# Group Walk Random Graphs

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*Bristol, 13/2/17*

*Partly joint work with J. Haslegrave and with V. Kaimanovich*

# Geometric Random Graphs Literature

[*Remco Van Der Hofstad. Random graphs and complex networks. Lecture Notes, 2013.*]

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Random planar graphs ...

Percolation theory ...

# How Mafia's grow

A network evolves in (continuous or discrete) time with the following rules:

- When a (Poisson) clock ticks, nodes split into two;
- When a node $x$ splits into two nodes $x', x''$, each of its existing edges gets inherited by $x'$ or $x''$ independently with probability 1/2;
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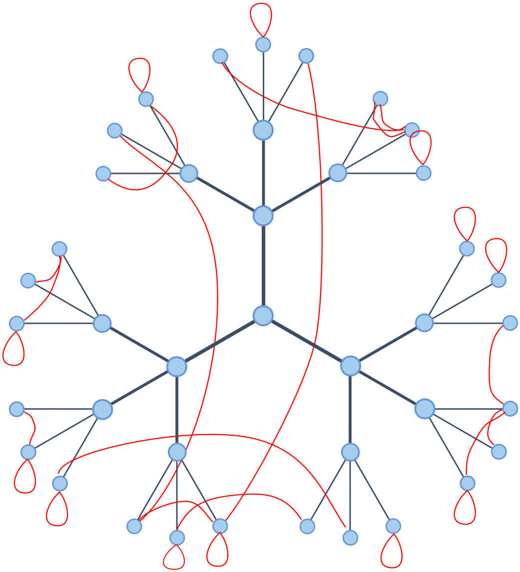
If it is finite, is its expected size finite or infinite?
If finite, how does it depend on k?

# Random Graphs from trees



$R^1_3(T)$

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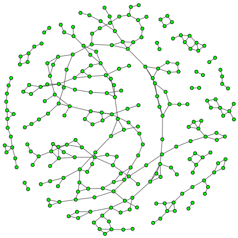

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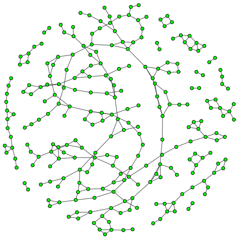
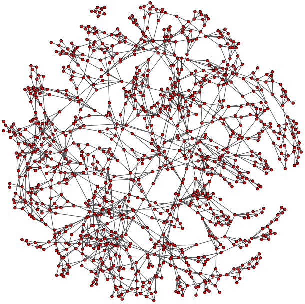
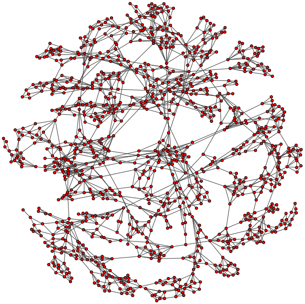
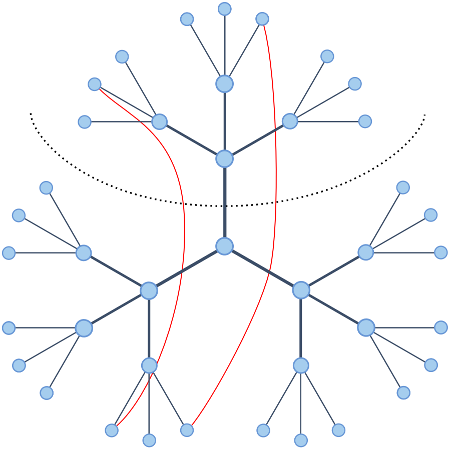






# A nice property



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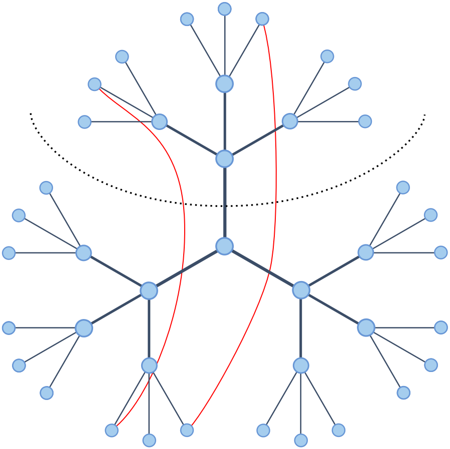

