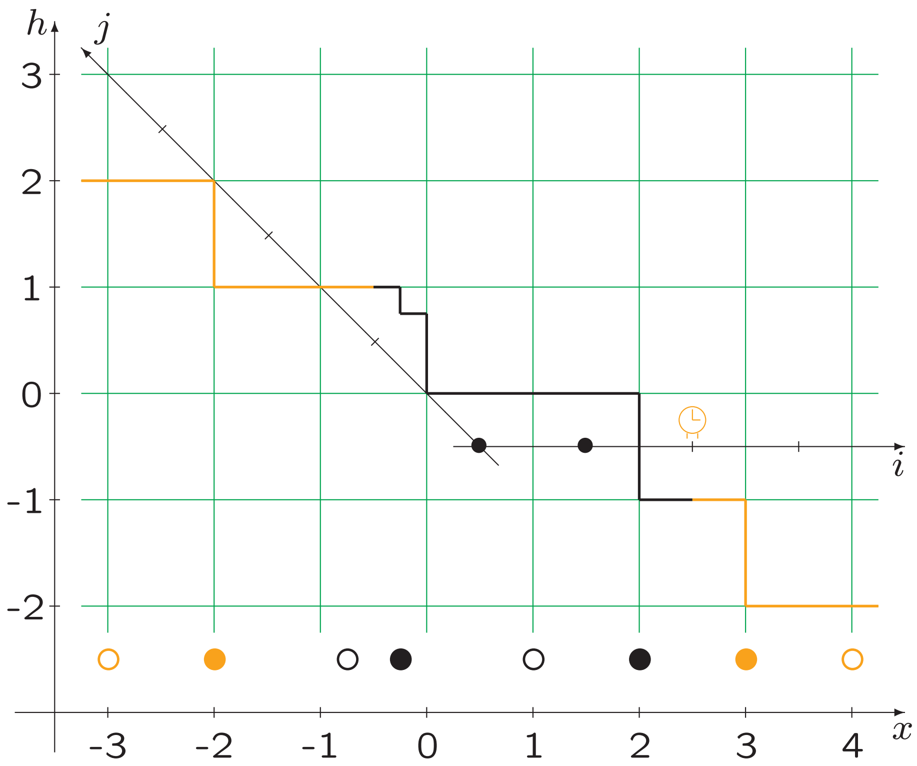
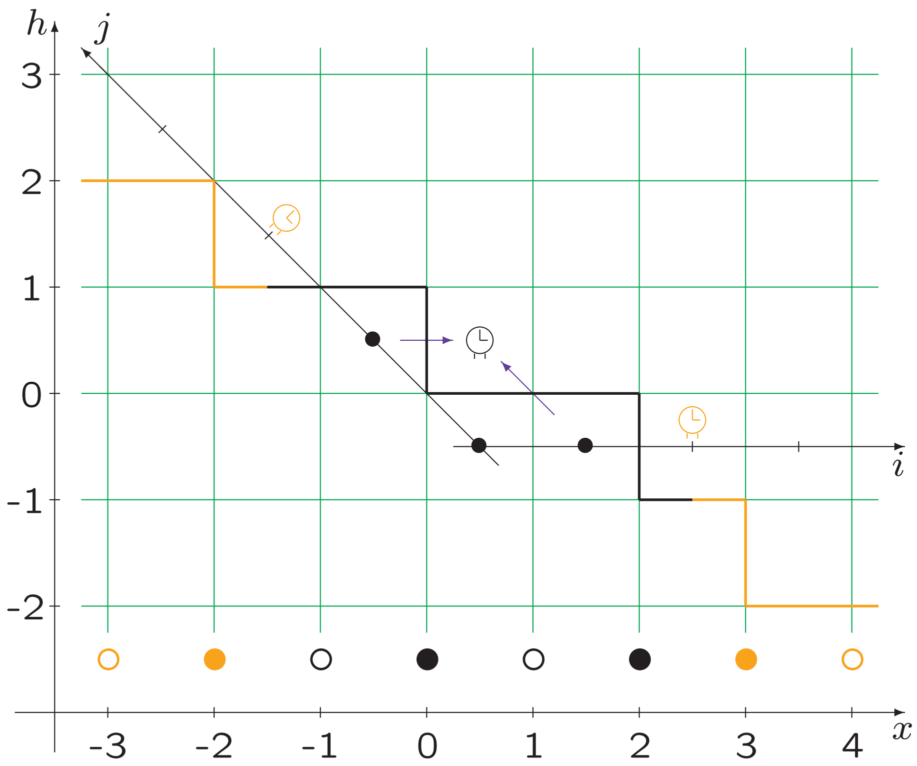
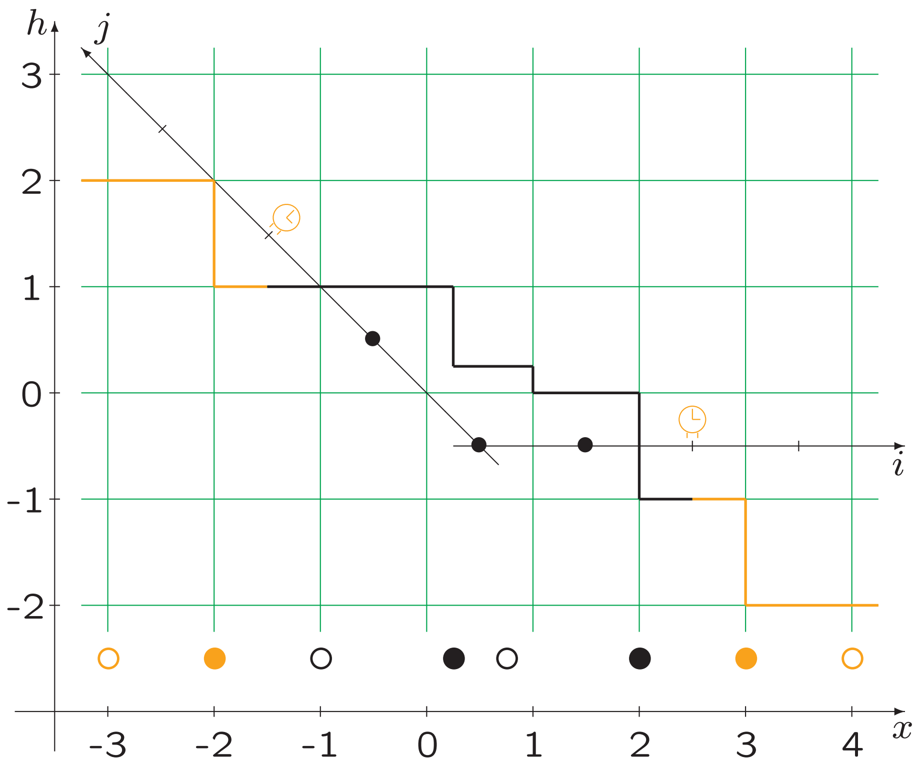