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$\mathbb{E}(\sharp$ *edges xy in* $R_n$ *with x in X and y in Y*$)$

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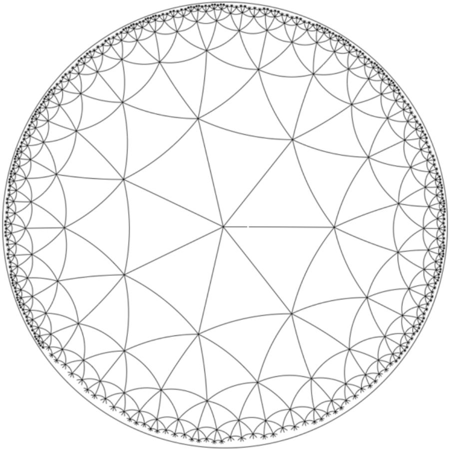

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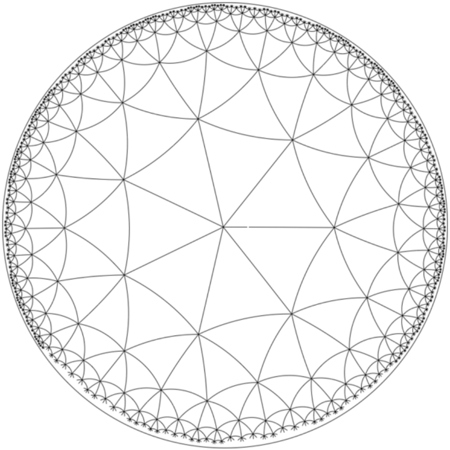

**Proposition**

*$\mathbb{E}(\sharp$ edges $xy$ in $R_n$ with $x$ in $X$ and $y$ in $Y)$*

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The classical Poisson formula

$$h(z) = \int_0^1 \hat{h}(\theta) P(z,\theta) d\theta$$

where $P(z,\theta) := \frac{1-|z|^2}{|e^{2\pi i\theta}-z|^2}$,
recovers every continuous harmonic function $h$ on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

# The Poisson integral representation formula

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- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;
- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$.

# The Poisson-Furstenberg boundary

Selected work on the Poisson boundary

- Introduced by Furstenberg to study semi-simple Lie groups [*Annals of Math. '63*]
- Kaimanovich & Vershik give a general criterion using the entropy of random walk [*Annals of Probability '83*]
- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria [*Annals of Math. '00*]

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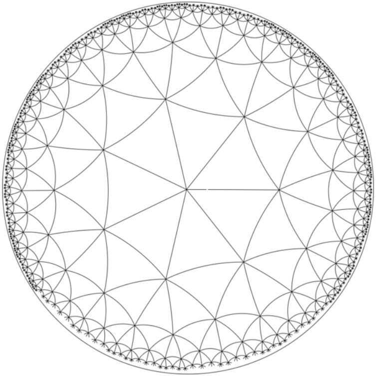

**Proposition**

*For every two measurable subsets $X, Y$ of the Poisson (or Martin) boundary $\partial G$,*

$$\mathbb{E}(\sharp \text{ edges } xy \text{ in } R_n \text{ with } x \text{ 'close to' } X \text{ and } y \text{ 'close to' } Y)$$

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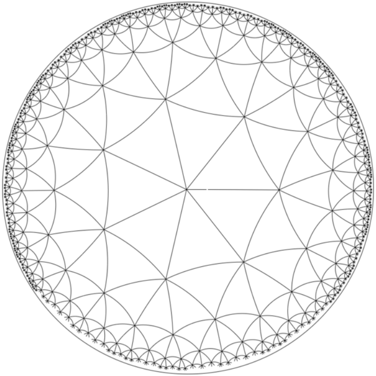