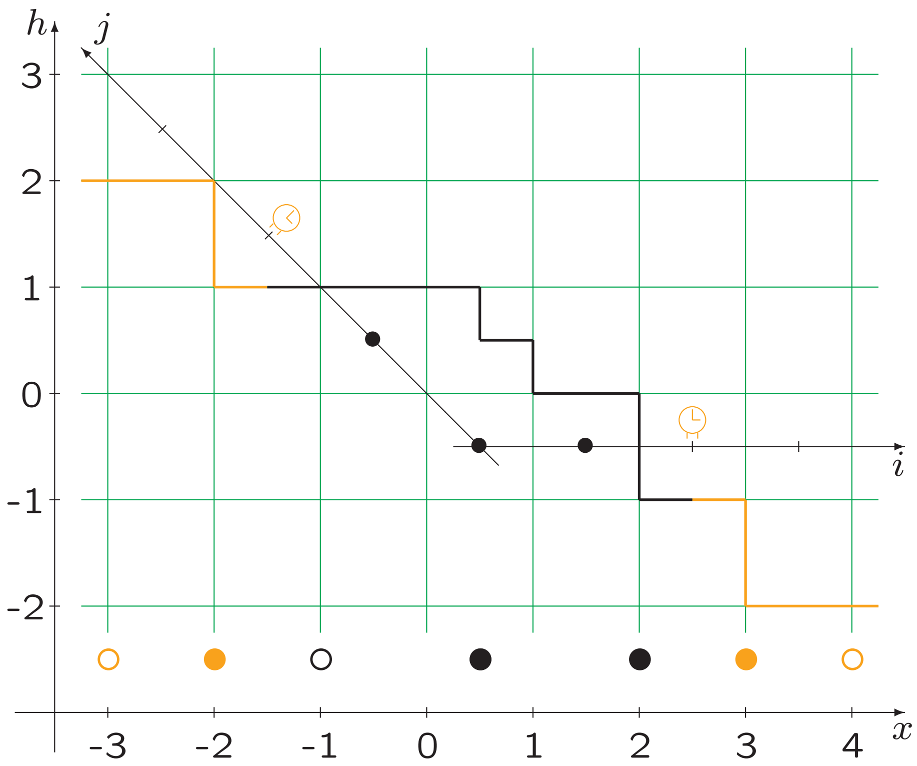
TASEP: Last passage percolation



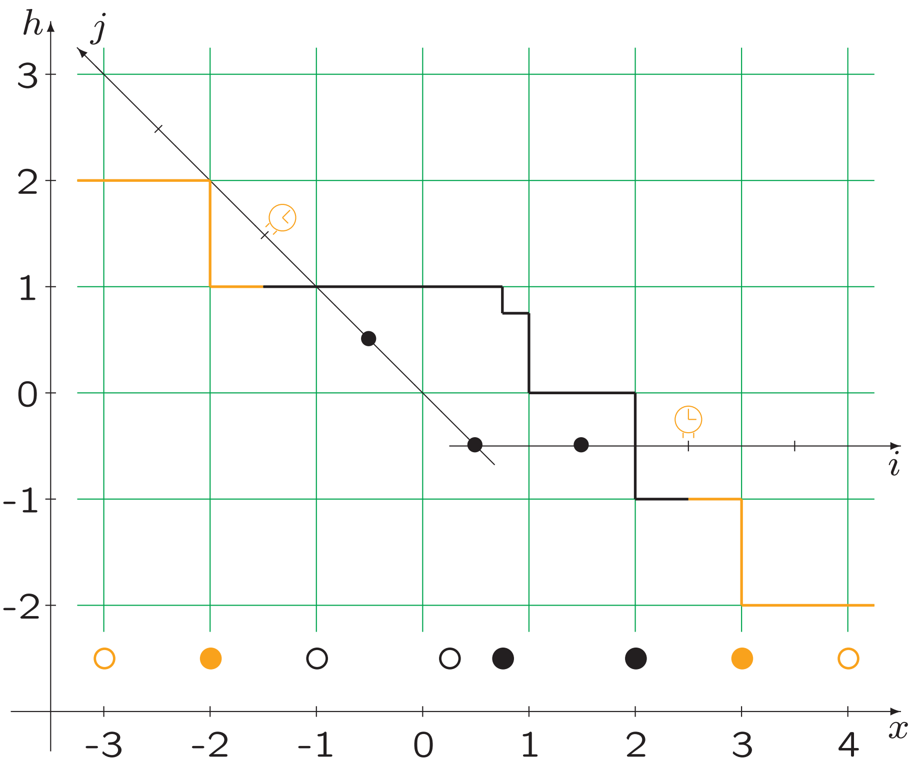
TASEP: Last passage percolation



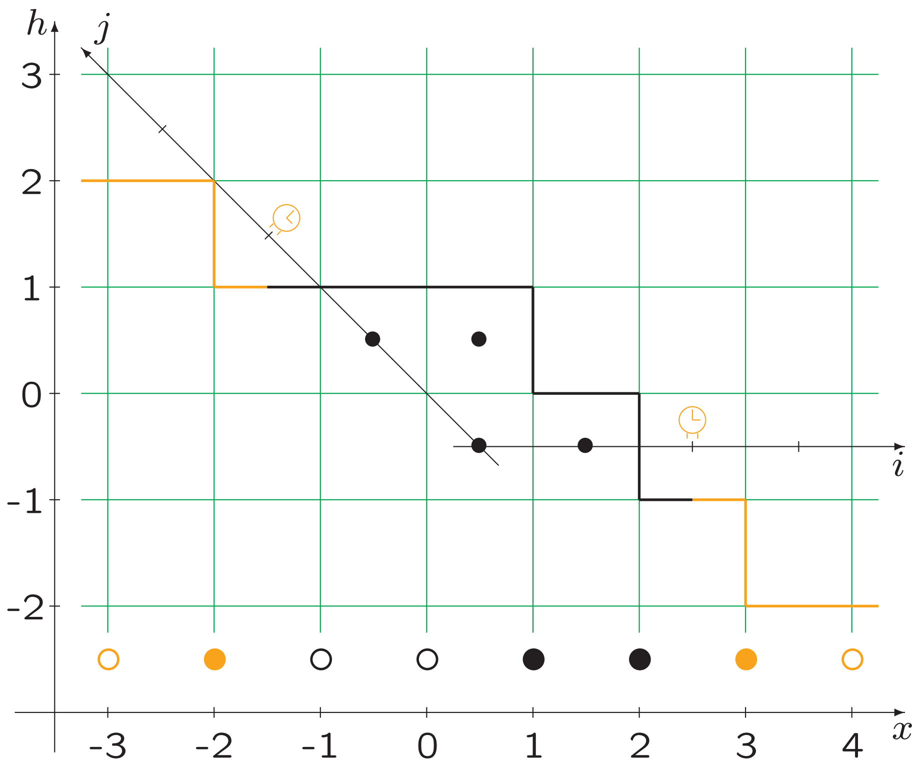
TASEP: Last passage percolation



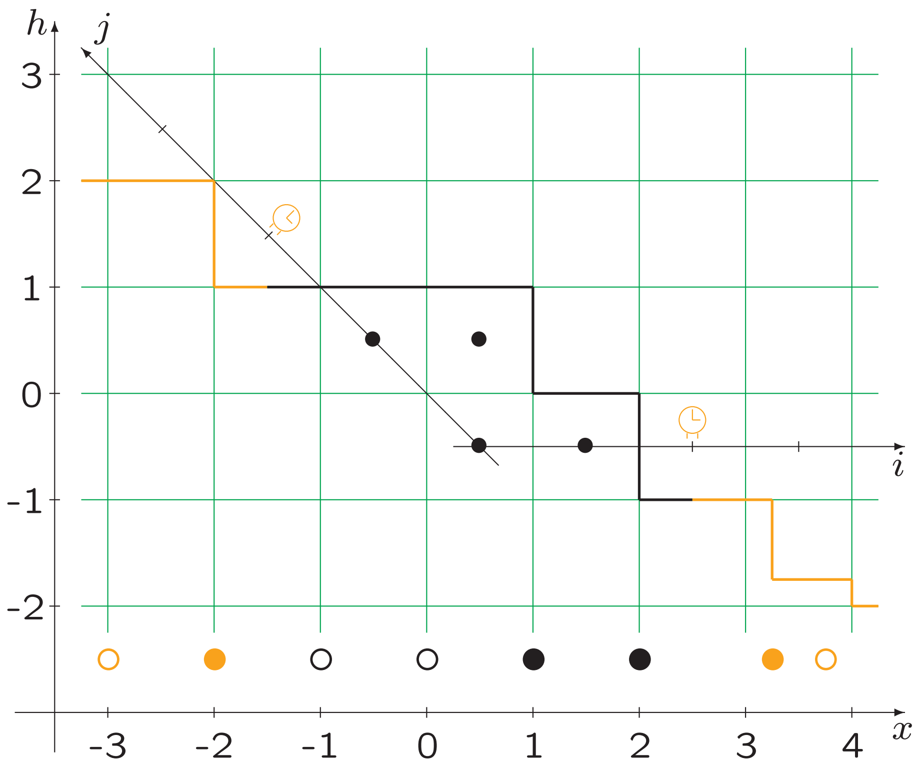
TASEP: Last passage percolation



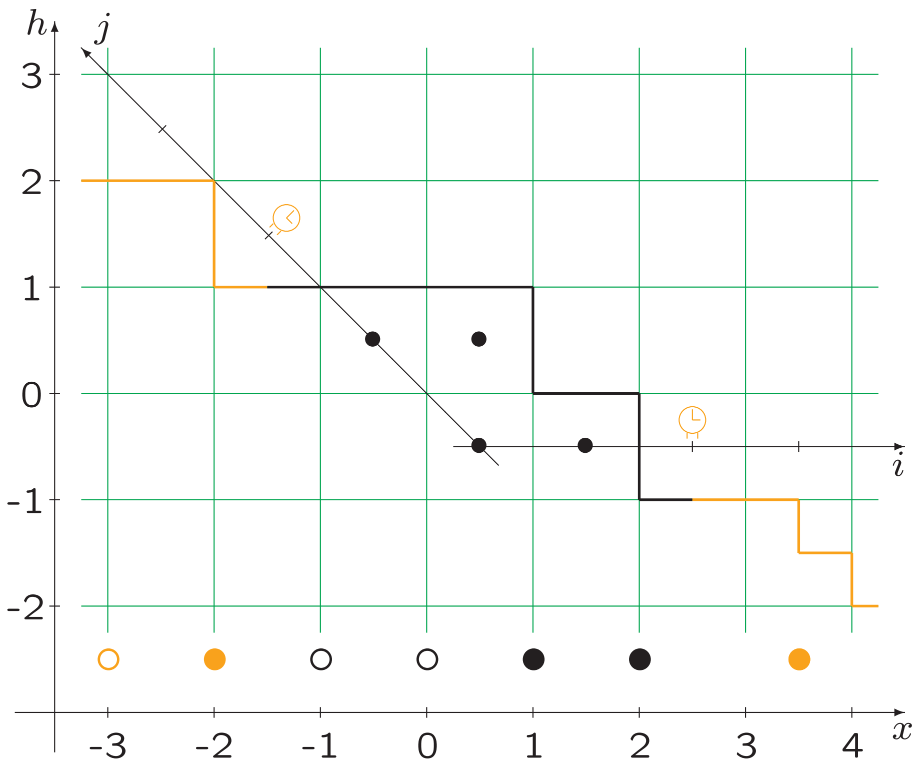
TASEP: Last passage percolation



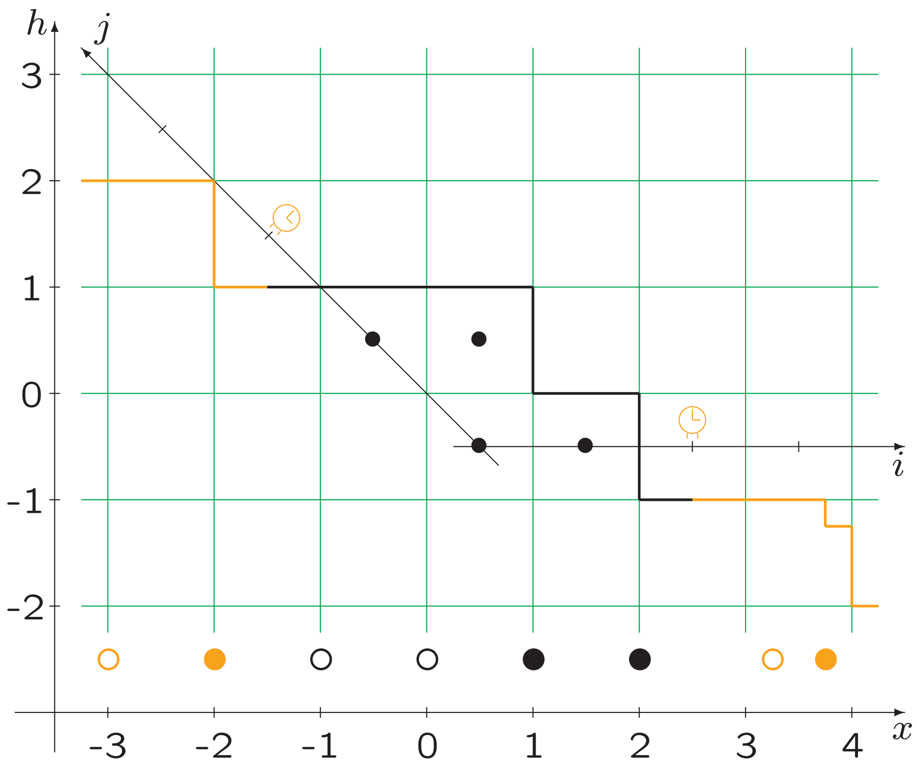
TASEP: Last passage percolation



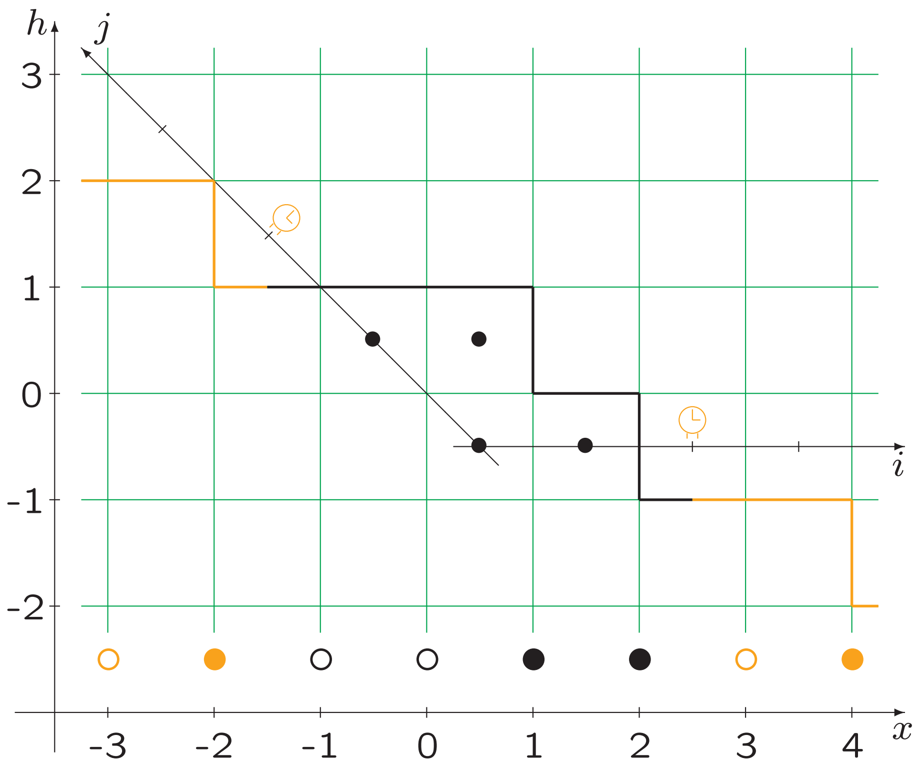
TASEP: Last passage percolation



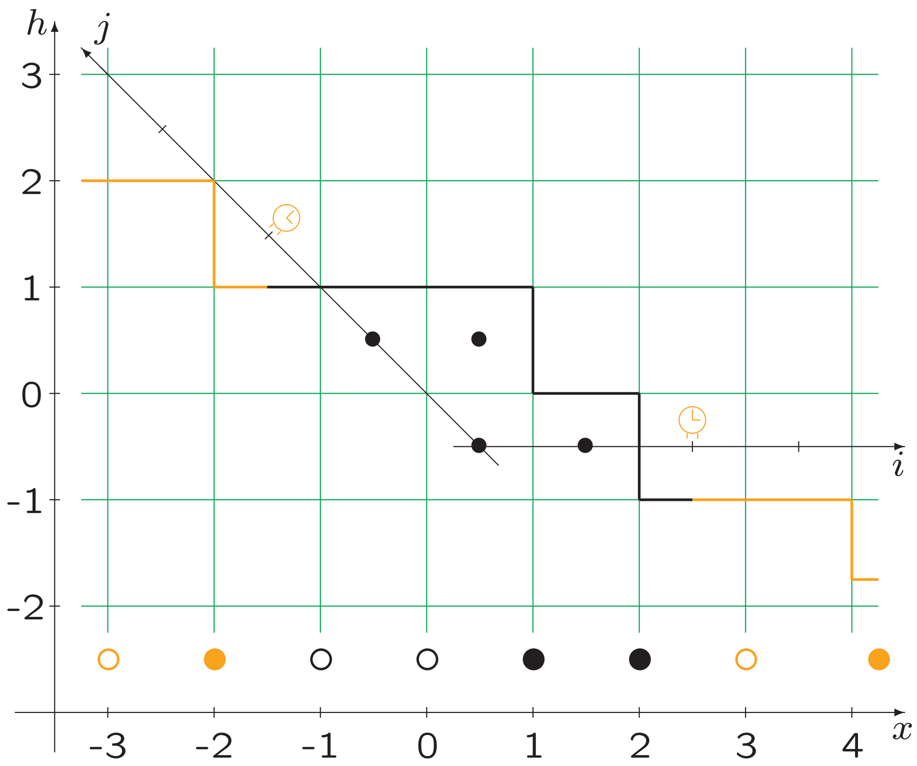
TASEP: Last passage percolation



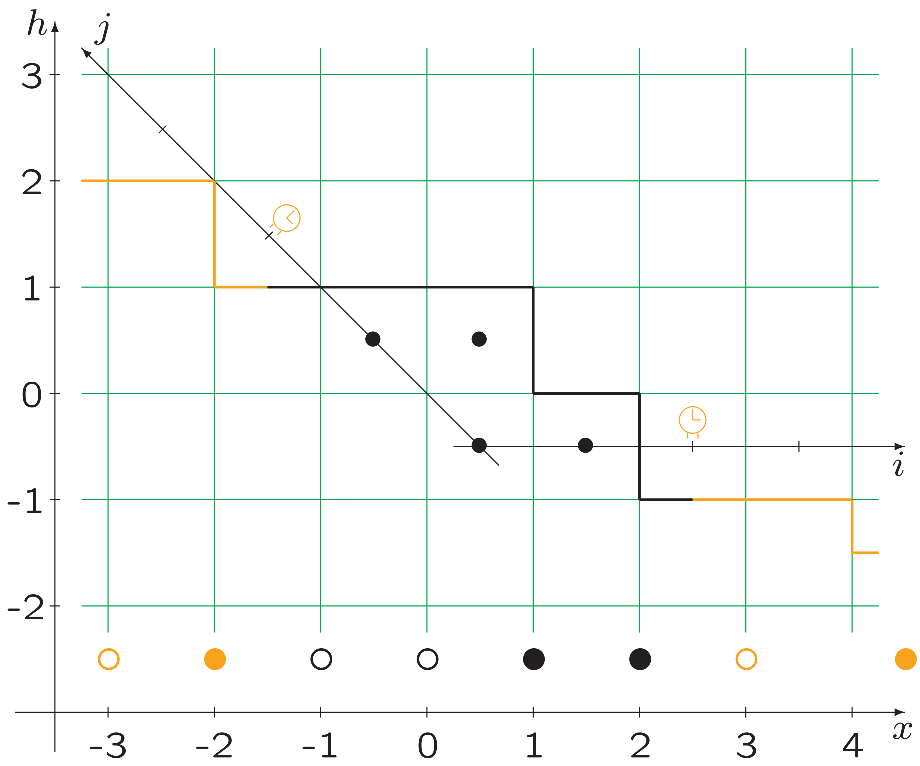
TASEP: Last passage percolation



TASEP: Last passage percolation



TASEP: Last passage percolation



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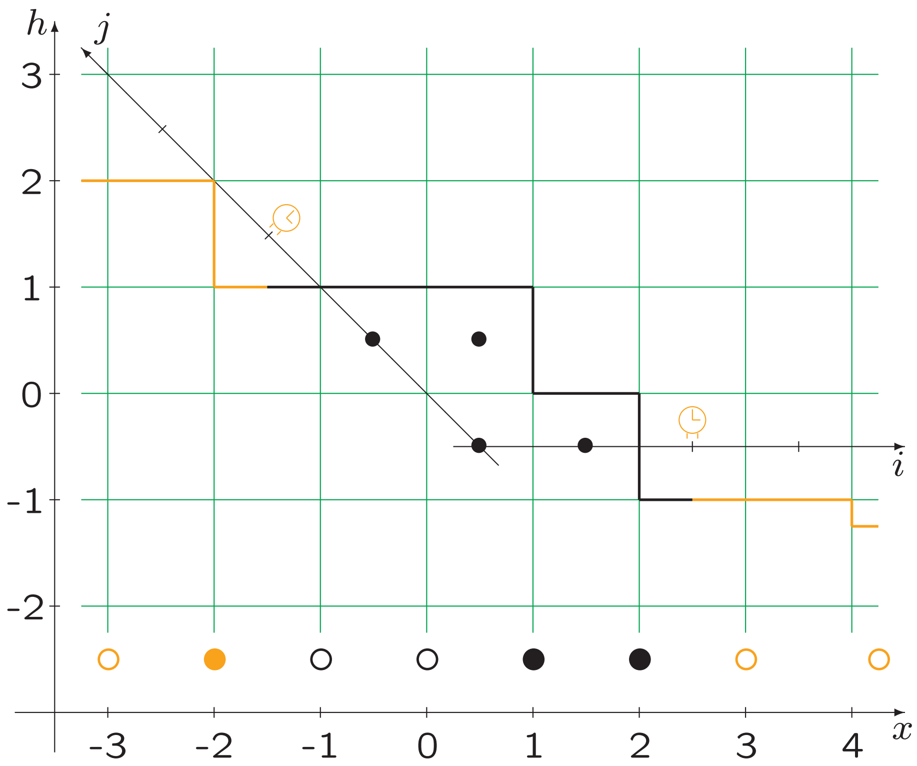
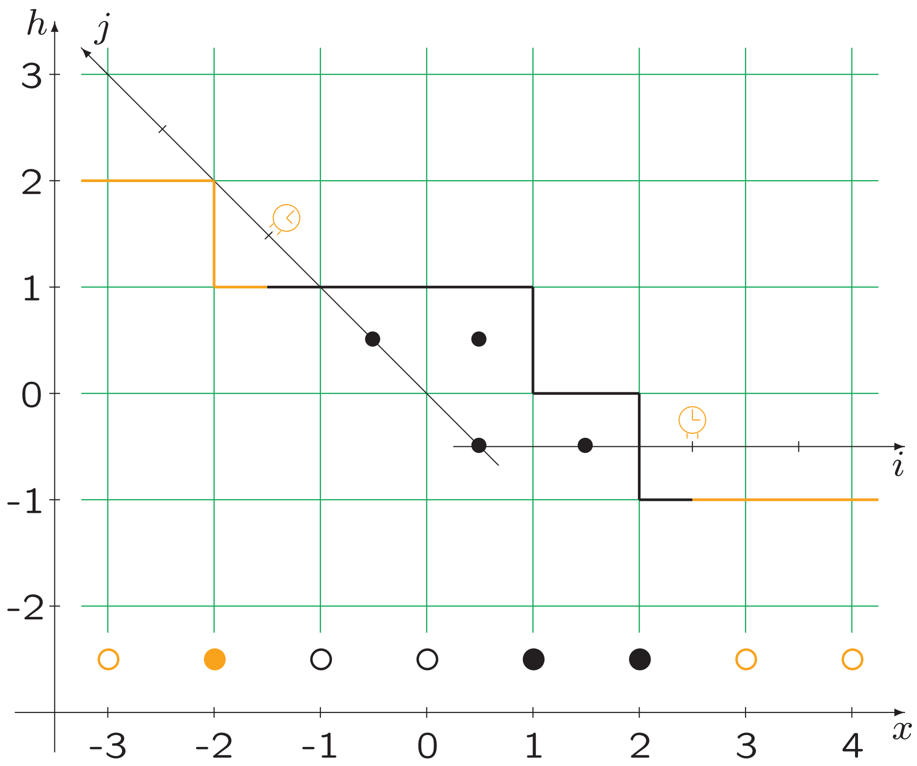
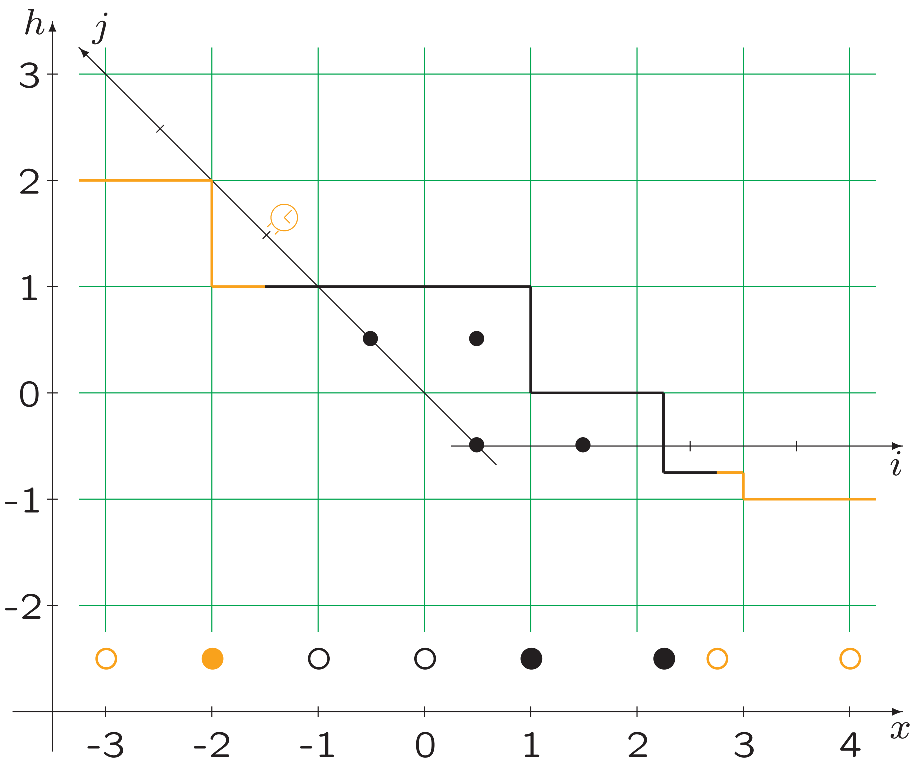


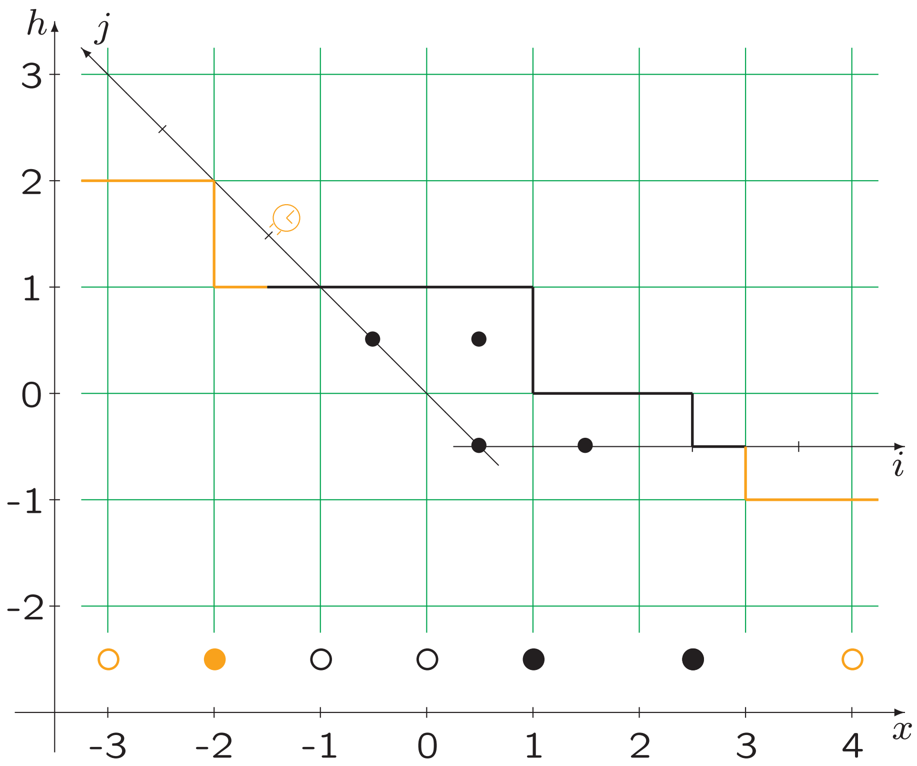
TASEP: Last passage percolation



TASEP: Last passage percolation



TASEP: Last passage percolation



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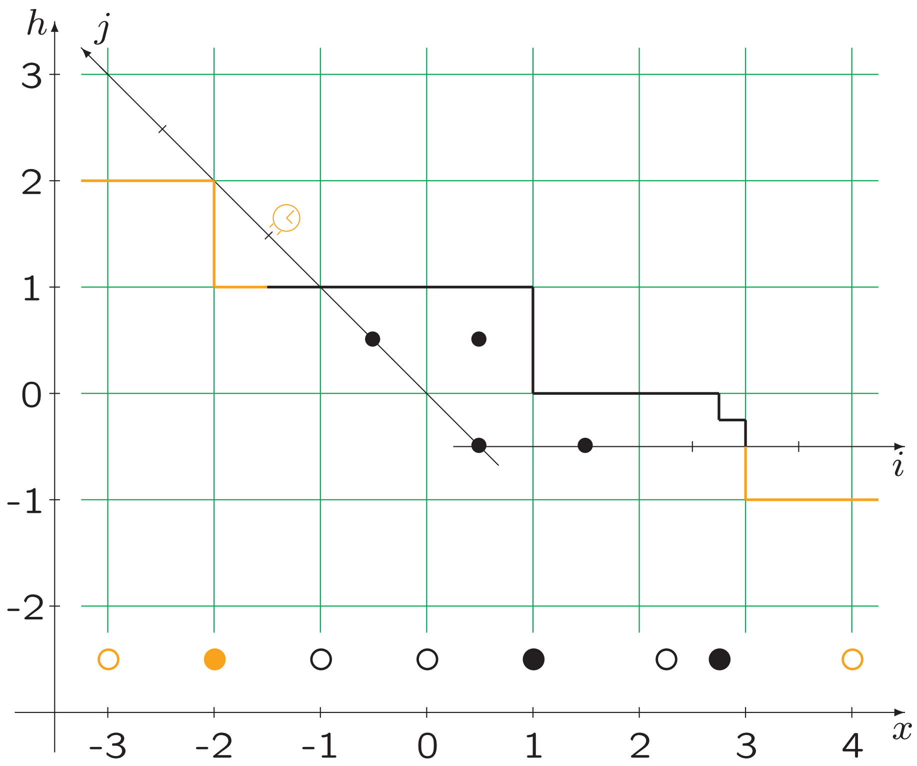

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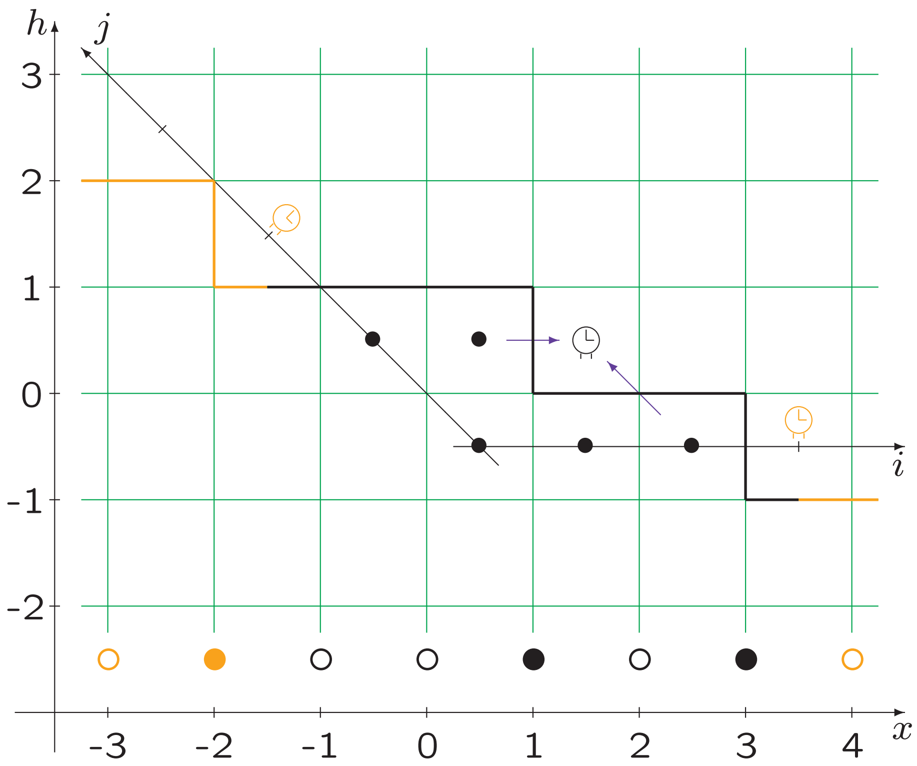

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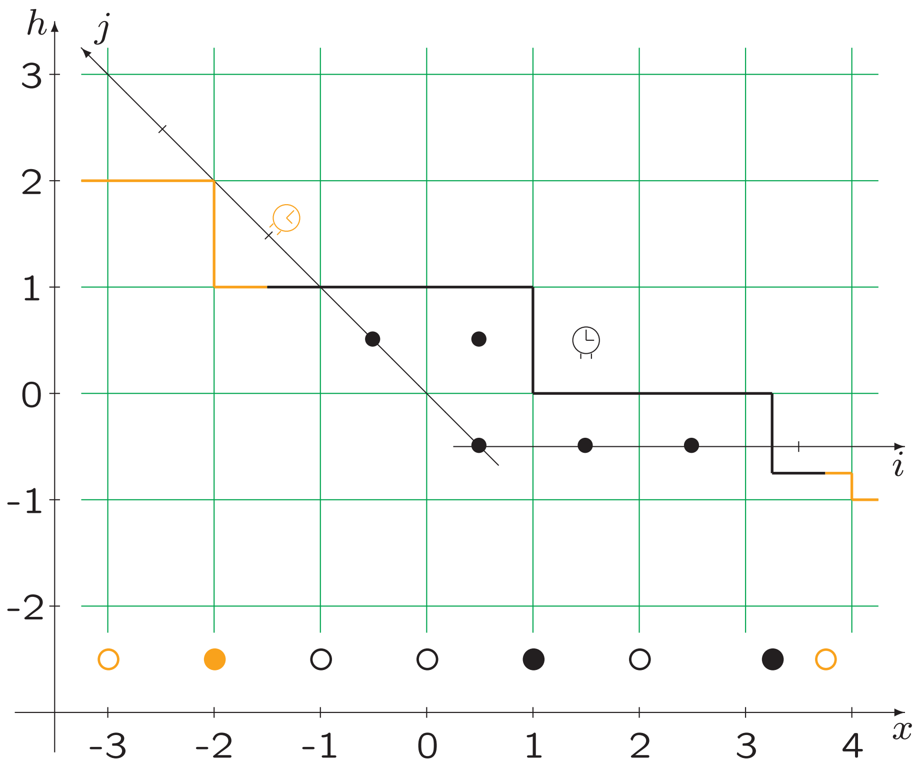

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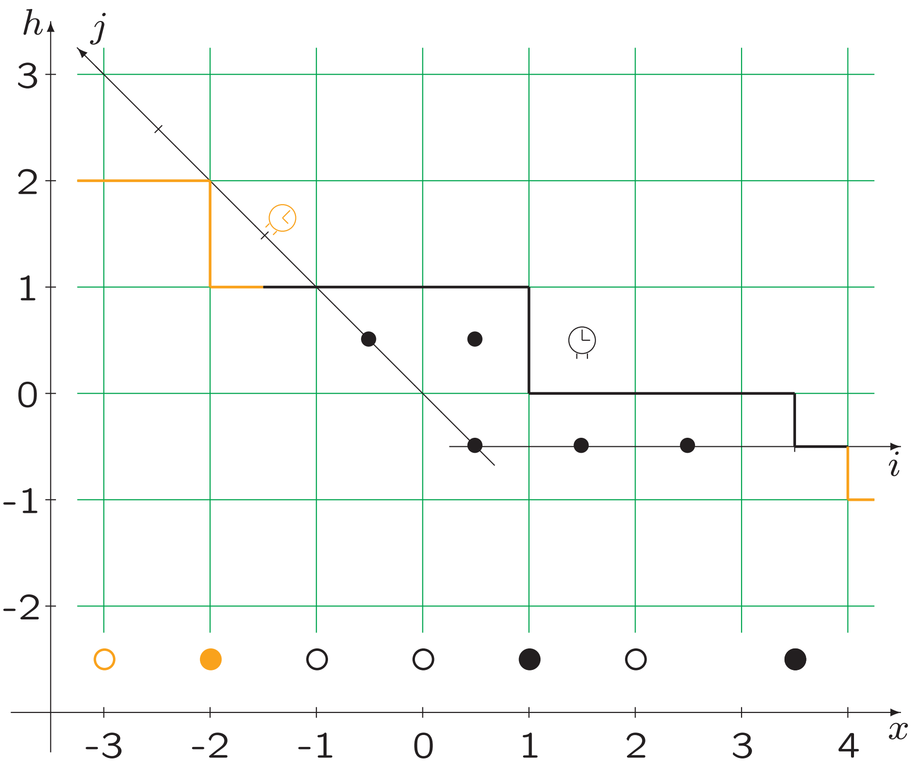

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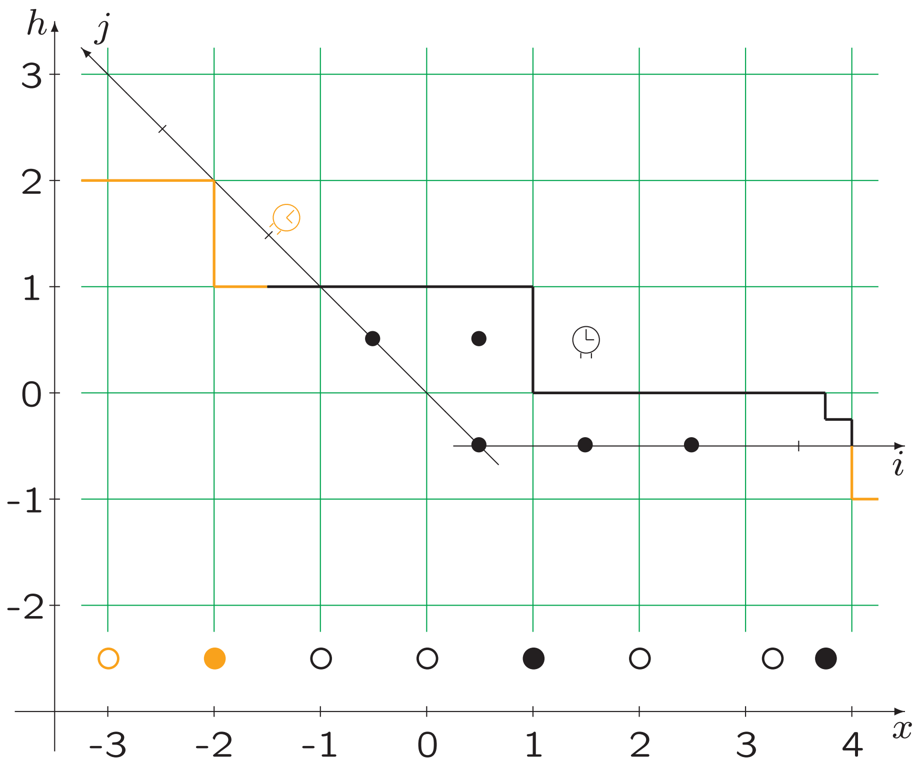

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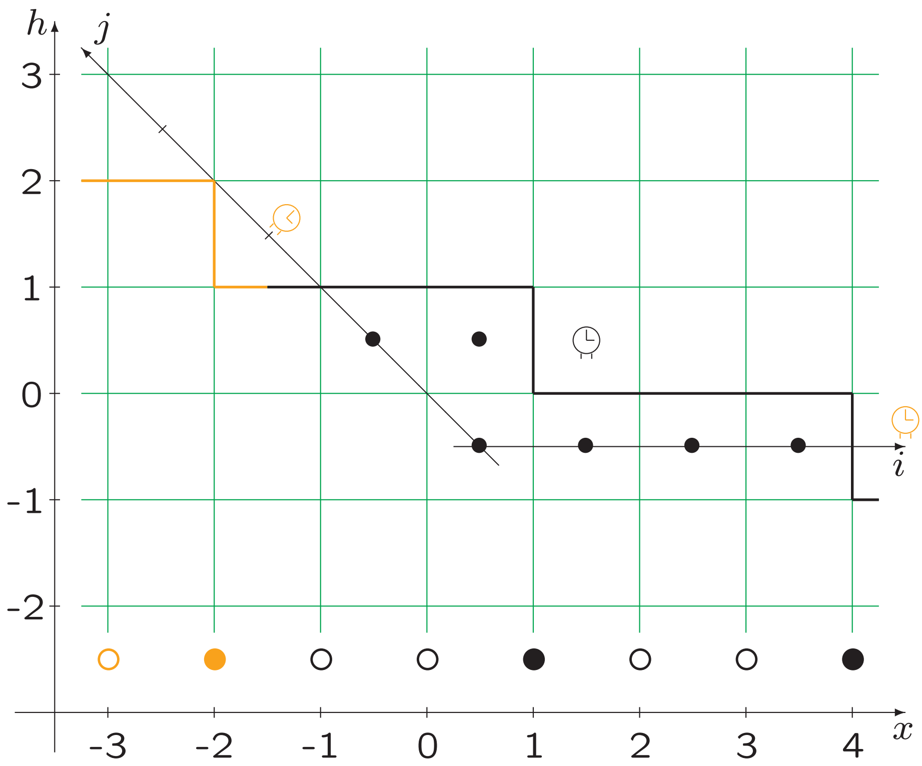

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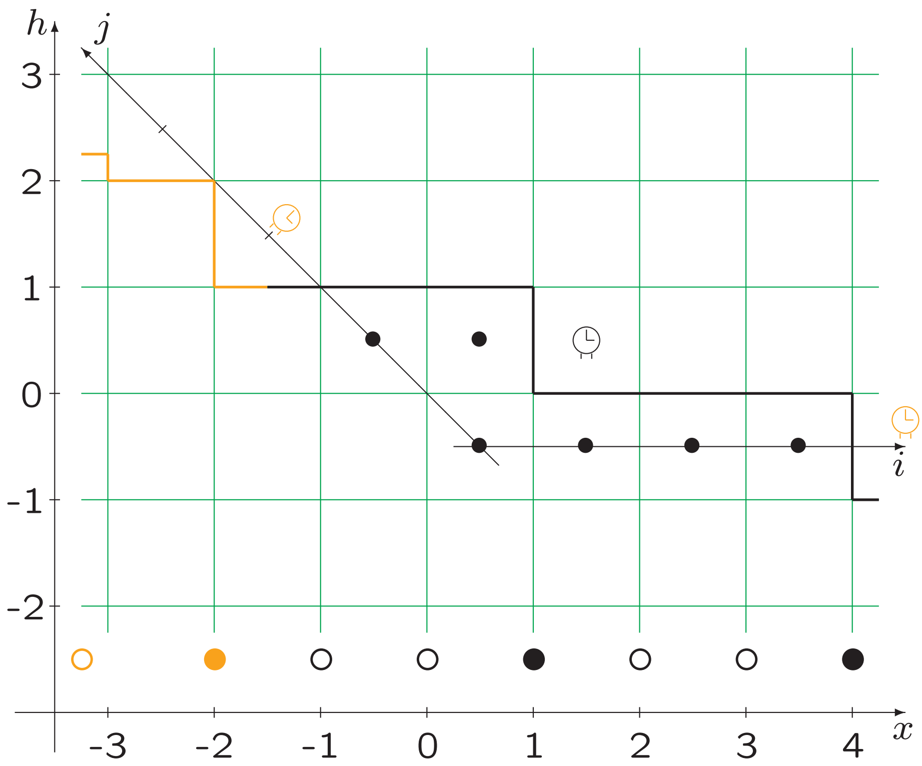

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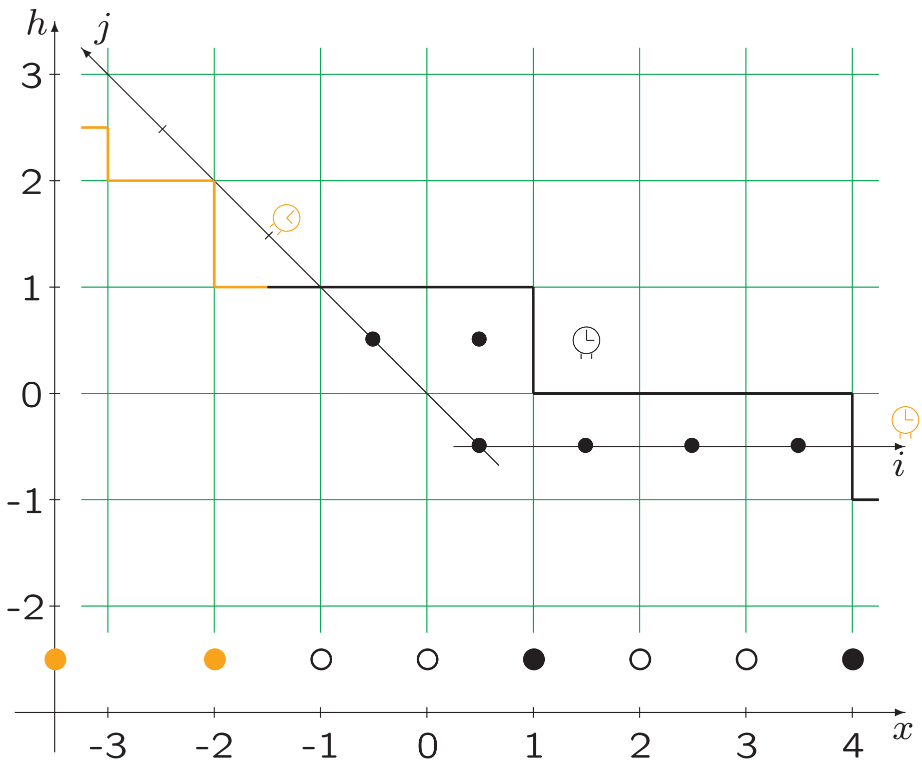

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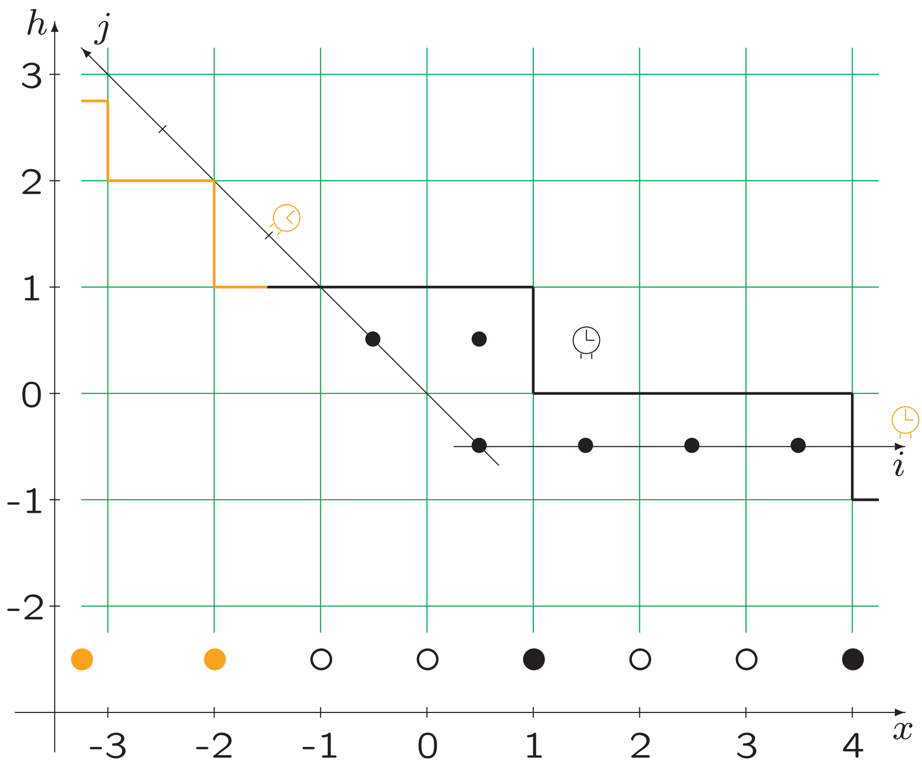

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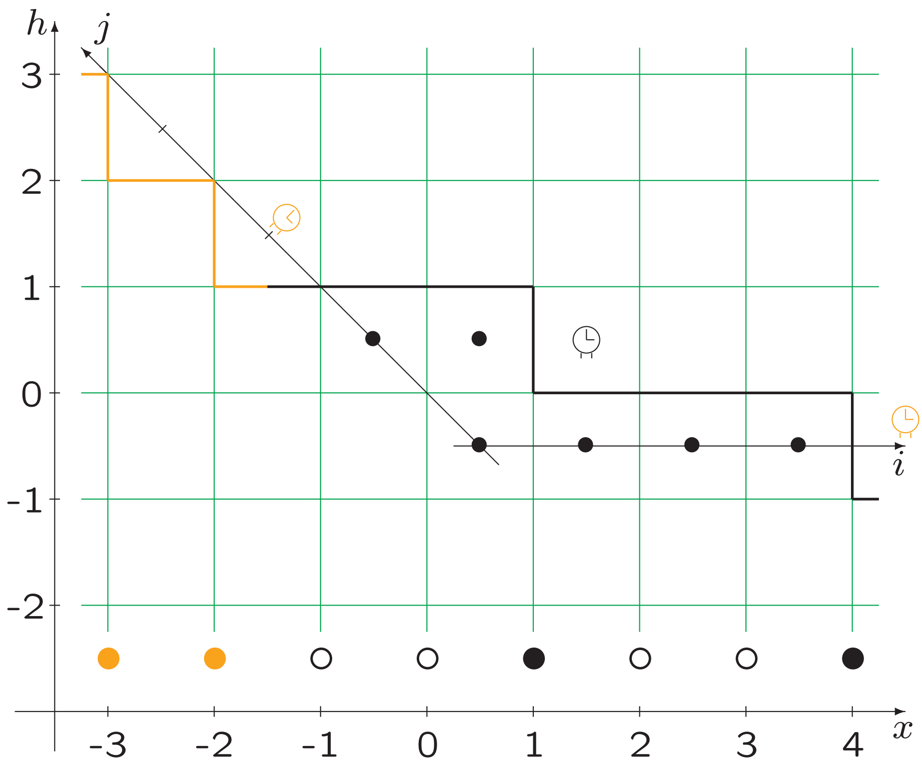

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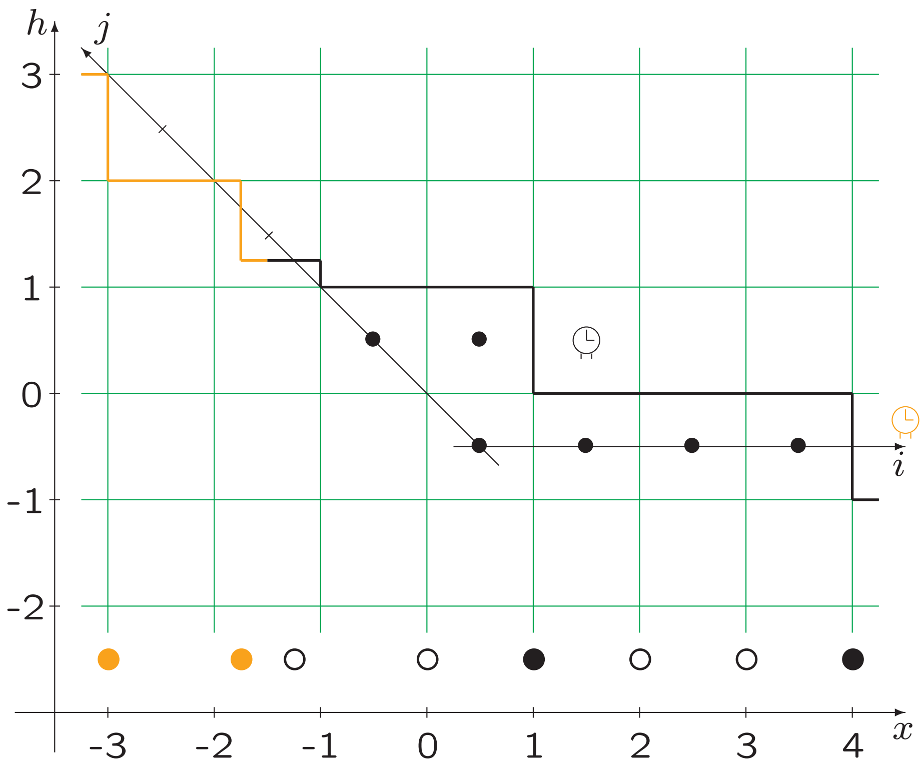

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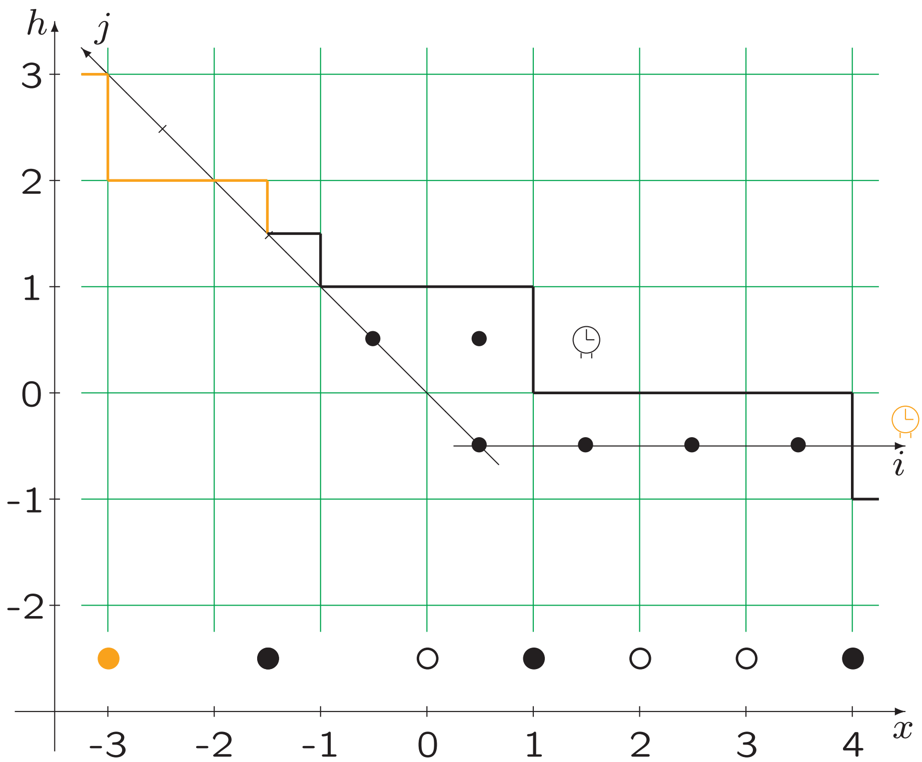

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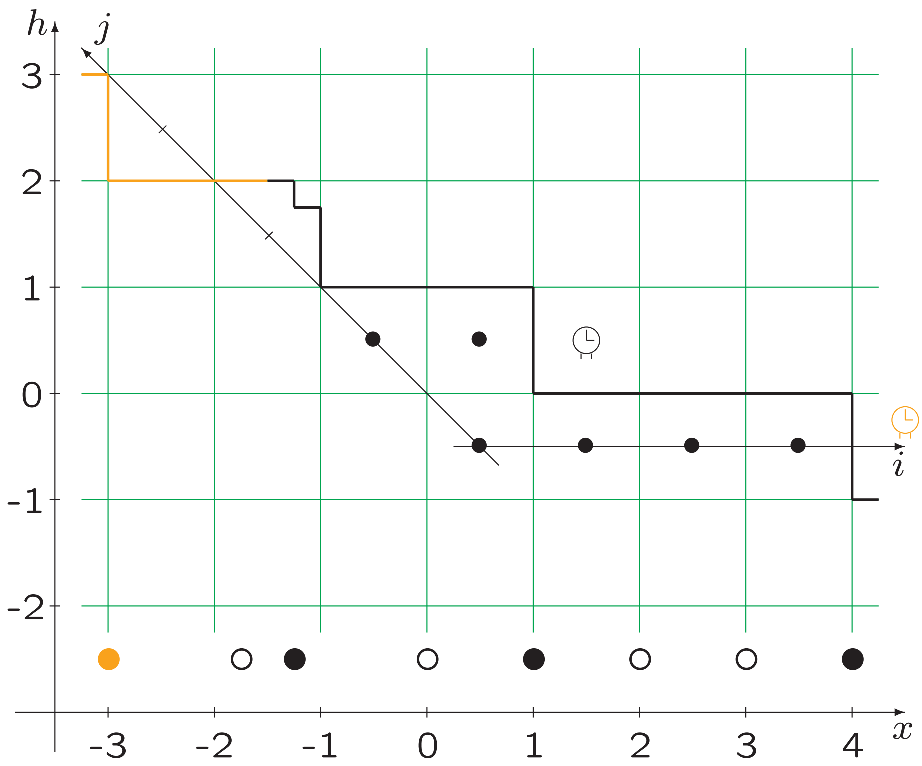

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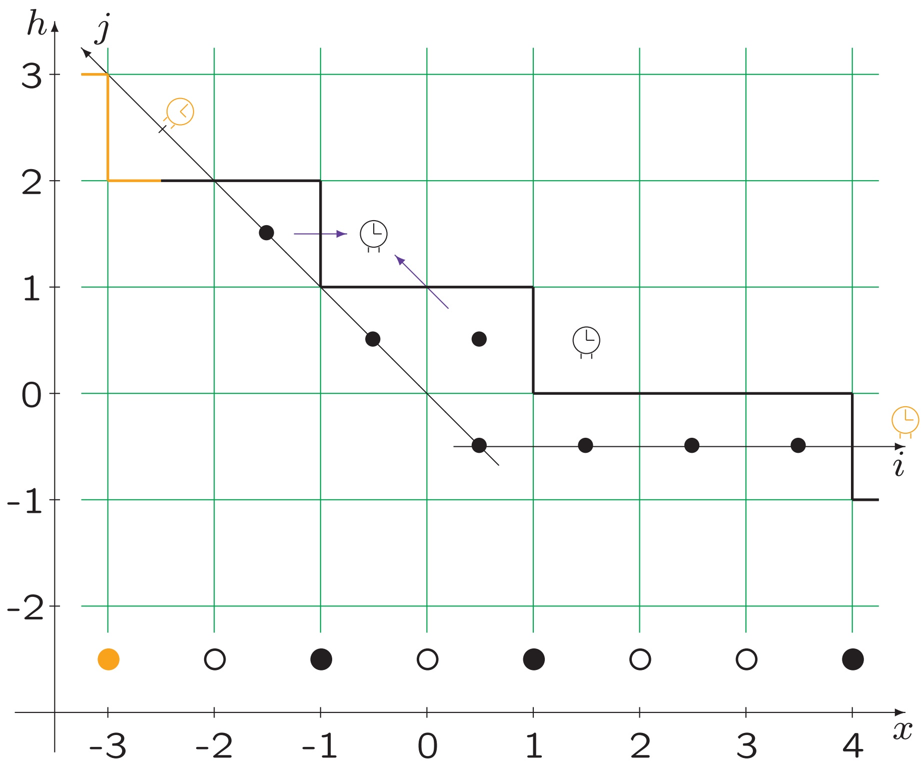

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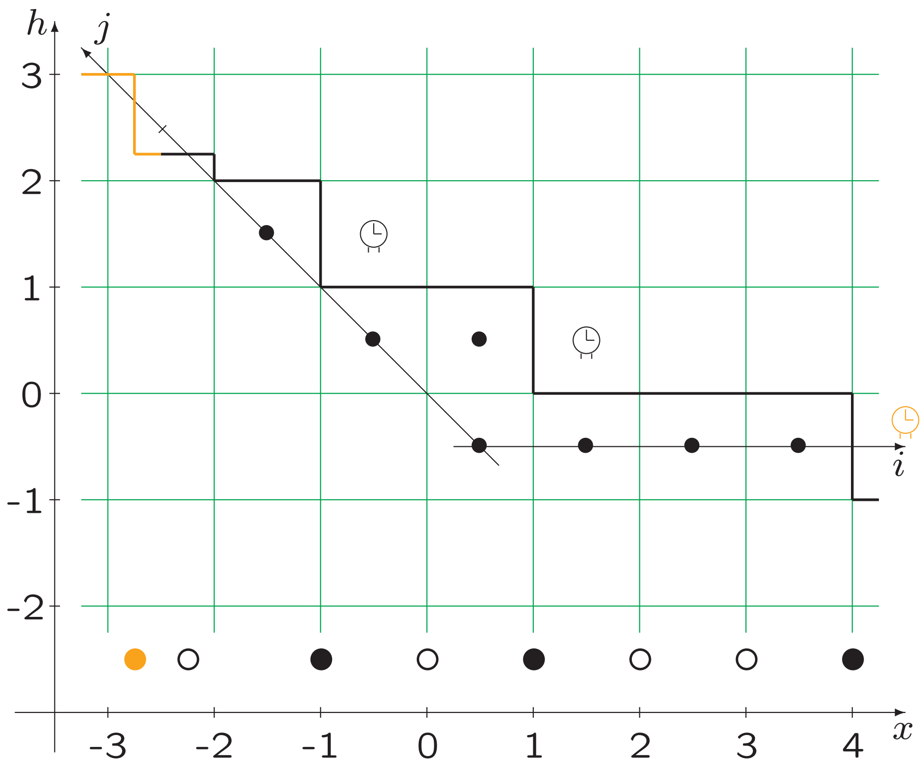
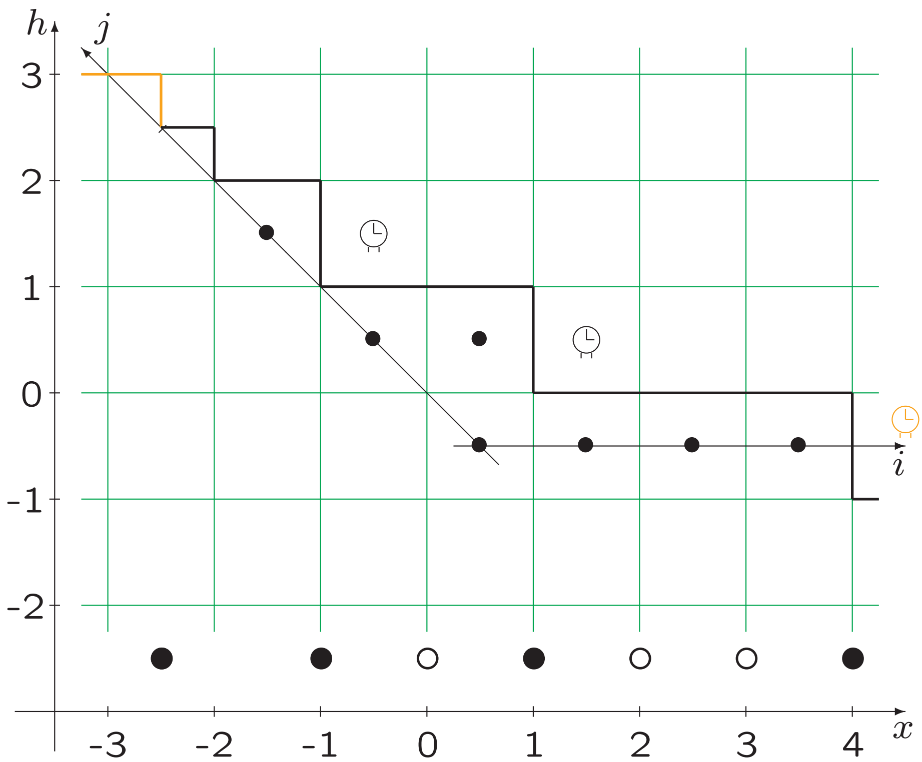
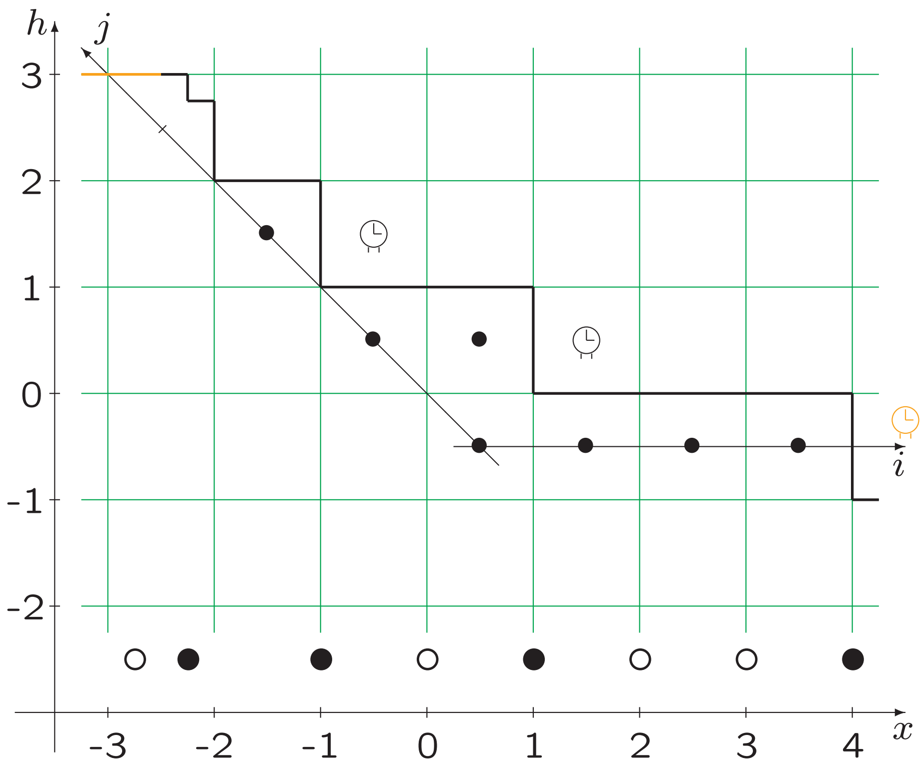