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We use the limit to define a measure on $\partial G \times \partial G$ via

$$C(X, Y) := lim \mathbb{E}(\sharp \text{ edges } \ldots)$$

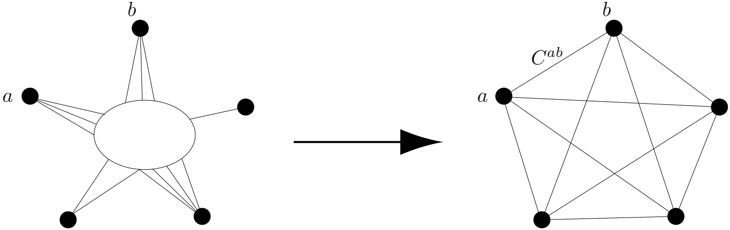
# Energy and Douglas' formula

The classical Douglas formula [Douglas '31]

$$E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(\zeta, \eta) d\eta d\zeta$$

calculates the (Dirichlet) energy of a harmonic function $h$ on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

# Energy in finite electrical networks

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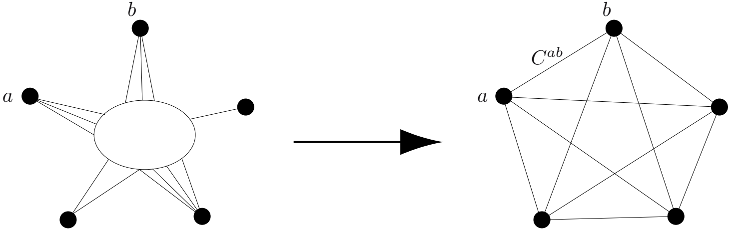


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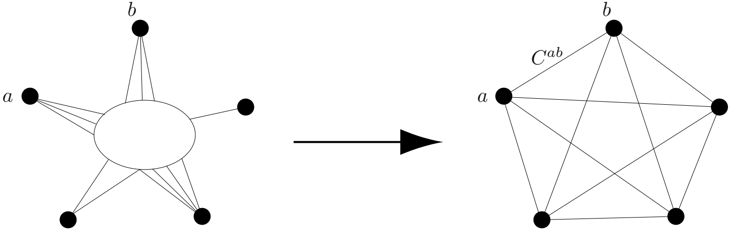


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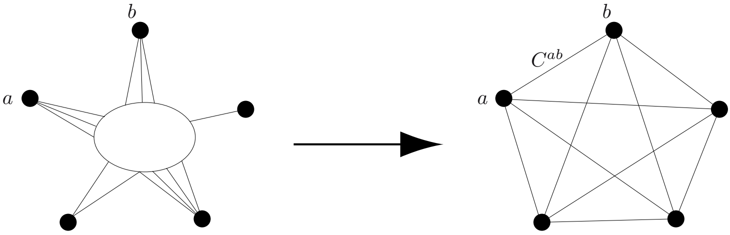


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How can we generalise this to an arbitrary domain?
To an infinite graph?

# Effective conductance

We call $C$ the *effective conductance measure*, because

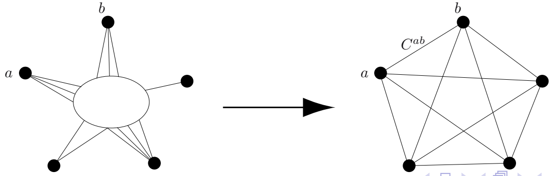
**Theorem (G & V. Kaimanovich '12-'17+)**

*For every locally finite network G, and every harmonic function h, we have*

$$E(h) = \int_{\partial G \times \partial G} \left(\widehat{h}(\eta) - \widehat{h}(\zeta)\right)^2 dC(\eta, \zeta).$$

History: Douglas '31, Naim '57, Doob '62, Silverstein '74

Finite version: $E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 C_{ab}$



# The Naim Kernel

Doob's formula:

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where the **Naim Kernel** $\Theta$ is defined as

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**Problem:** Let $(z_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ be independent simple random walks from $o$. Then $\lim_{n,m\to\infty}\Theta(z_n,y_m)$ exists almost surely.

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**Theorem (G & Kaimanovich '17+)**

*For every transient, locally finite graph G,*

$$C(X, Y) = \nu(1_{XY} W^*).$$

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How large is $R_\infty^\lambda(T)$?

# The expected size of the TWRG

Let $C_o^\lambda$ denote the component of a uniformly random vertex of $R_n^\lambda(T)$ (or $R_\infty^\lambda(T)$).

**Theorem (G & Haslegrave, state of the art 2/17)**

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Conjecture:

$$\mathbb{E}(|C_o^\lambda|) \sim \lambda^\lambda$$

(backed by simulations)

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