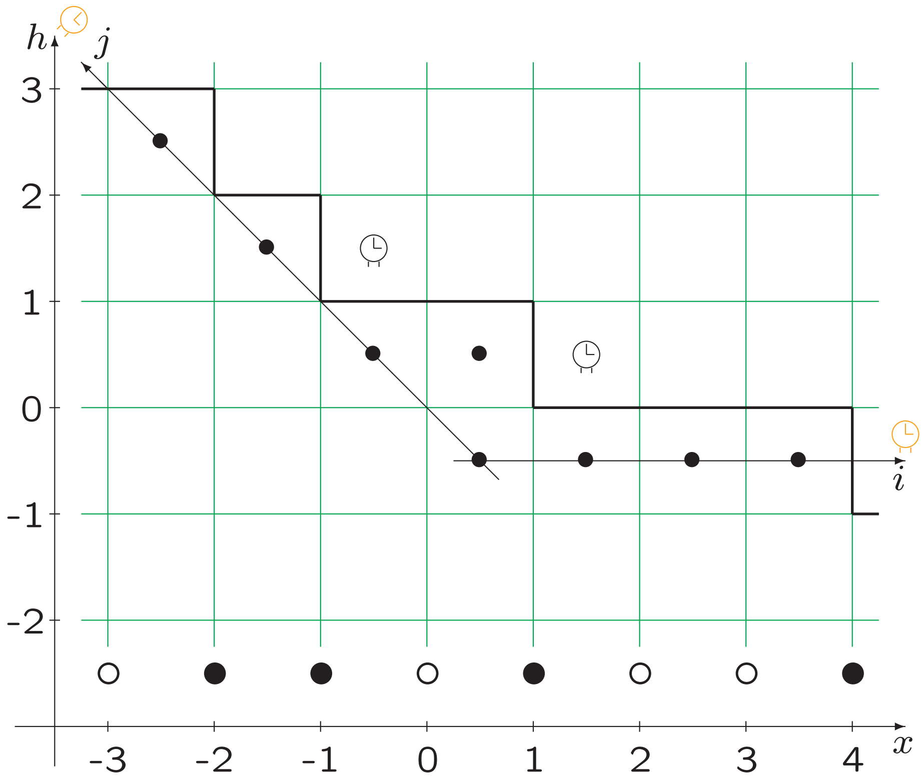
TASEP: Last passage percolation



TASEP: Last passage percolation



TASEP: Last passage percolation



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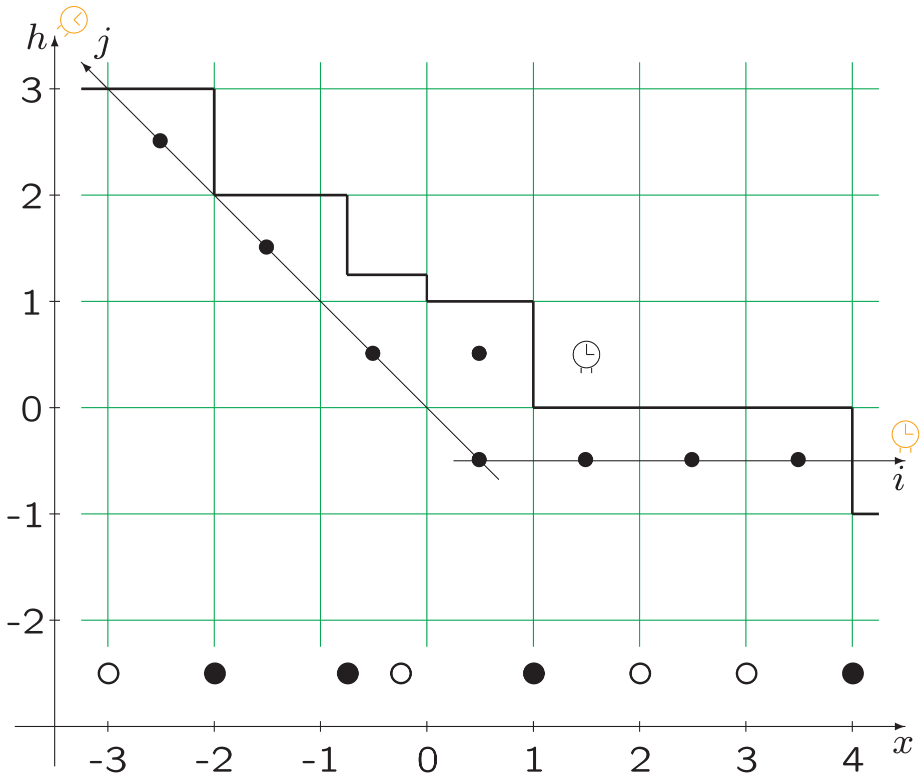
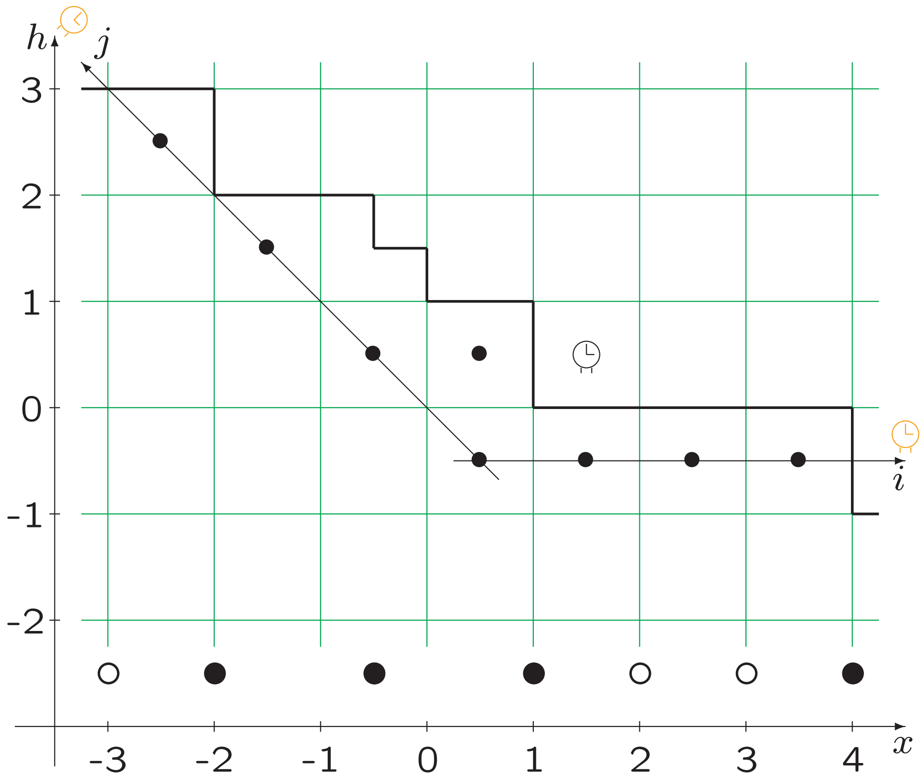


TASEP: Last passage percolation



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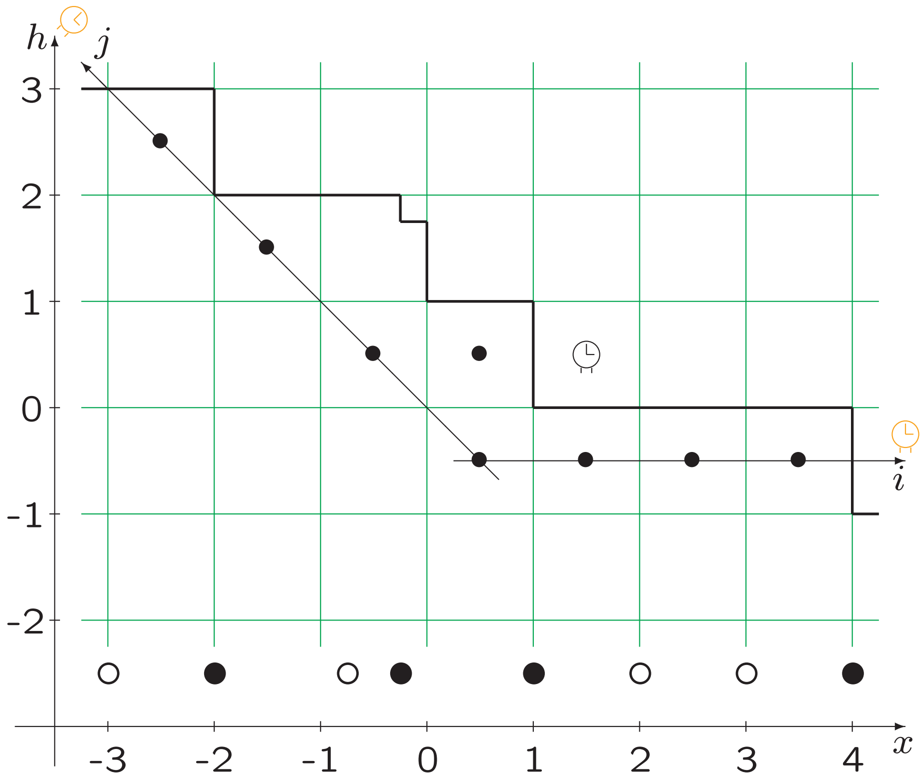

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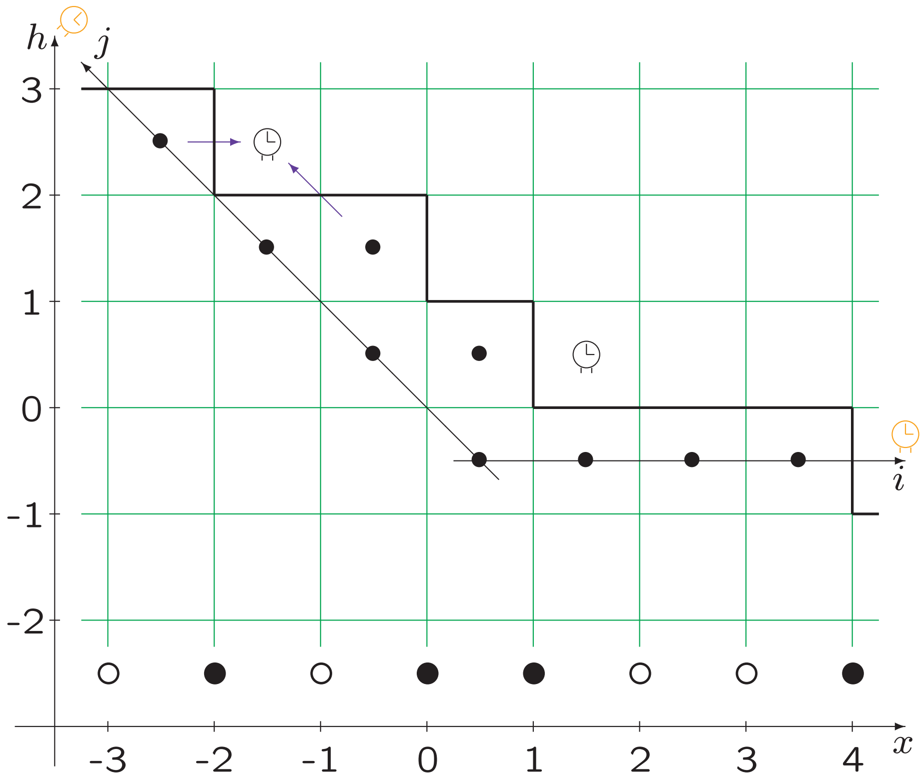

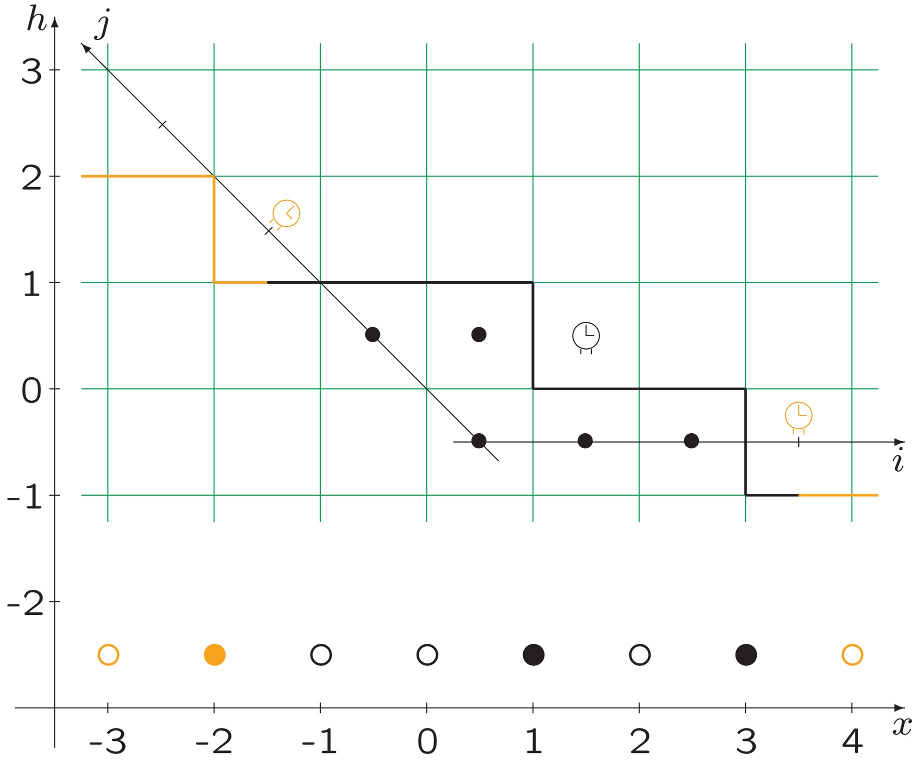
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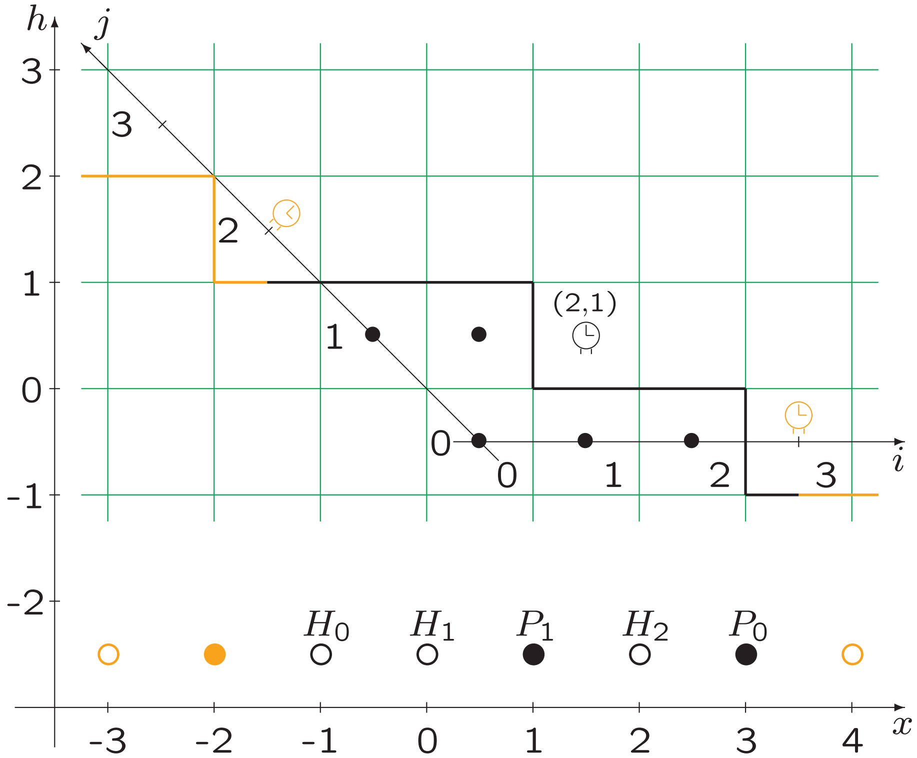

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H_0
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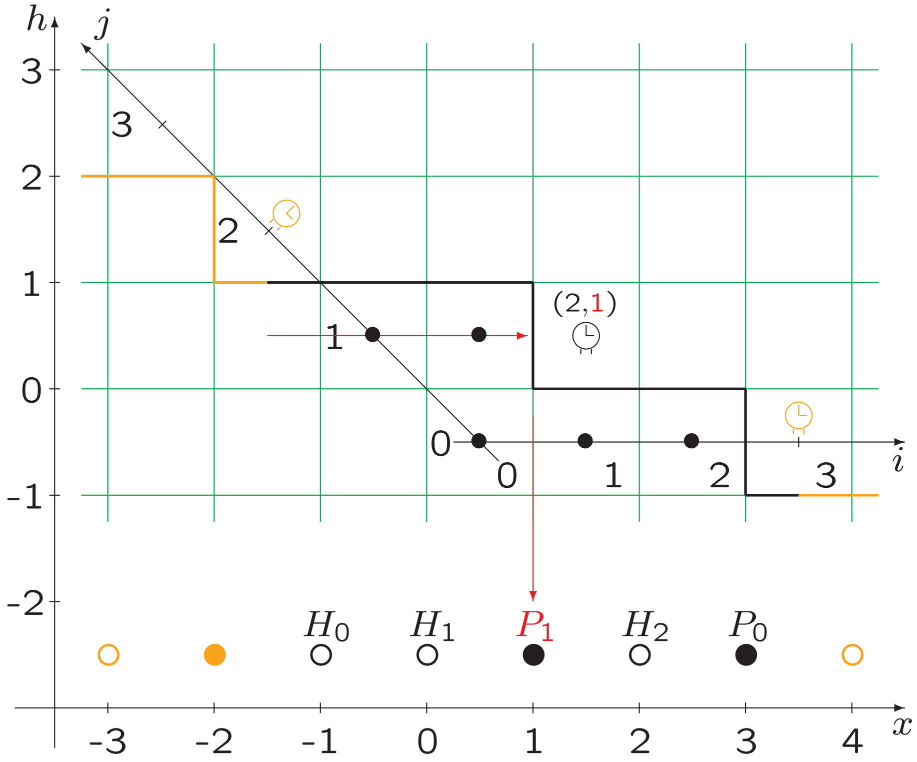
P_1
H_2
P_0



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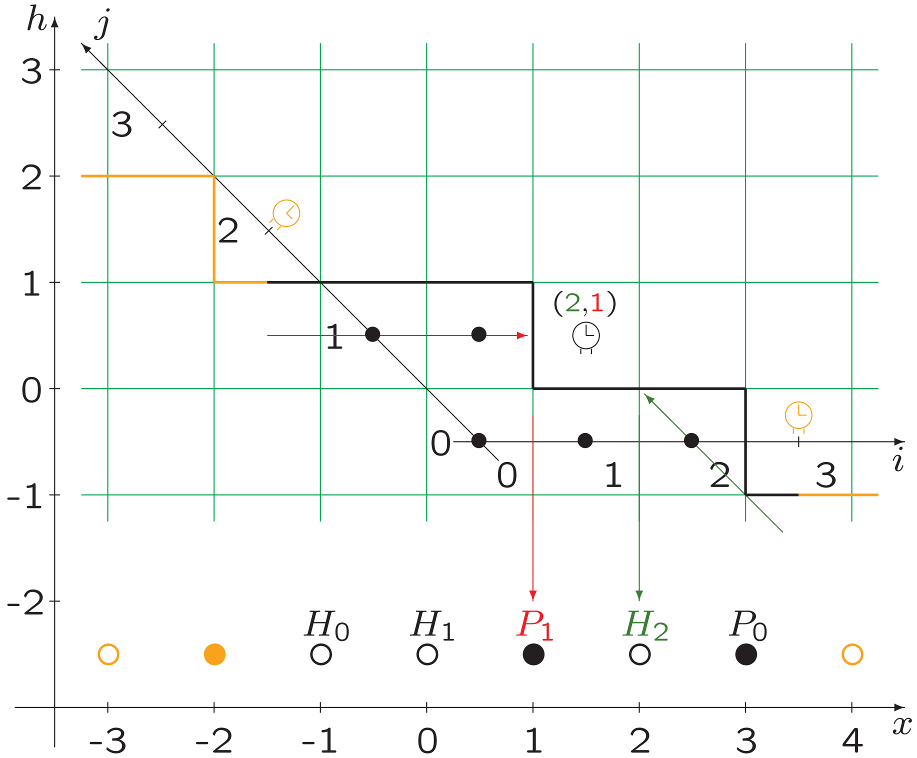

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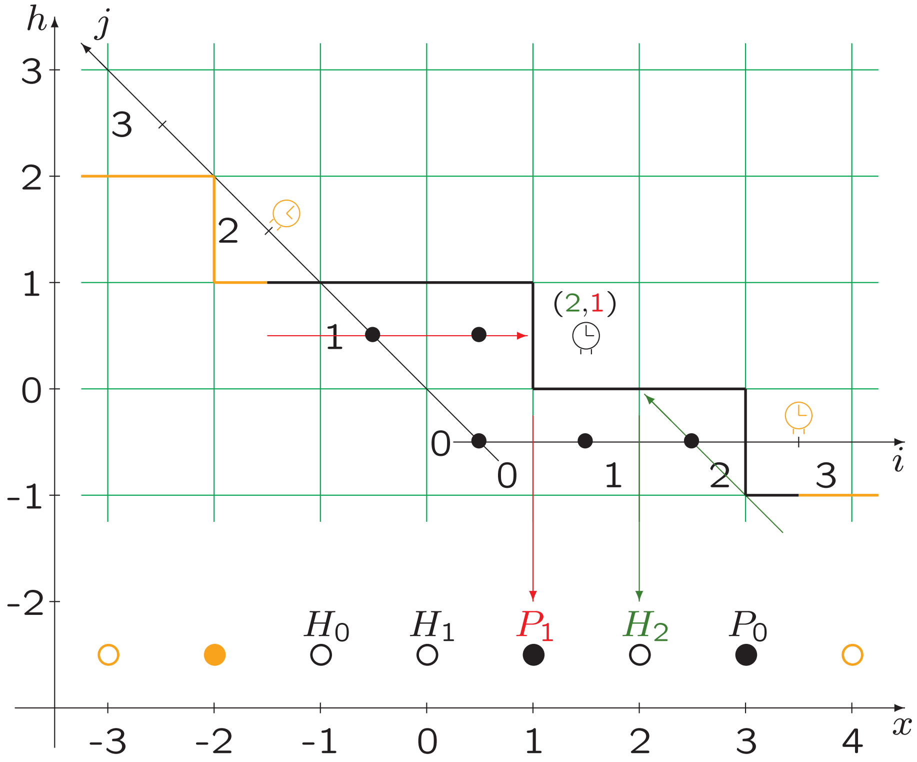
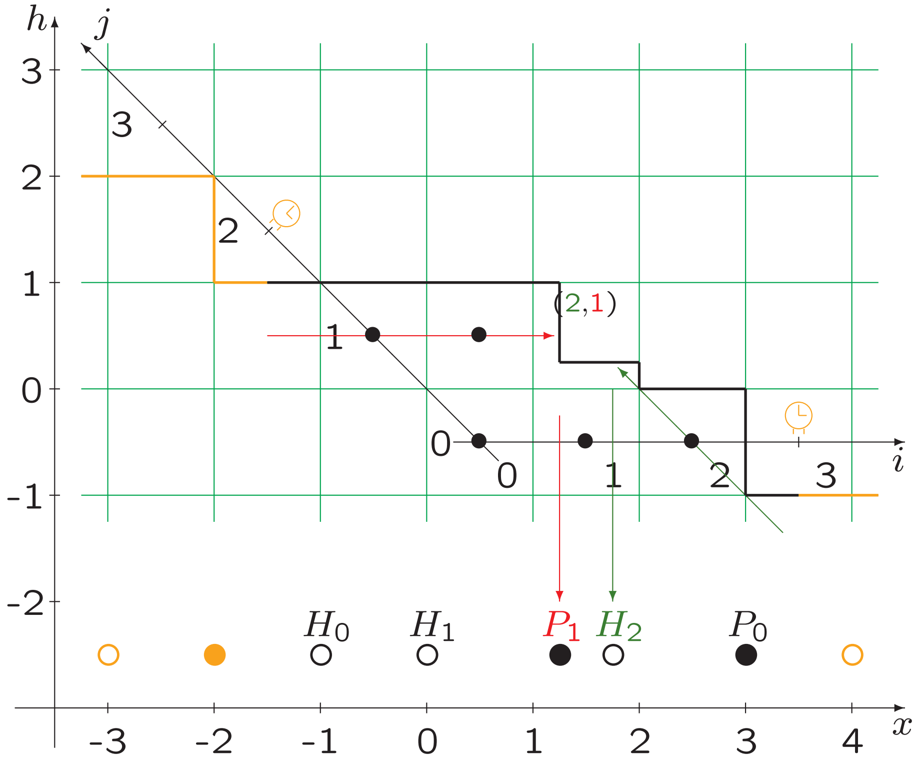


Occupation of (i, j) = jump of P_j over H_i.

Occupation of $(2, 1)$ = jump of P_1 over H_2.



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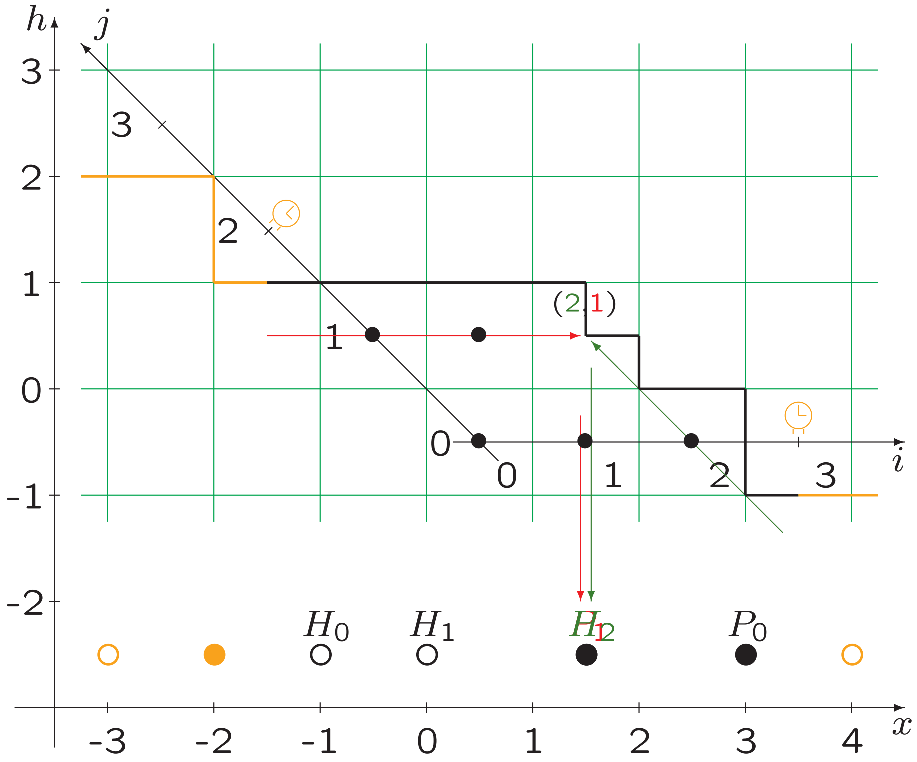

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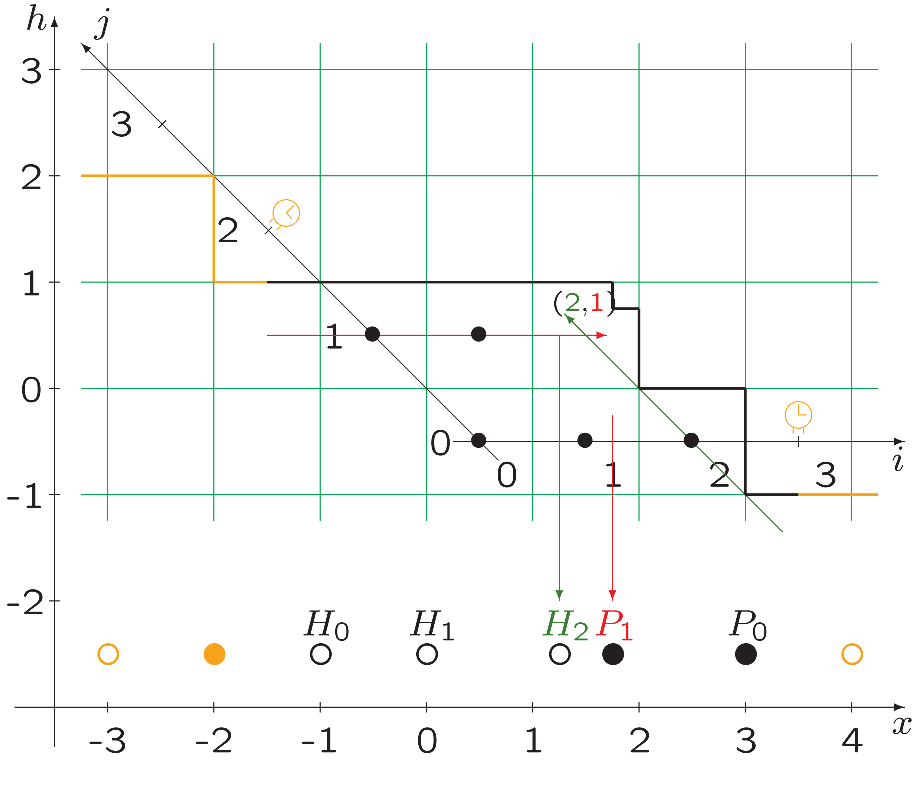

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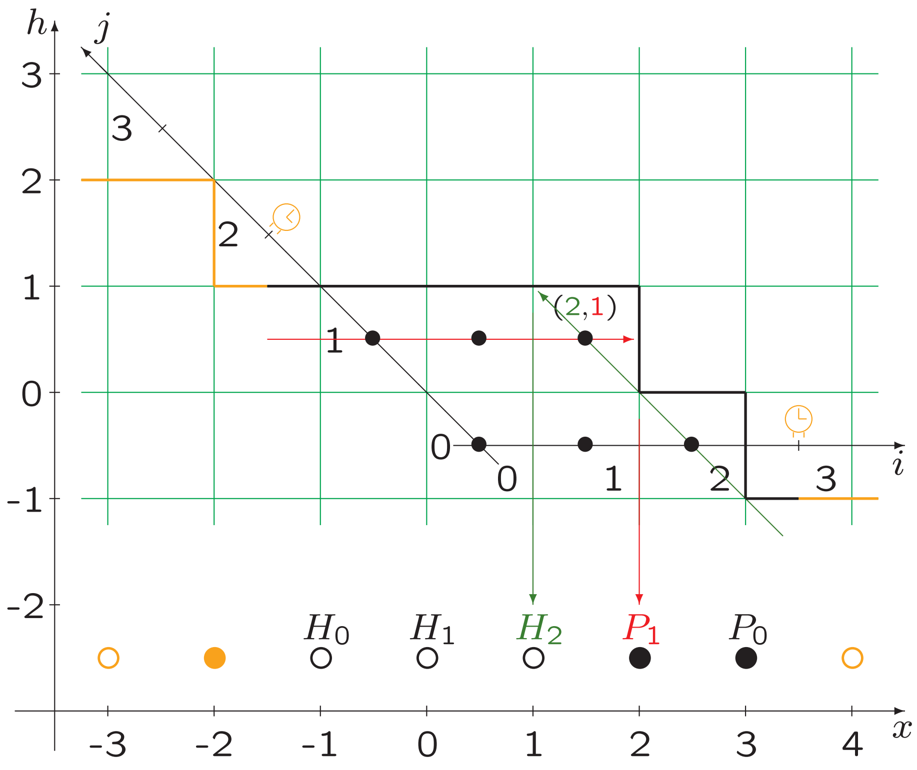

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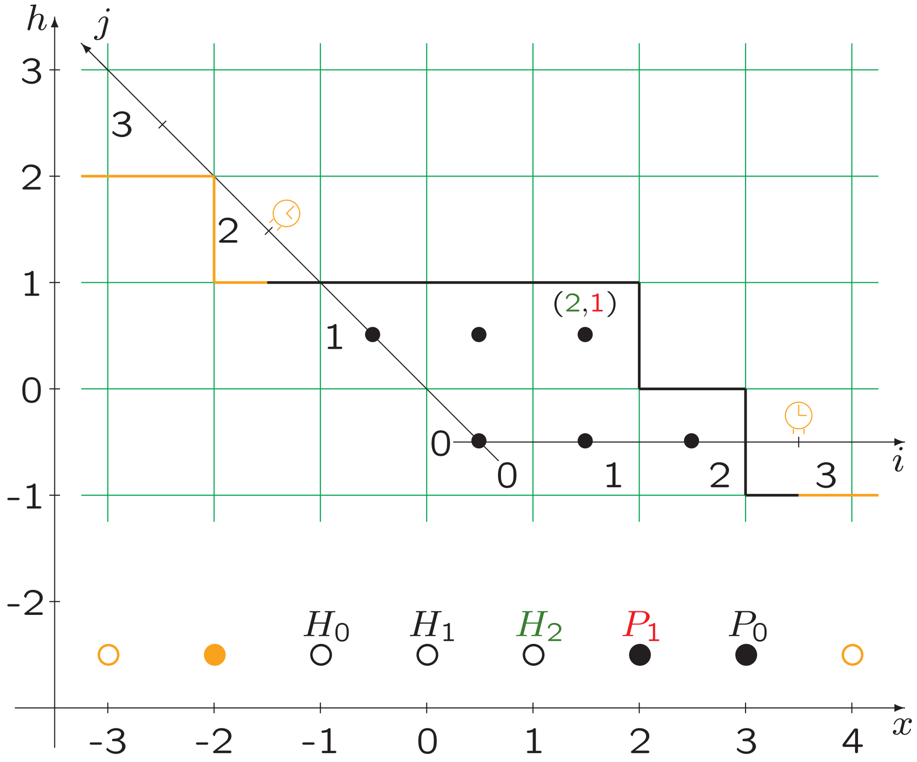
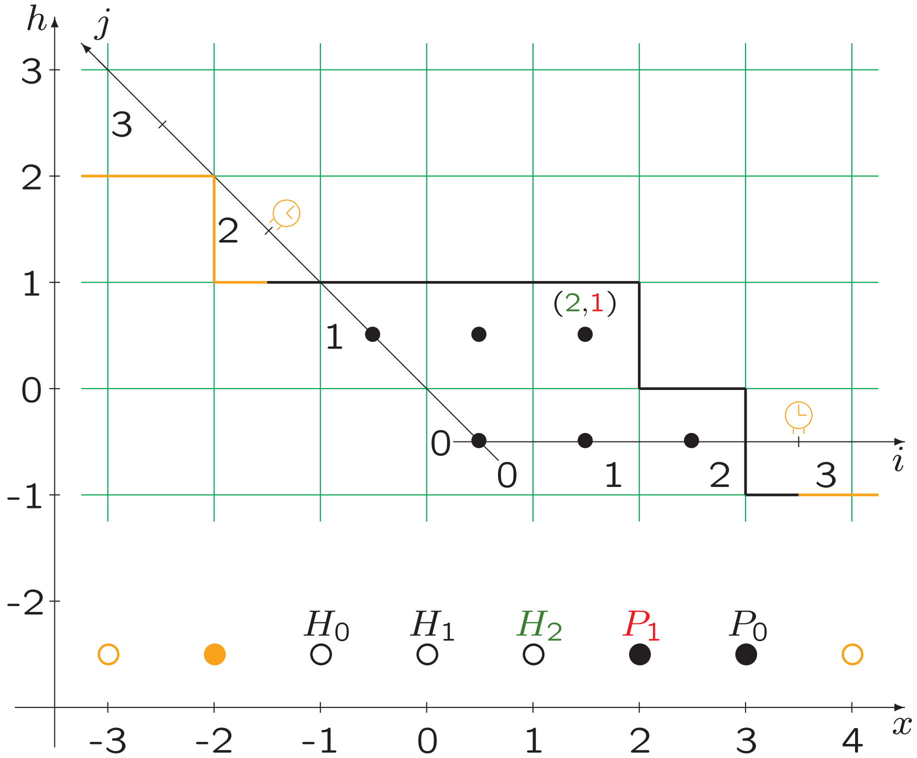


Occupation of (i, j) = jump of P_j over H_i.

Occupation of $(2, 1)$ = jump of P_1 over H_2.

The time when this happens $=: G_{ij}$.



Occupation of (i, j) = jump of P_j over H_i.

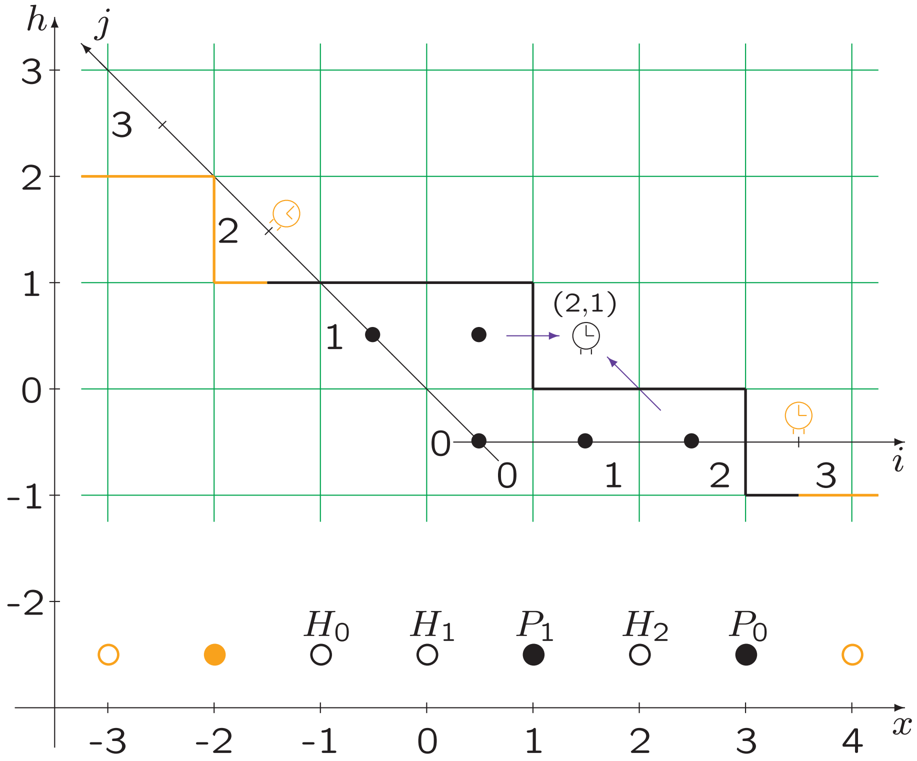
Occupation of $(2, 1)$ = jump of P_1 over H_2.

The time when this happens $=: G_{ij}$.

The characteristic speed $V = C(\varrho)$ translates to

$$m := (1 - \varrho)^2 t \text{ and } n := \varrho^2 t.$$

Will present results on G_{mn}.



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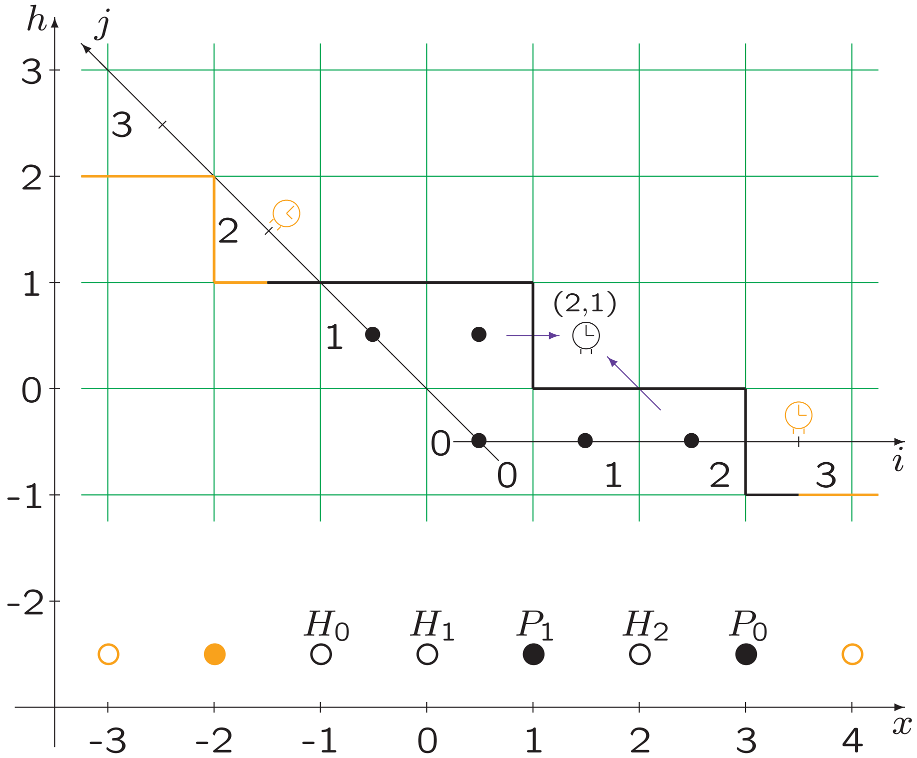
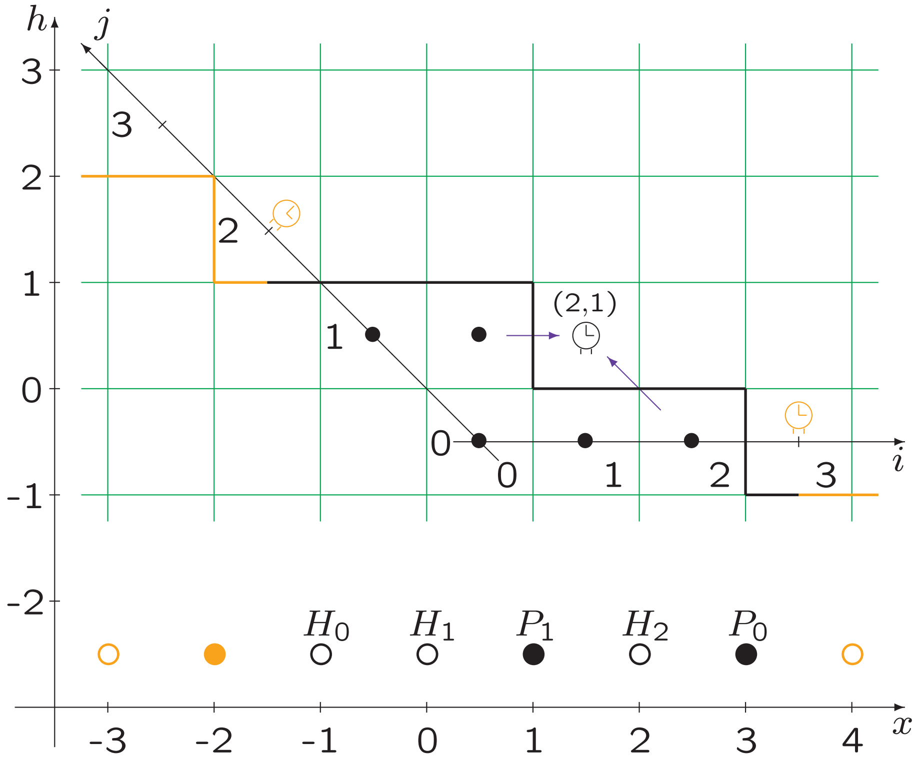


<u>Burke's Theorem:</u>

P_0 jumps according to a Poisson$(1-\varrho)$ process,

governed by the right orange part



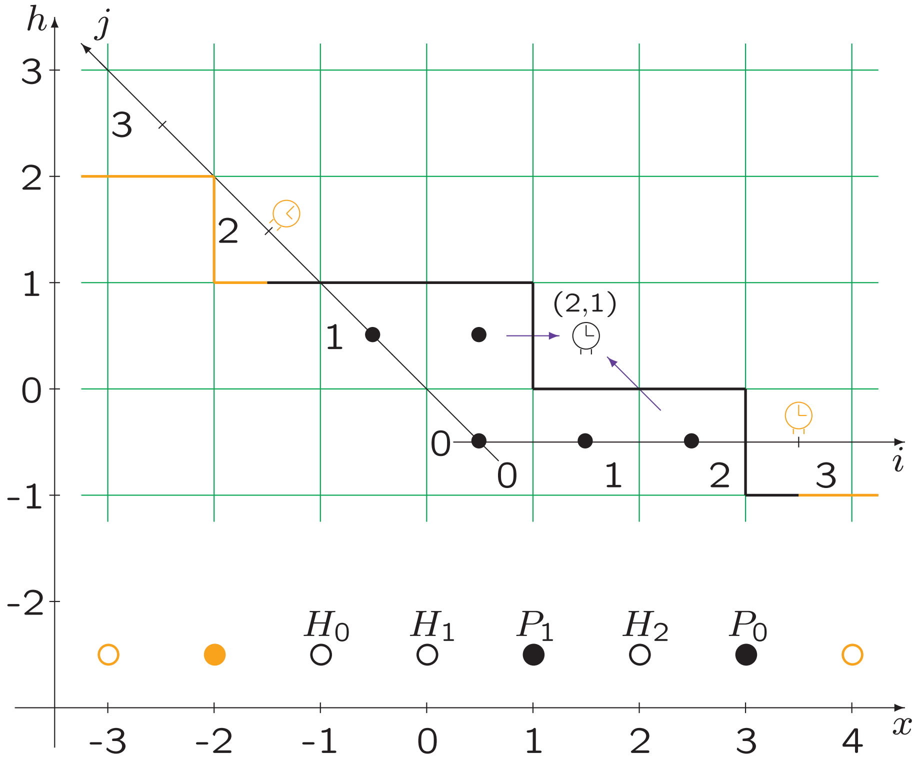
Burke's Theorem:

P_0 jumps according to a Poisson($1-\varrho$) process,

governed by the right orange part

H_0 jumps according to a Poisson(ϱ) process,

governed by the left orange part



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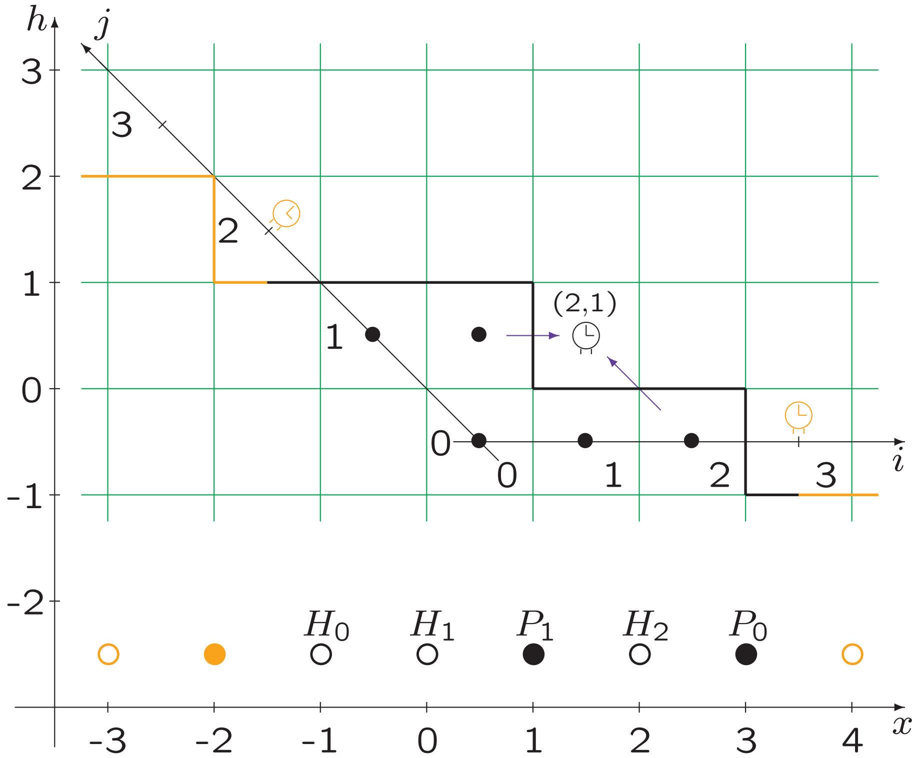
H_0 jumps according to a Poisson(ϱ) process,

governed by the left orange part

independently of the ◷'s.

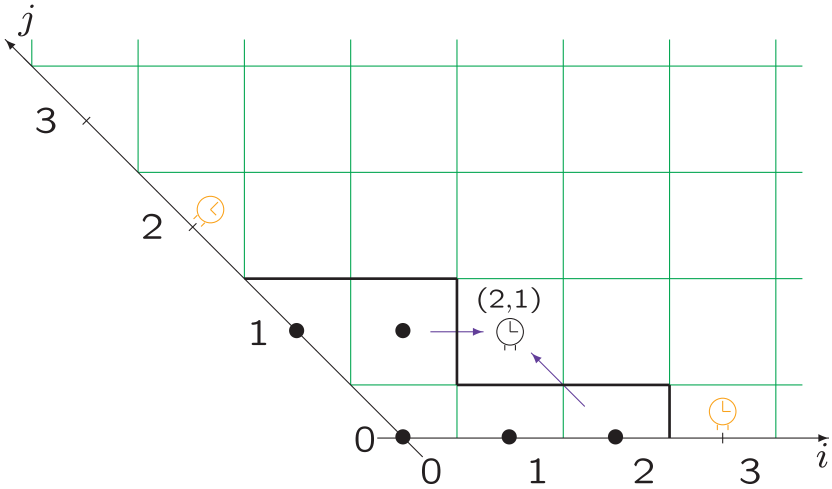


Burke’s Theorem:

P_0 jumps according to a Poisson($1-\varrho$) process,
governed by the right orange part
H_0 jumps according to a Poisson(ϱ) process,
governed by the left orange part
independently of the ◷’s.

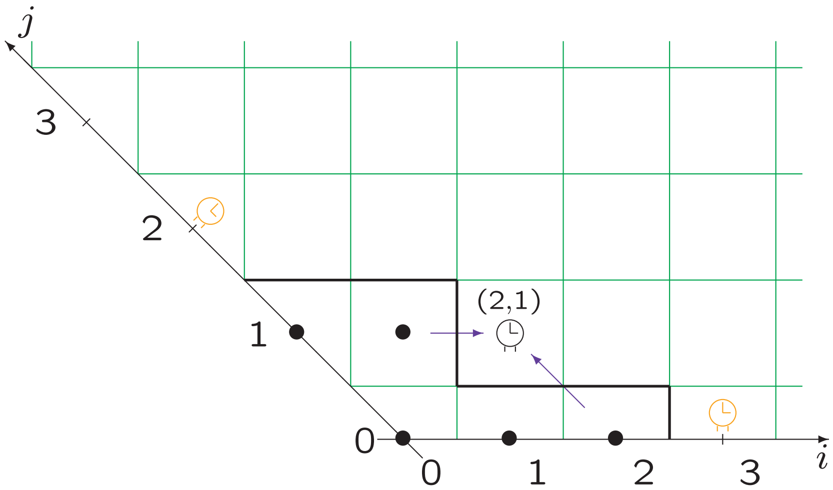
Therefore:

$$\left.\begin{array}{l} ◷ \sim \text{Exponential}(1-\varrho) \\ ◷ \sim \text{Exponential}(\varrho) \\ ◷ \sim \text{Exponential}(1) \end{array}\right\} \text{independently}$$



(orange clock) $\sim$ Exponential$(1 - \varrho)$
(orange tilted clock) $\sim$ Exponential(ϱ) } independently
(black clock) $\sim$ Exponential(1)

The last passage model



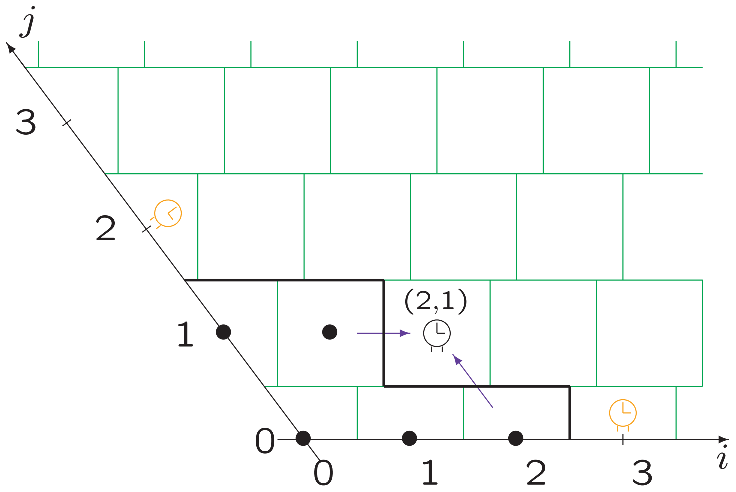
$\sim$ Exponential($1-\varrho$)

$\sim$ Exponential(ϱ)

$\sim$ Exponential(1)

} independently

The last passage model



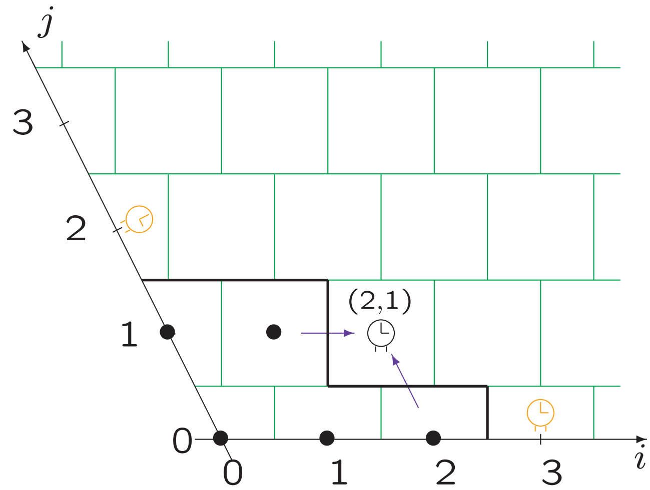
$\sim$ Exponential$(1-\varrho)$

$\sim$ Exponential(ϱ)

$\sim$ Exponential(1)

} independently

The last passage model



$\sim$ Exponential$(1-\varrho)$

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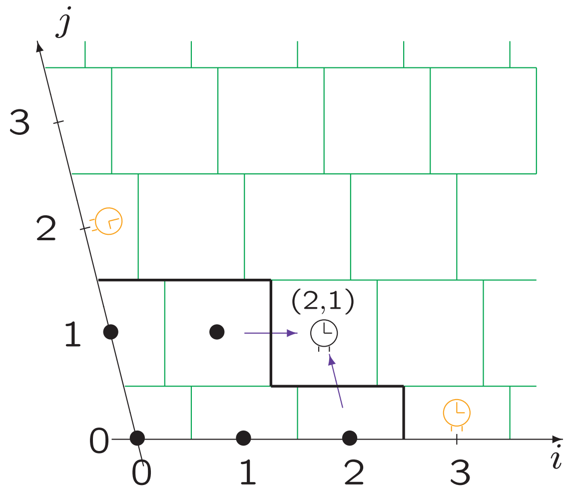

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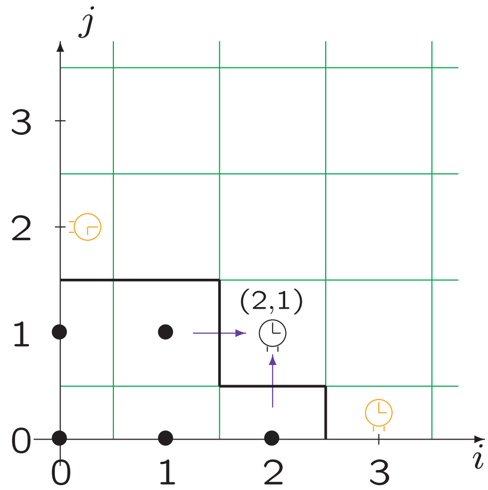
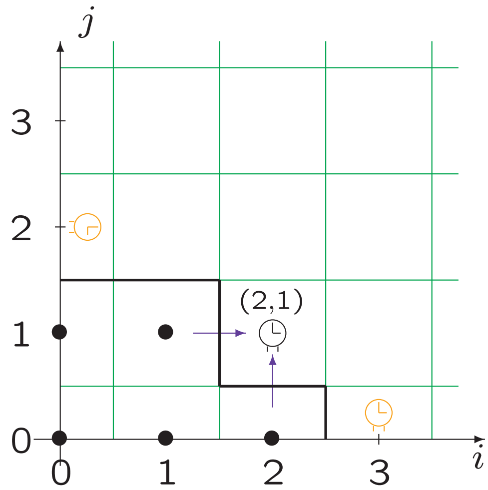
} independently

The last passage model



$$\left.\begin{array}{l} \sim \text{Exponential}(1-\varrho) \\ \sim \text{Exponential}(\varrho) \\ \sim \text{Exponential}(1) \end{array}\right\}$$ independently

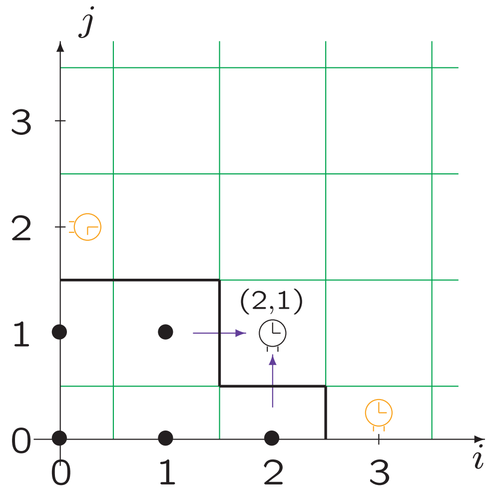
The last passage model



$\sim$ Exponential$(1 - \varrho)$ } independently
$\sim$ Exponential(ϱ)
$\sim$ Exponential(1)

starts ticking when its west neighbor becomes occupied

The last passage model

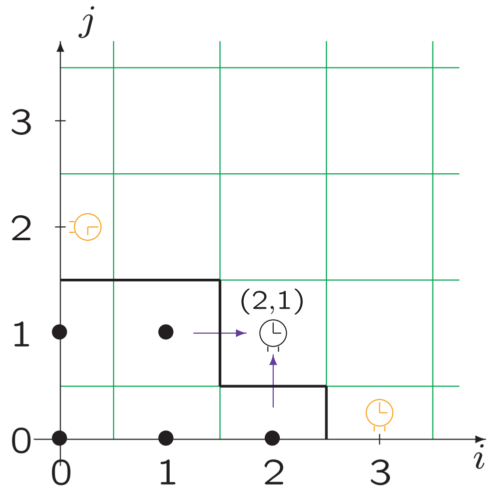


$\sim$ Exponential$(1-\varrho)$
$\sim$ Exponential(ϱ) } independently
$\sim$ Exponential(1)

starts ticking when its west neighbor becomes occupied

starts ticking when its south neighbor becomes occupied

The last passage model



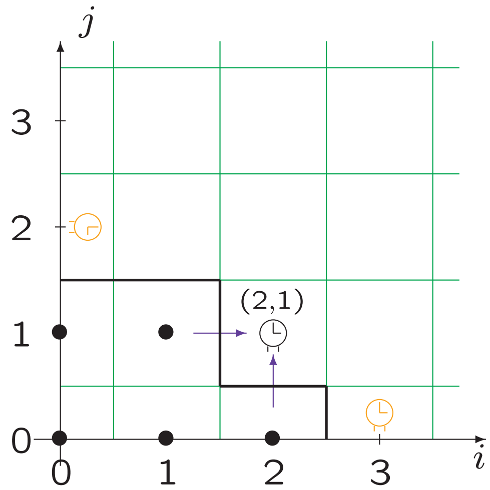
$\sim$ Exponential$(1-\varrho)$
$\sim$ Exponential(ϱ) — independently
$\sim$ Exponential(1)

starts ticking when its west neighbor becomes occupied

starts ticking when its south neighbor becomes occupied

starts ticking when both its west and south neighbors become occupied

The last passage model



M. Prähofer and H. Spohn 2002

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$\sim$ Exponential(1)

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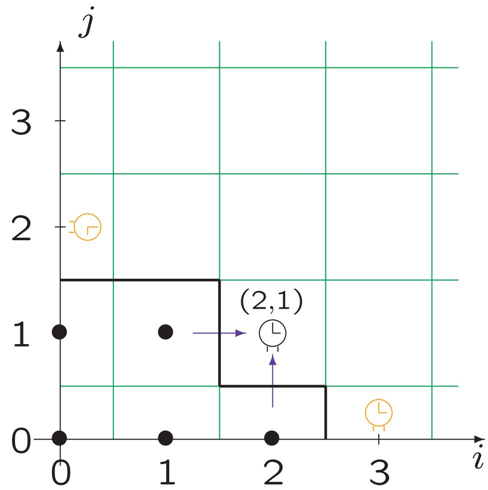

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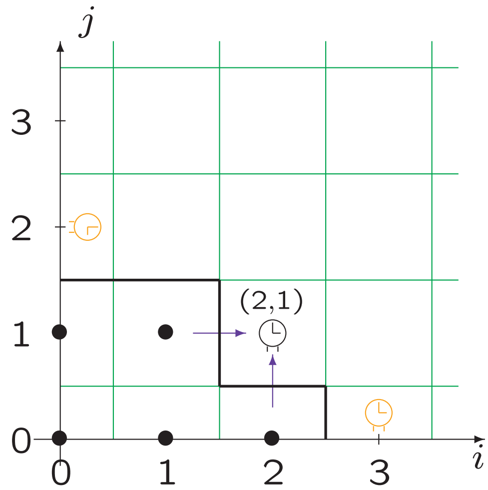
G_{ij} = the occupation time of (i, j)

The last passage model



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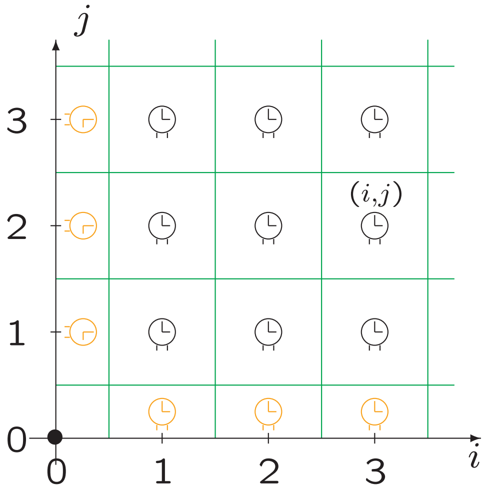
G_{ij} = the occupation time of (i, j)

G_{ij} = the maximum weight collected by a north-east path from $(0, 0)$ to (i, j).

The last passage model



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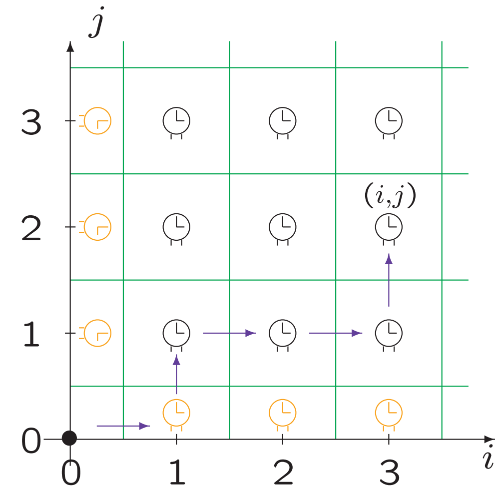

M. Prähofer and H. Spohn 2002

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G_{ij} = the occupation time of (i,j)

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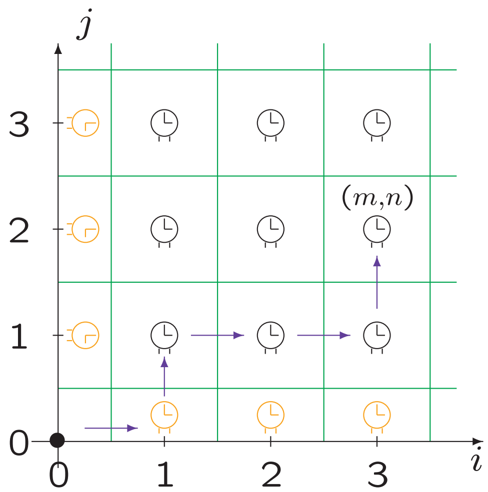
Results



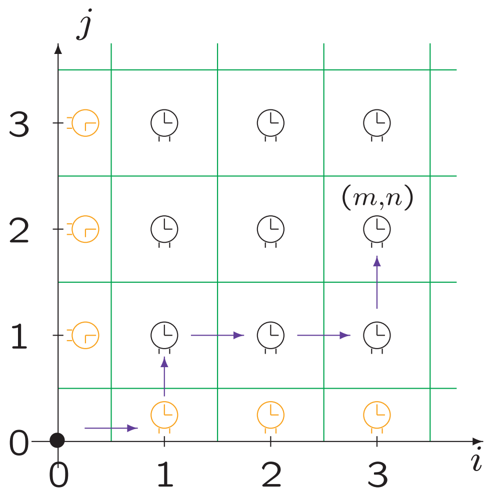
On the characteristics

$$m := (1-\varrho)^2 t \text{ and } n := \varrho^2 t,$$

Theorem:

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(G_{mn})}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(G_{mn})}{t^{2/3}} < \infty.$$

Results



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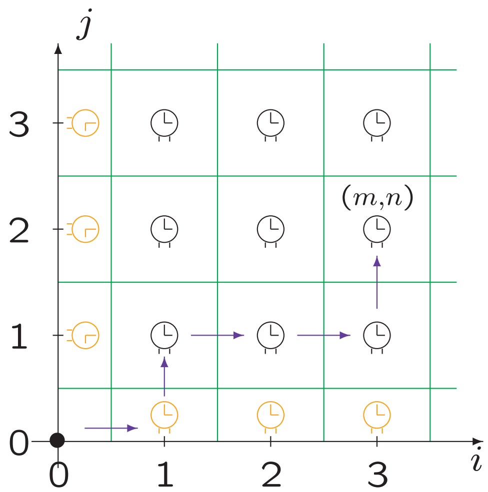
Johansson (2000) identifies the limiting distribution of G_{mn} in terms of Tracy-Widom GUE distributions, when  and  $\sim$ Exponential(1).

Results



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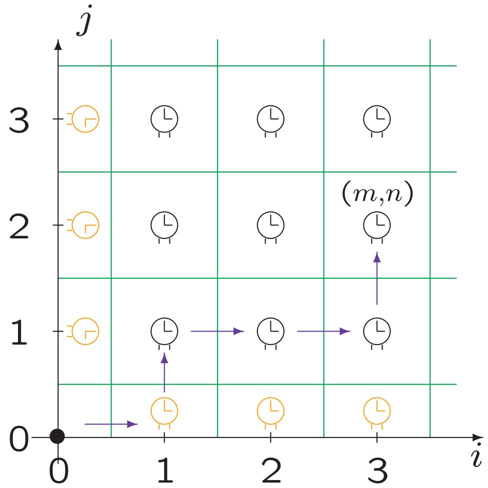
P. L. Ferrari and H. Spohn (2005) identify the limiting distribution off the characteristics by $t^{1/3}$.

Results



On the characteristics

$$m := (1 - \varrho)^2 t \text{ and } n := \varrho^2 t,$$

Theorem:

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(G_{mn})}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(G_{mn})}{t^{2/3}} < \infty.$$

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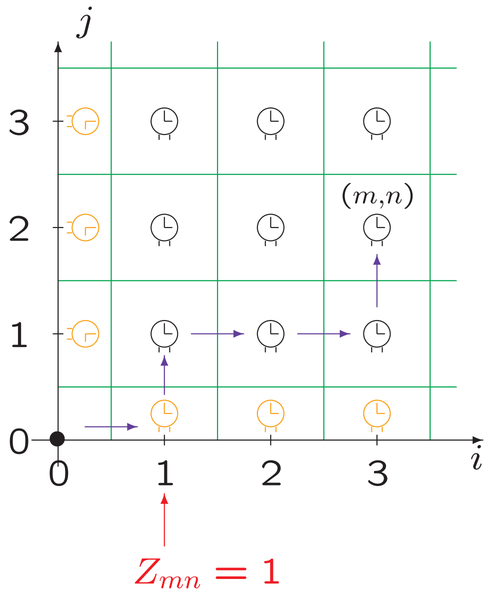
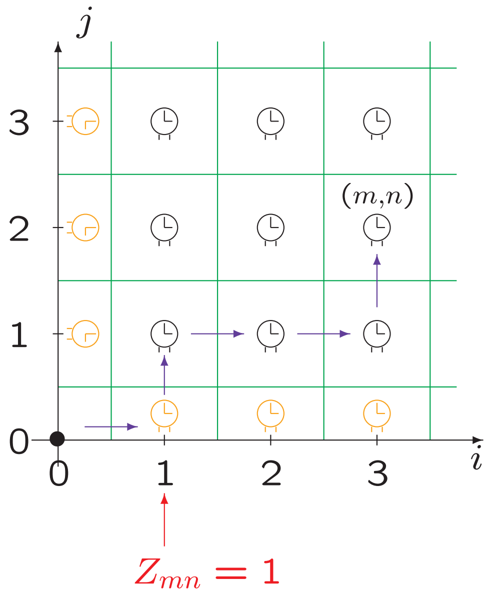
P. L. Ferrari and H. Spohn (2005) identify the limiting distribution off the characteristics by $t^{1/3}$.

Their method: RSK correspondence, random matrices.



Z_{mn} is the exit point of the longest path to

$$(m,\ n) = ((1-\varrho)^2 t,\ \varrho^2 t).$$



Z_{mn} is the exit point of the longest path to

$$(m,\, n) = ((1-\varrho)^2 t,\, \varrho^2 t).$$

Theorem:

For all large t and all $a > 0$,

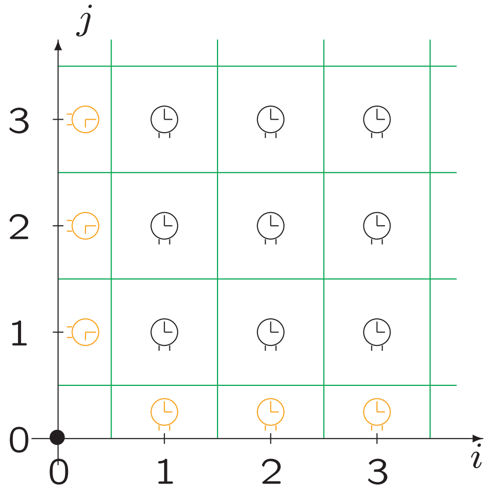
$$\mathbf{P}\{Z_{mn} \geq a t^{2/3}\} \leq C a^{-3}.$$

Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mathbf{P}\{1 \leq Z_{mn} \leq \delta t^{2/3}\} \leq \varepsilon$$

for all large t.

Last passage equilibrium



Equilibrium:

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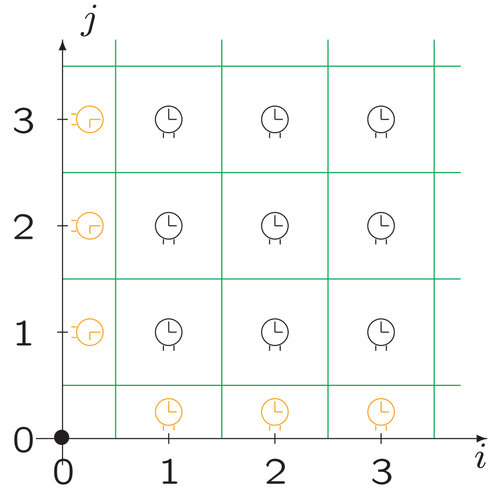
$\sim$ Exponential(ϱ)

$\sim$ Exponential(1)

independently

Last passage equilibrium



Equilibrium:

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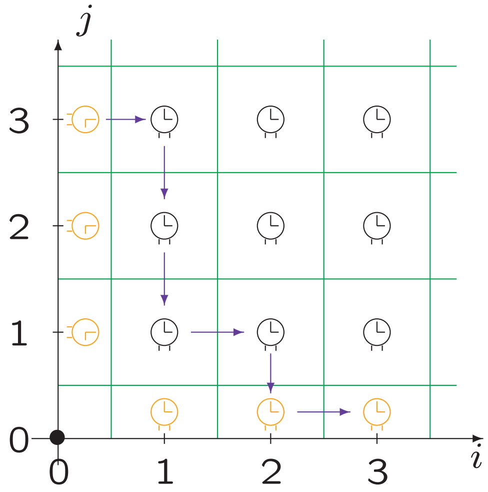
} independently

G-increments:

$$I_{ij} := G_{ij} - G_{\{i-1\}j} \quad \text{for } i \geq 1,\ j \geq 0, \qquad \text{and}$$

$$J_{ij} := G_{ij} - G_{i\{j-1\}} \quad \text{for } i \geq 0,\ j \geq 1.$$

Last passage equilibrium



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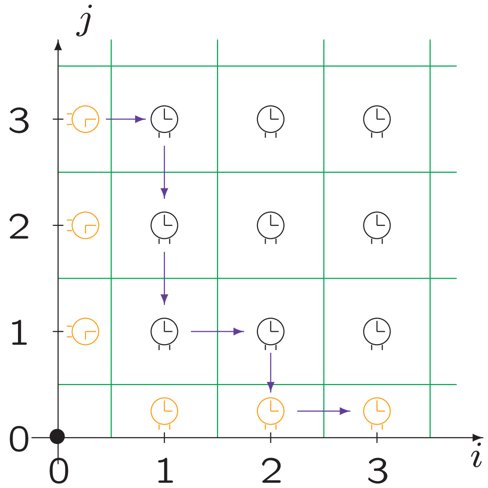
$\rightsquigarrow$ Any fixed southeast path meets *independent* increments

$$I_{ij} \sim \text{Exponential}(1-\varrho) \quad \text{and}$$
$$J_{ij} \sim \text{Exponential}(\varrho).$$

Last passage equilibrium



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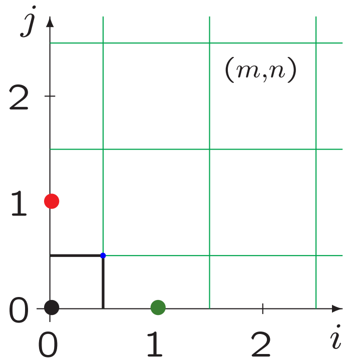
Of course, this doesn't help directly with G_{mn}.

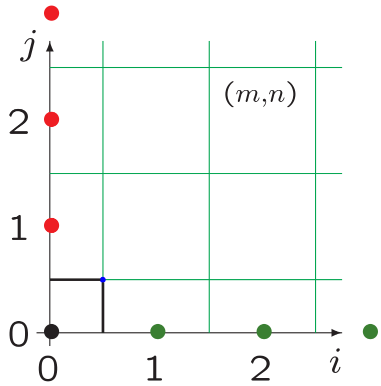
The competition interface



Ferrari, Martin, Pimentel (2005)

Which squares are infected via $(1, 0)$ and via $(0, 1)$?

The competition interface



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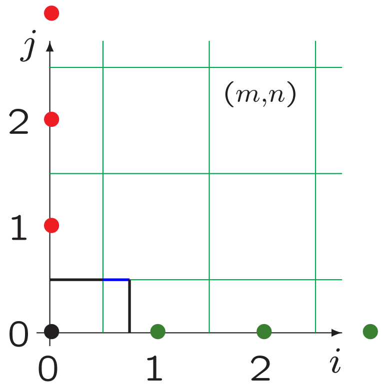

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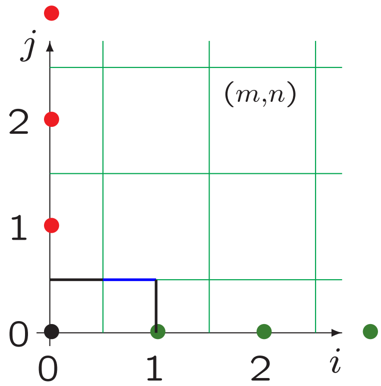

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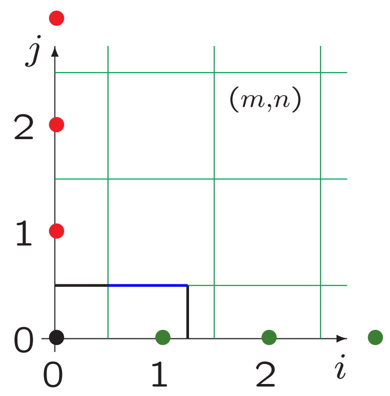

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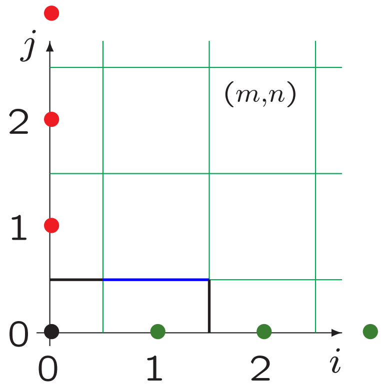

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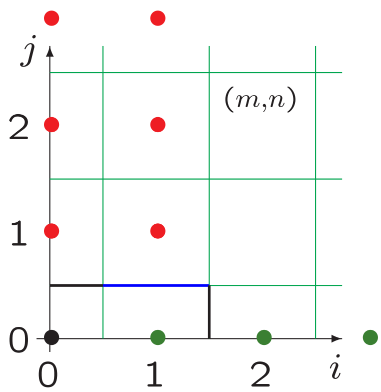

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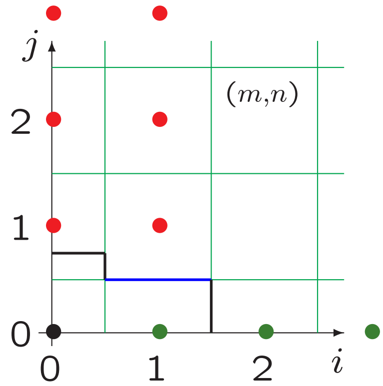

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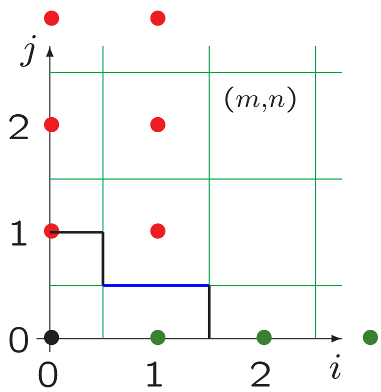

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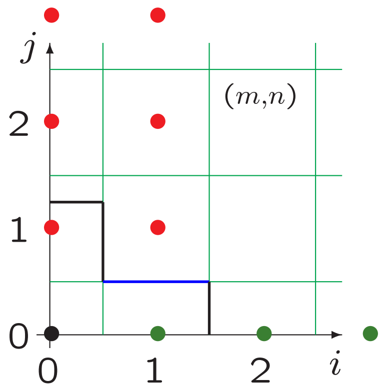

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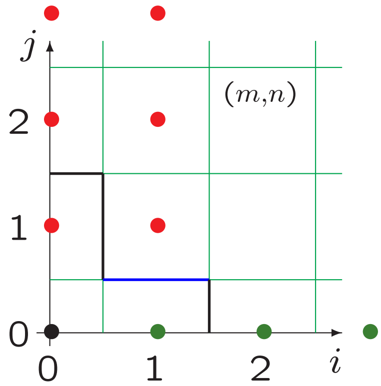

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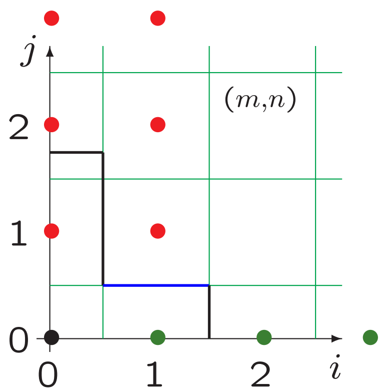

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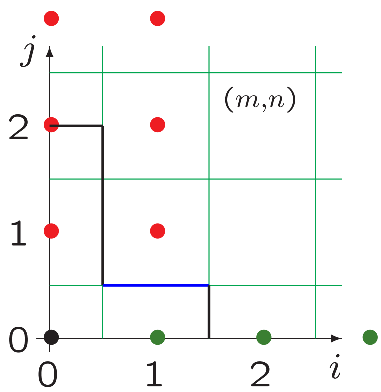

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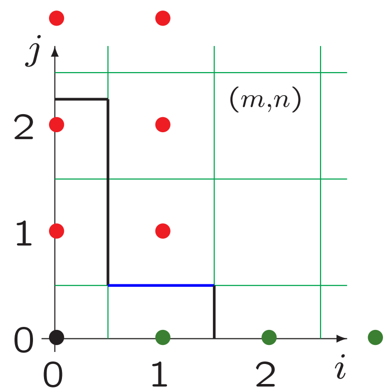

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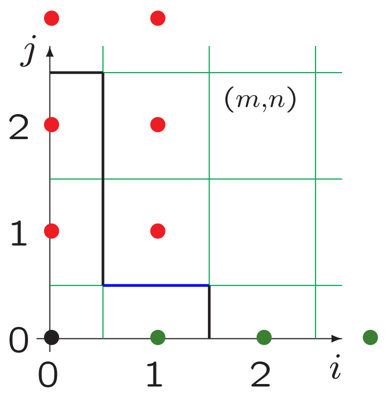

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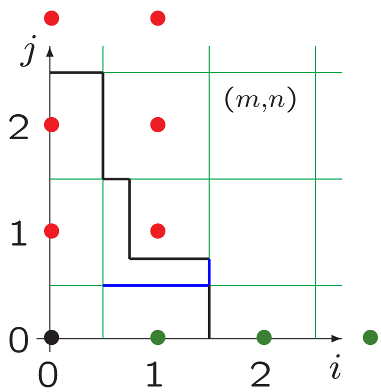

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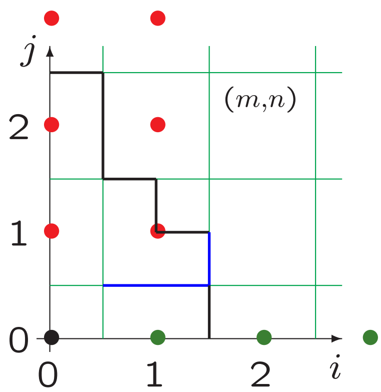

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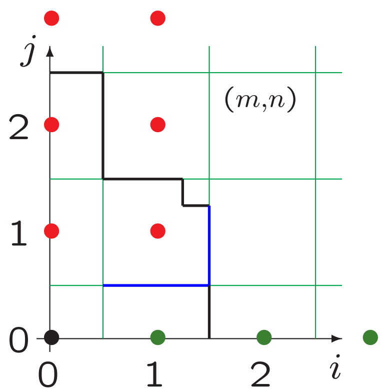

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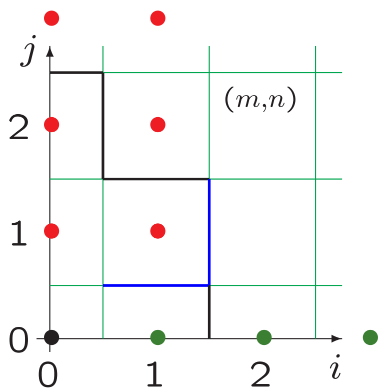

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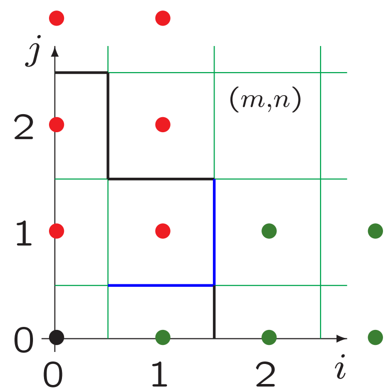

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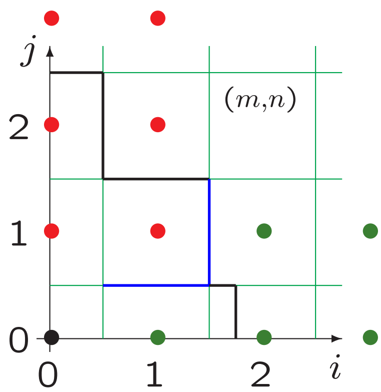
The competition interface



Ferrari, Martin, Pimentel (2005)

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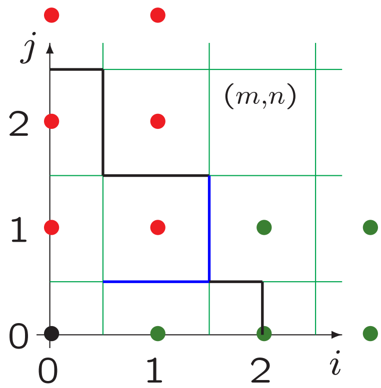

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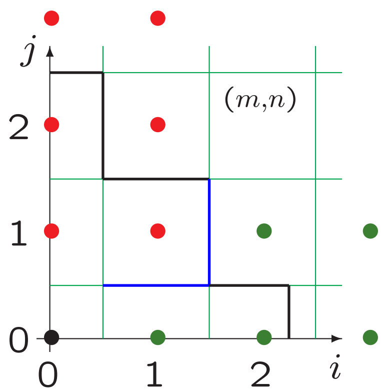

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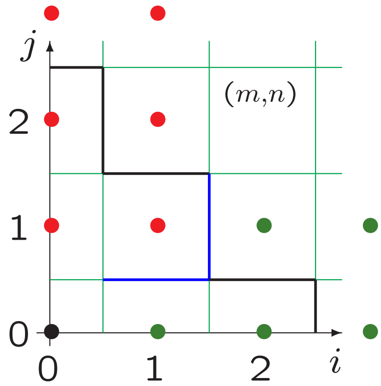

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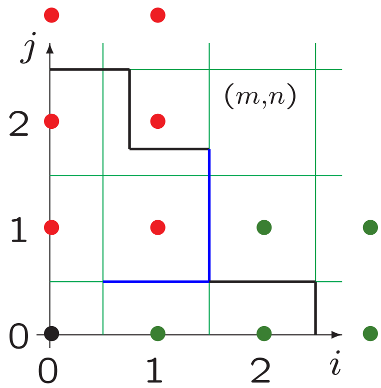

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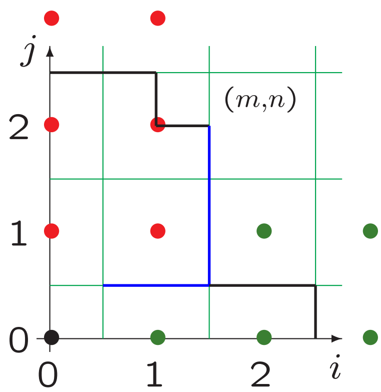

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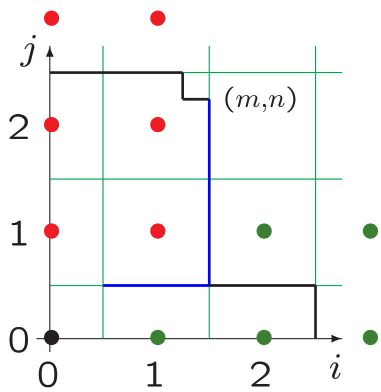

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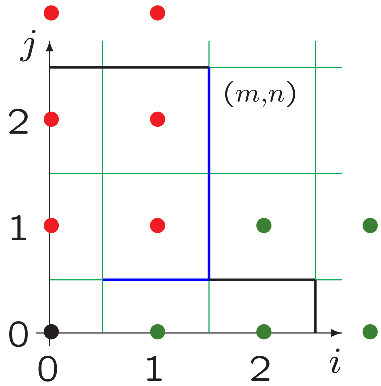

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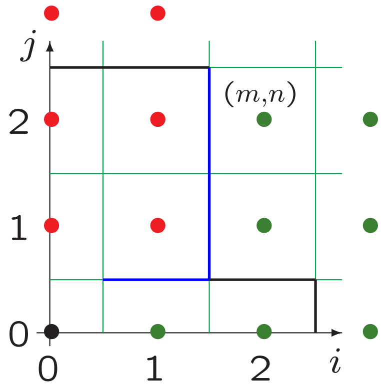

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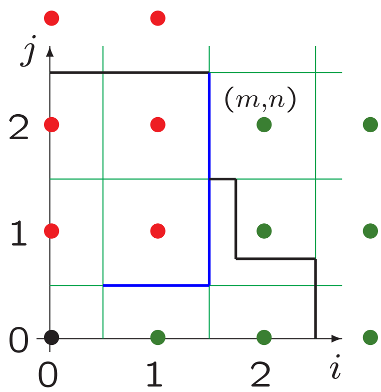

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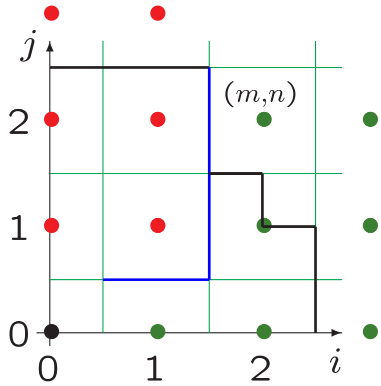

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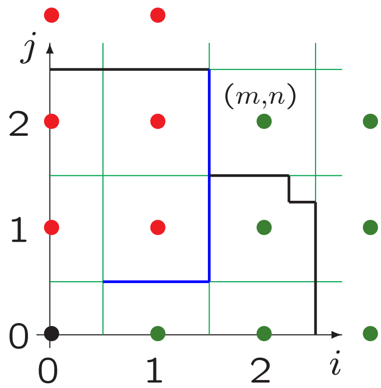

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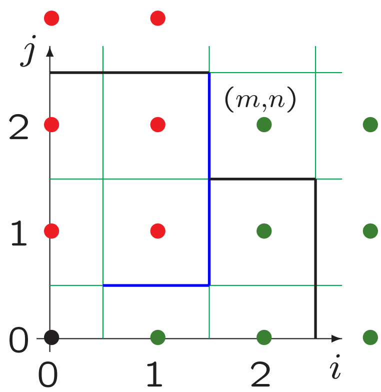

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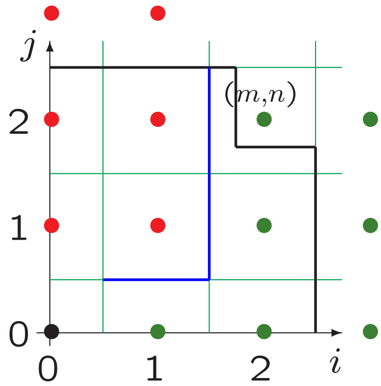

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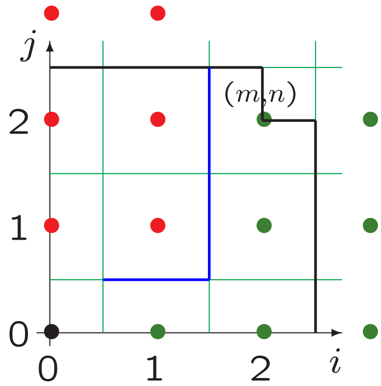

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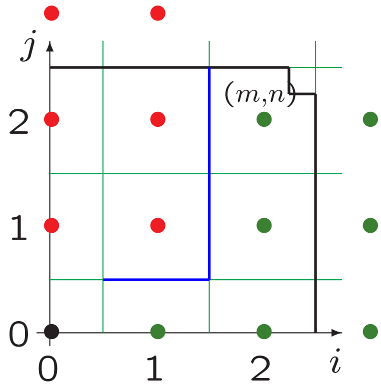

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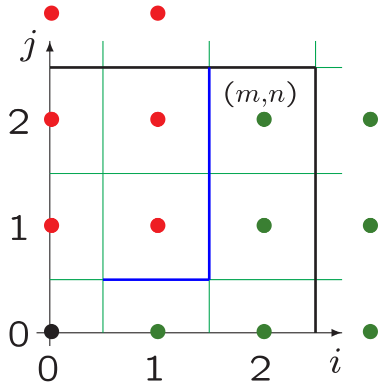

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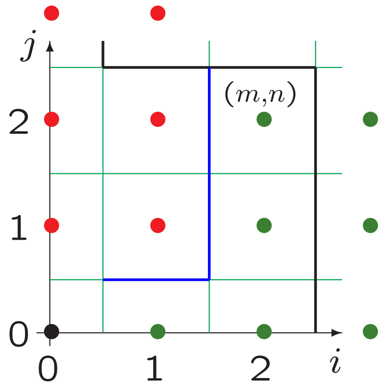

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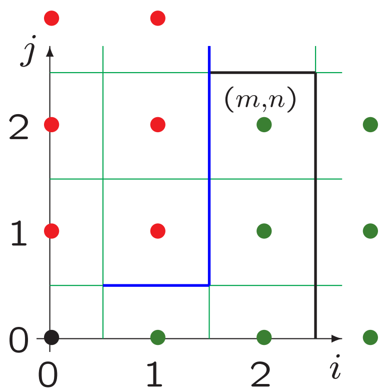

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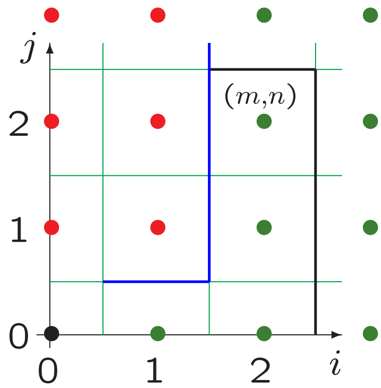

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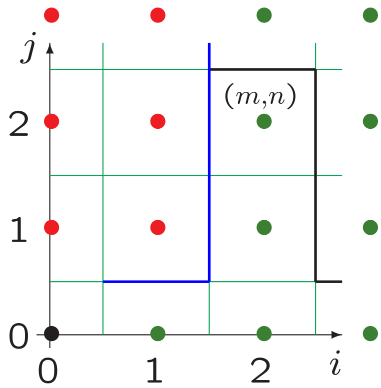

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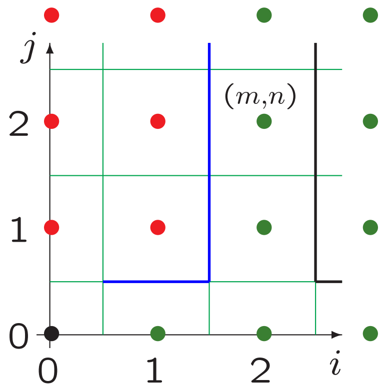

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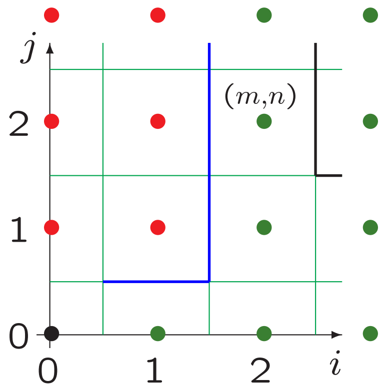

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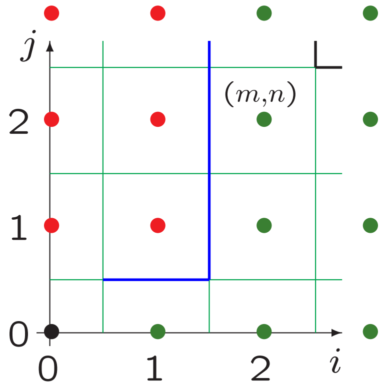

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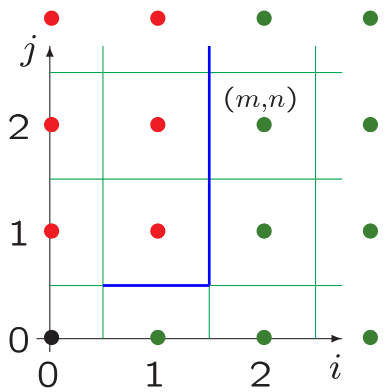

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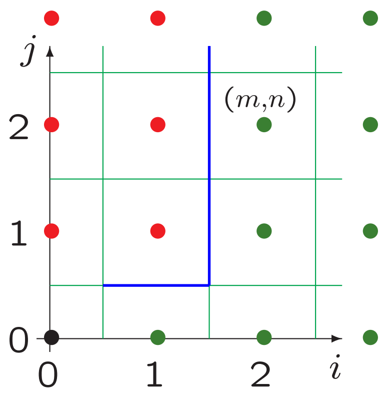

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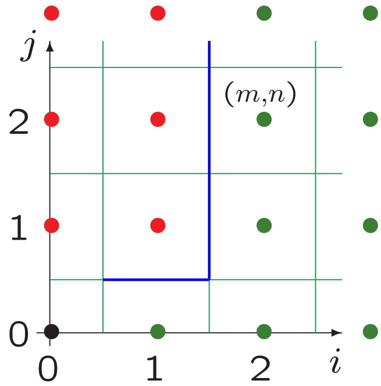
The competition interface follows the same rules as the *second class particle* of simple exclusion.

The competition interface



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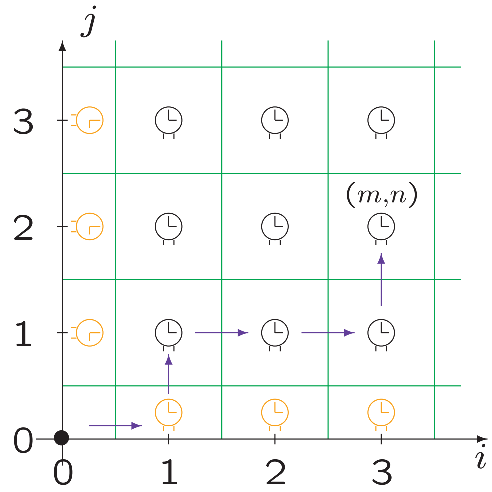
Which squares are infected via $(1,0)$ and via $(0,1)$?

The competition interface follows the same rules as the *second class particle* of simple exclusion.

If it passes left of (m,n), then G_{mn} is not sensitive to decreasing the weights on the j-axis.

If it passes below (m,n), then G_{mn} is not sensitive to decreasing the weights on the i-axis.

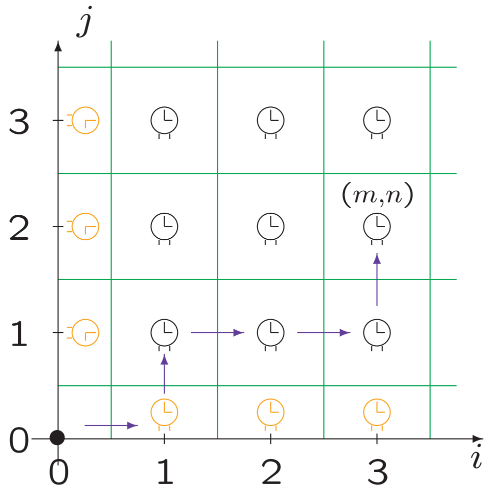
Upper bound (E. Cator and P. Groeneboom)



G^ϱ: weight collected by the longest path.

Z^ϱ: exit point of the longest path.

Upper bound (E. Cator and P. Groeneboom)



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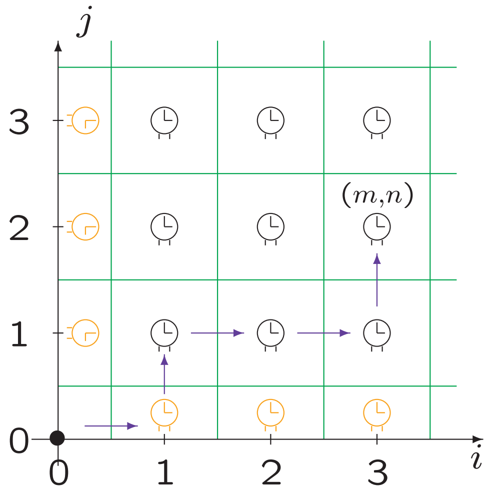

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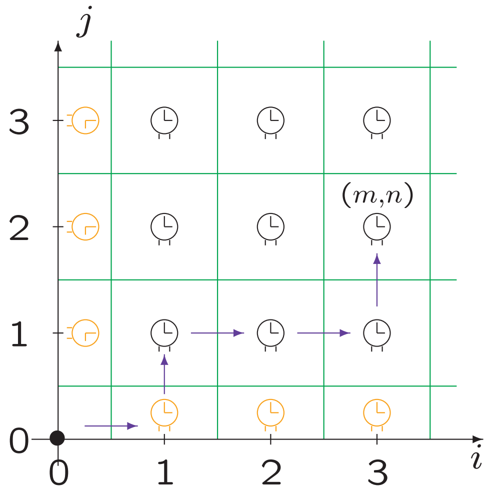

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Step 1:

$$U_z^\lambda + A_z \leq G^\lambda$$

for any z, any $0 < \lambda < 1$.

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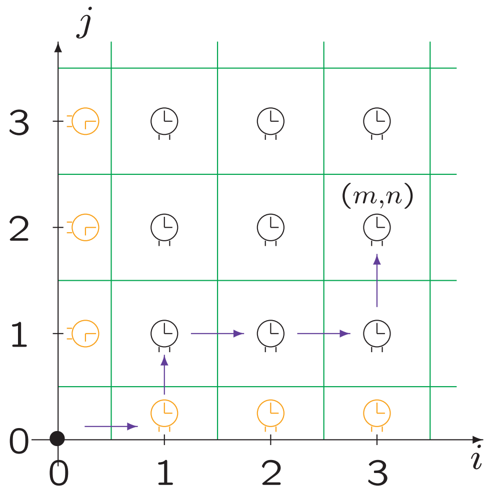

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for any z, any $0 < \lambda < 1$. Fix $u \ge 0$ and $\lambda \ge \varrho$,

$$\mathbf{P}\{Z^\varrho > u\} = \mathbf{P}\{\exists z > u \, : \, U_z^\varrho + A_z(t) = G^\varrho\}$$

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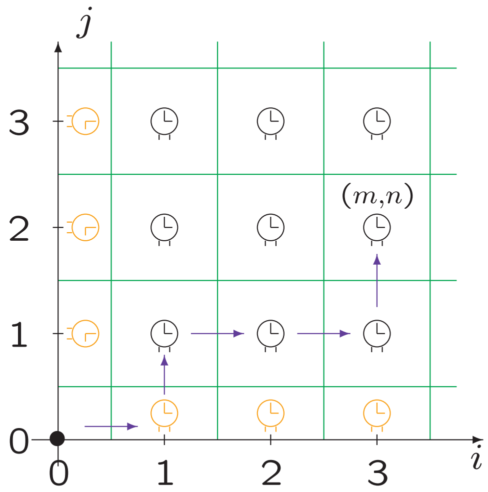

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$$U_z^{\lambda} + A_z \le G^{\lambda}$$

for any z, any $0 < \lambda < 1$. Fix $u \ge 0$ and $\lambda \ge \varrho$,

$$\begin{aligned}\mathbf{P}\{Z^{\varrho} > u\} &= \mathbf{P}\{\exists z > u \,:\, U_z^{\varrho} + A_z(t) = G^{\varrho}\}\\ &\le \mathbf{P}\{\exists z > u \,:\, U_z^{\varrho} - U_z^{\lambda} + G^{\lambda} \ge G^{\varrho}\}\end{aligned}$$

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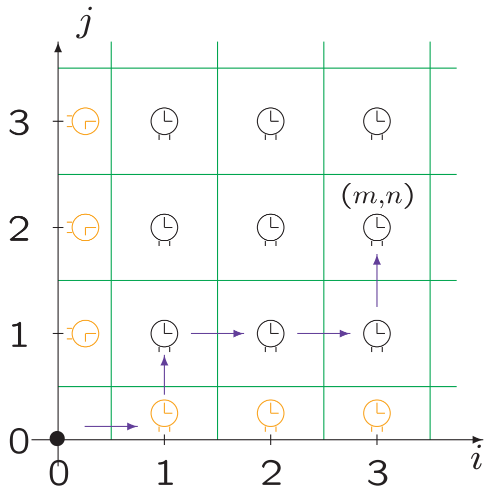

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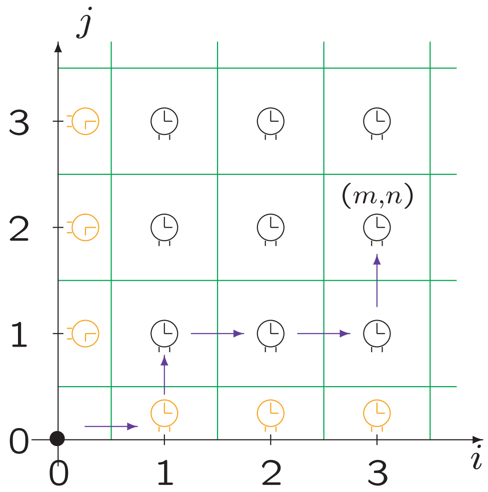

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A_z: largest weight of a path from z to (m, n).

Step 1:

$$U^{\lambda}_{z} + A_z \le G^{\lambda}$$

for any z, any $0 < \lambda < 1$. Fix $u \ge 0$ and $\lambda \ge \varrho$,

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Step 2:

Optimize λ so that $\mathbf{E}(U_u^{\lambda} - G^{\lambda})$ be maximal. (The equilibrium makes it possible to compute the expectation.) This makes the estimate sharp.

$$\mathbf{P}\{Z^{\varrho} > u\} \leq \mathbf{P}\{U_u^{\lambda} - U_u^{\varrho} \leq G^{\lambda} - G^{\varrho}\}.$$

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Apply Chebyshev's inequality on the right-hand side. **Var**(U_u) is elementary.

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Prove, by a perturbation argument, that **Var**(G) is related to $\mathbf{E}(U_{Z+})$.

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Step 5:

A large deviation estimate connects $\mathbf{P}\{Z^\varrho > y\}$ and $\mathbf{P}\{U^\varrho_{Z^\varrho+} > y\}$.

$$\rightsquigarrow \mathbf{P}\{U^\varrho_{Z+} > y\} \le C\Big(\frac{t^2}{y^4} \cdot \mathbf{E}(U^\varrho_{Z^\varrho+}) + \frac{t^2}{y^3}\Big)$$

$$\mathbf{P}\{Z^\varrho > u\} \le \mathbf{P}\{U_u^\lambda - U_u^\varrho \le G^\lambda - G^\varrho\}.$$

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Optimize λ so that $\mathbf{E}(U_u^\lambda - G^\lambda)$ be maximal. (The equilibrium makes it possible to compute the expectation.) This makes the estimate sharp.

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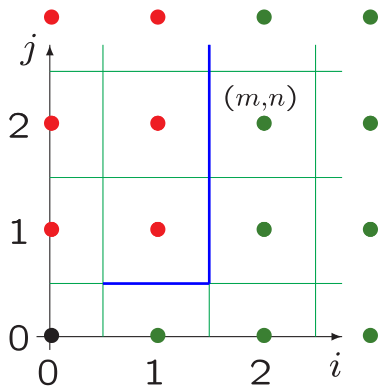
$$\leadsto \mathbf{P}\{U^\varrho_{Z+} > y\} \le C\Big(\frac{t^2}{y^4} \cdot \mathbf{E}(U^\varrho_{Z^\varrho+}) + \frac{t^2}{y^3}\Big)$$

Conclude

$$\limsup_{t\to\infty} \frac{\mathbf{E}(U^\varrho_{Z^\varrho+})}{t^{2/3}} < \infty, \qquad \limsup_{t\to\infty} \frac{\mathbf{Var}(G^\varrho)}{t^{2/3}} < \infty.$$

Time-reversal and the lower bound

(E. Cator and P. Groeneboom)



$\rightsquigarrow$ Z-probabilities are connected to competition interface-probabilities.

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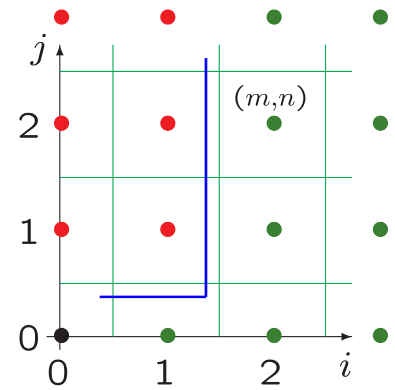

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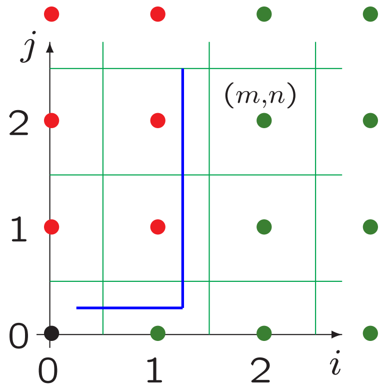

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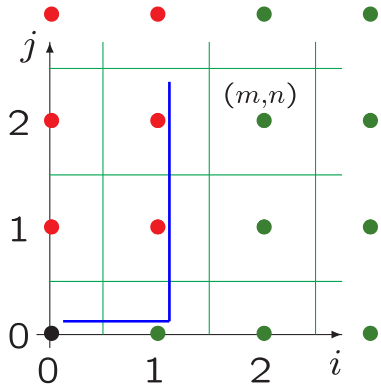

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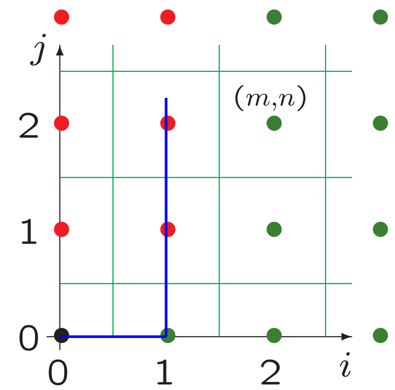

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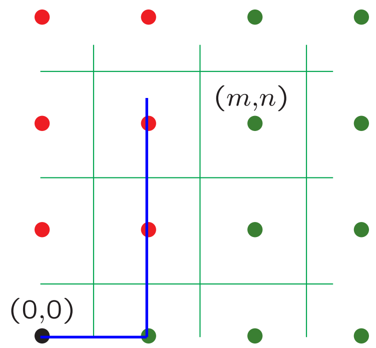

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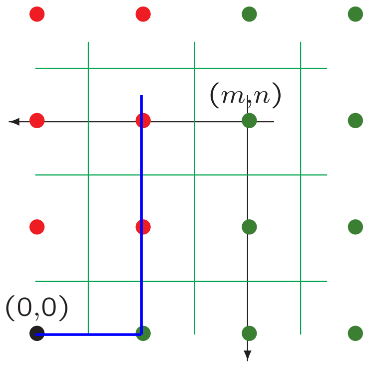

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Time-reversal and the lower bound

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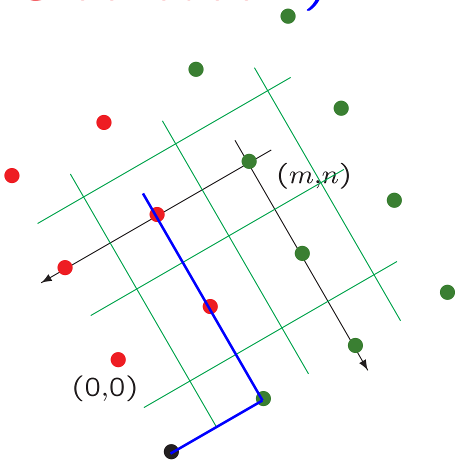

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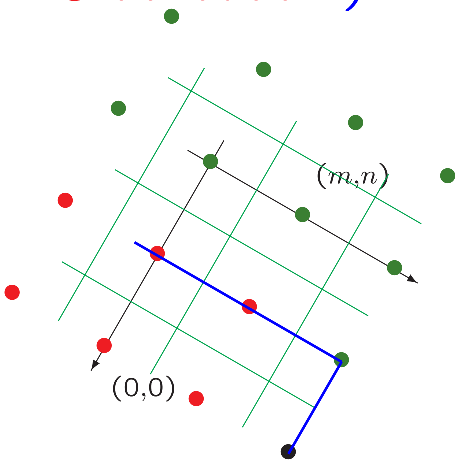

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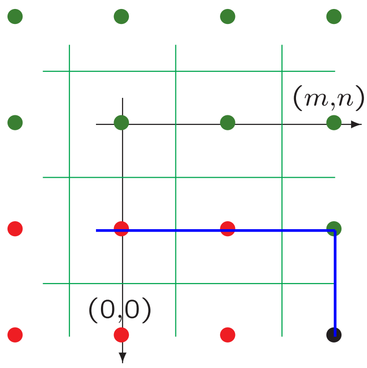

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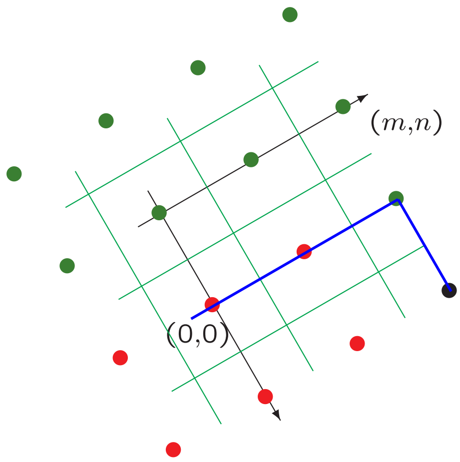

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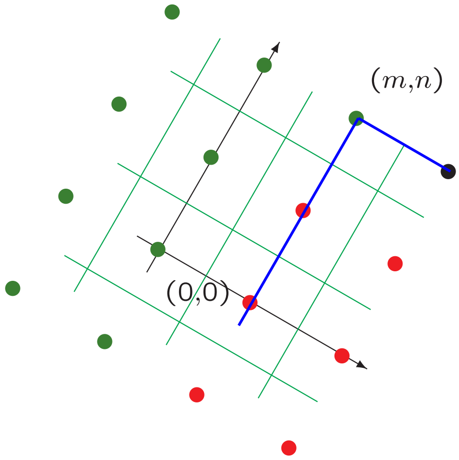

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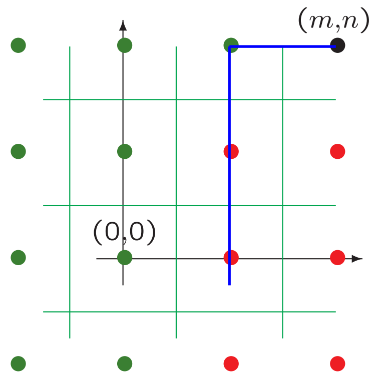

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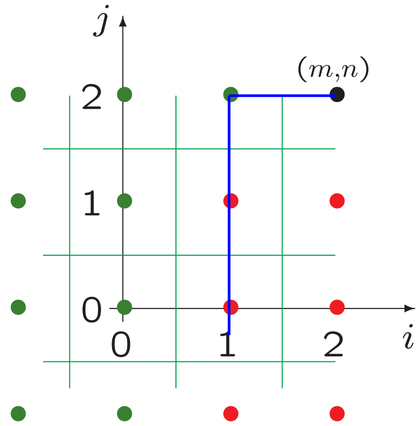

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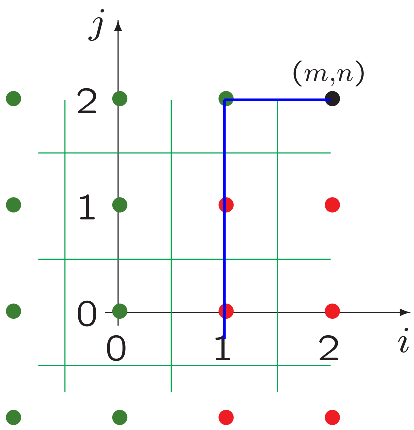

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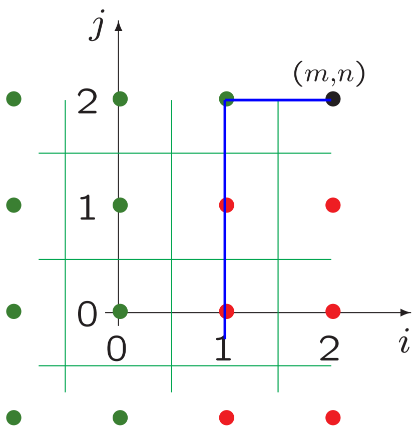

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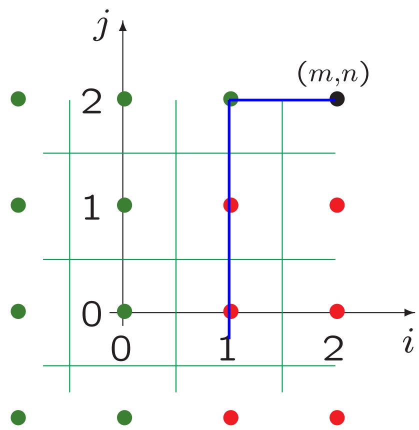

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Conclude

$$\liminf_{t\to\infty} \frac{\mathbf{E}(U^{\varrho}_{Z^{\varrho}+})}{t^{2/3}} > 0, \qquad \liminf_{t\to\infty} \frac{\mathbf{Var}(G^{\varrho})}{t^{2/3}} > 0.$$

Further directions

We managed to drop the last passage picture and repeat these arguments directly in the asymmetric simple exclusion process.

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Thank you.